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ON CERTAIN EVEN CANONICAL SURFACES

KAZUHIRO KONNO

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Abstract. We classify even canonical surfaces on the Castelnuovo lines, and show that the moduli space is non-reduced in many cases. We show that, in most cases, the rational map associated with a semi-canonical bundle induces a linear pencill of nonhyperelliptic curves of genus three, and that a nonsingular rational curve with self-intersection number -2 appears as a fixed component of the semi-canonical system. By the latter, we can apply a result of Burn and Wahl to show that they are obstructed surfaces.

Introduction. According to [8], we call a minimal surface a *canonical surface* if the canonical map induces a birational map onto its image. Canonical surfaces with $c_1^2=3p_g-7$ and $3p_g-6$ were studied in our previous papers [1] and [10] (see also [4] and [8]). These are regular surfaces whose canonical linear system $|K|$ has neither fixed components nor base points.

In this article, we list up those which are even surfaces in order to supplement [1] and [10]. Here, we call a compact complex manifold of dimension 2 an *even surface* if its second Steifel-Whitney class w_2 vanishes ([8]). This topological condition implies the existence of a line bundle L with $K=2L$. In a recent paper [9], Horikawa classified all the even surfaces with $p_g=10$, $q=0$ and $K^2=24$ (numerical sextic surfaces). Following [9], we consider the rational map Φ_L associated with $|L|$ also in the remaining cases. Recall that most canonical surfaces with $c_1^2=3p_g-7$, $3p_g-6$ have a pencil $|D|$ of nonhyperelliptic curves of genus 3. Therefore, it is naturally expected that Φ_L should be composed of such a pencil. We show that this is the case, except for numerical sextic surfaces. Let $f\colon S\to \boldsymbol{P}^1$ be the corresponding fibration. It turns out that the fact that S is an even surface forces $f_*\mathcal{O}(K)$ to be very special (Lemmas 1.2 and 2.2). Using this, we can determine the fixed part Z of $|L|$. The remaining problem is to write down the equation of the canonical model. When $K^2=3p_g-7$, we have no difficulty in doing this, since the (relative) canonical image itself is the canonical model. On the other hand, when $K^2=3p_g-6$, we need to study the bi-graded ring $\bigoplus H^0(\alpha D+\beta Z)$ as in [9]. The calculation after Lemma 2.3 is a verbatim translation of [9].

As a by-product, we find that the moduli space is non-reduced in many cases (Theorems 1.5 and 2.5). The point is the presence of a (-2)-curve contained in Z. Then a general result of Burns and Wahl [3] can be applied to show that the Kuranishi space is everywhere singular. As far as surfaces of general type are concerned, such pathological

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examples were first obtained by Horikawa [7] and, later, by Miranda [11]. These two are put together in a remarkable paper of Catanese [5], where we can find many other obstructed surfaces.

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1. The case $c_1^2=3p_g-7$. To simplify the notation, for any divisor Z on a surface S, we write $H^i(Z)$ instead of $H^i(S, \mathcal{O}([Z]))$ and put $h^i(Z)=\dim H^i(Z)$. Divisors and line bundles will be treated interchangeably.

In this section, let S denote an even canonical surface with $c_1^2=3p_g-7$. Recall that it is a regular surface whose canonical system $|K|$ is free from base points [1, §1]. Since $w_2=0$, we can find a line bundle L on S which satisfies $K=2L$. Since L^2 is a positive even integer and since $K^2=4L^2=3p_g-7$, there exists a positive integer n safisfying

$$L^2=6n+2\,, \qquad p_g=8n+5\,. \tag{1.1}$$

Since $K=2L$, it follows from the Riemann-Roch theorem and the Serre duality that

$$2h^0(L)-h^1(L)=-\frac{1}{2}L^2+\chi(\mathcal{O}_S)=5n+5\,. \tag{1.2}$$

In particular, we get

$$h^0(L)\geq\frac{5}{2}(n+1)\,. \tag{1.3}$$

We put $m=h^0(L)-1$ and consider the rational map $\Phi_L: S\to \boldsymbol{P}^m$ induced by the complete linear system $|L|$.

LEMMA 1.1. *Φ_L is composed of a pencil of nonhyperelliptic curves of genus* 3.

PROOF. Suppose that Φ_L induces a generically finite map onto its image V. Since V is a nondegenerate surface in $\boldsymbol{P}^m$, we have $\deg V\geq m-1$. We consider Φ_L as a rational map of S onto V. Then, it follows from (1.3) that

$$6n+2=L^2\geq \deg V \deg \Phi_L \geq \frac{1}{2}(5n+1)\deg \Phi_L\,.$$

Therefore, we get $\deg\Phi_L\leq 2$. If $\deg\Phi_L=1$, then we have $p_g=h^0(2L)\geq 4h^0(L)-6$ (see, e.g., [6, Proposition 3.1]). This is impossible by (1.1) and (1.3). If $\deg\Phi_L=2$, then we have $\deg V\leq L^2/2=3n+1$. It follows from (1.3) that $\deg V<2m-2$. Therefore, V is birationally equivalent to a ruled surface by [2, Lemma 1.4]. This is impossible, since S is a canonical surface. Therefore, Φ_L is composed of a pencil. Since S is a regular surface, it is a linear pencil.

Put $|L|=|mD|+Z$, where $|D|$ is an irreducible pencil and Z is the fixed part of $|L|$. Since we have

$$6n+2=L^2=mLD+LZ\geq mLD\geq\frac{1}{2}(5n+3)LD\ ,$$

we get $LD\leq 2$. Since $LD=mD^2+DZ$ and $m\geq 4$, it follows that $D^2=0$. If $LD=1$, then $|D|$ is a pencil of curves of genus 2. This contradicts the assumption that S is canonical. Therefore, we have $LD=2$ and see that $|D|$ is a pencil of curves of genus 3 which must be of nonhyperelliptic type. q.e.d.

Let $f\colon S\to\boldsymbol{P}^1$ denote the holomorphic map induced by $|D|$. Put $f_*\mathcal{O}(K)=\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c)$, where a, b, c are integers satisfying

(1.4) $$0\leq a\leq b\leq c\ ,\qquad a+b+c=p_g-3\ .$$

These integers can be characterized as follows: Let D be a general fiber of f, and consider the restriction map

$$\rho_i\colon H^0(K-iD)\to H^0(D, K_D)$$

for any integer i. Then a is the greatest integer among those i's such that ρ_i is surjective. Note that ρ_0 is surjective since S is a canonical surface. Therefore, $a\geq 0$. c is the greatest integer among those i's such that ρ_i is a nonzero map, and b is the greatest integer among those i's such that we can find three sections $x_0\in H^0(K-aD)$, $x_1\in H^0(K-iD)$ and $x_2\in H^0(K-cD)$ which induce a basis for $H^0(K_D)$.

LEMMA 1.2. *$a=n-1$, $b=2n$ and $c=2m=5n+3$. In particular, $h^0(L)=5(n+1)/2$ and $h^1(L)=0$.*

PROOF. From [1, Claim III], we see that a, b and c further satisfy

(1.5) $$a+c\leq 3b+2\ ,\qquad b\leq 2a+2\ .$$

We have $K=2L=[2mD+2Z]$. Since c is the greatest integer with $H^0(K-cD)\neq 0$, we have $c\geq 2m$. Hence, it follows from (1.3), (1.4) and (1.5) that

$$a=n-1\ ,\quad b=2n\ ,\quad c=2m=5n+3\ .$$

In particular, the equality holds in (1.3). Then we get $h^1(L)=0$ by (1.2). q.e.d.

By this lemma, we know that n is an odd integer. Therefore, we can find a positive integer k with $n=2k-1$. Then $L=[(5k-1)D+Z]$. Furthermore, we have

(1.6) $$LZ=2k-2\ ,\quad DZ=2\ ,\quad Z^2=-8k\ ,$$

by the proof of Lemma 1.1.

LEMMA 1.3. *$Z=2G$, where G is a nonsingular rational curve satisfying $DG=1$, $LG=k-1$, $G^2=-2k$.*

PROOF. Let T and F respectively denote a tautological divisor and a fiber of

$$W=\boldsymbol{P}(\mathcal{O}(2k-2)\oplus\mathcal{O}(4k-2)\oplus\mathcal{O}(10k-2))\to\boldsymbol{P}^1 .$$

We can choose sections X_0, X_1 and X_2 of $[T-(2k-2)F]$, $[T-(4k-2)F]$ and $[T-(10k-2)F]$, respectively, in such a way that they form a system of homogeneous coordinates on each fiber of $W\to\boldsymbol{P}^1$. We let (z_0, z_1) denote a system of homogeneous coordinates on the base curve $\boldsymbol{P}^1$. Since $|K|$ is free from base points, we get a natural holomorphic map $g: S\to W$ over $\boldsymbol{P}^1$, the (relative) canonical map, which satisfies $K=f^*T$. Put $S'=g(S)$. Then S' is linearly equivalent to $4T-(p_g-5)F$ (see [1,§1]). The equation of any member of $|4T-(p_g-5)F|$ can be written as

$$\begin{aligned}(1.7)\qquad &\phi X_1^4+X_2(\phi_0X_0^3+\phi_{2k}X_0^2X_1+\phi_{4k}X_0X_1^2+\phi_{6k}X_1^3+\phi_{8k}X_0^2X_2\\ &+\phi_{10k}X_0X_1X_2+\phi_{12k}X_1^2X_2+\phi_{16k}X_0X_2^2+\phi_{18k}X_1X_2^2+\phi_{24k}X_2^3)=0 ,\end{aligned}$$

where ϕ is a constant and ϕ_i is a homogeneous form of degree i in z_0, z_1. If it defines S', then it follows from the proof of Claim III in [1,§2] that ϕ and ϕ_0 are both nonzero constants. Furthermore, S' has at most rational double points ([1, §1]).

From the above equation, we know that S' contains a rational curve B defined in W by $X_1=X_2=0$. Note that, in a neighbourhood of B in S', S' is nonsingular, B is defined by $X_1=0$, and X_2 vanishes to the fourth order along B. We denote by G the inverse image of B by g. Since $K=[g^*T]=[(10k-2)D+g^*(X_2)]$, we have $2Z=g^*(X_2)$. Therefore, Z is of the form $Z=2G$. We clearly have $DG=1$. q.e.d.

If the coefficients ϕ's are sufficiently general, (1.7) defines a nonsingular minimal surface with $c_1^2=3p_g-7$ which is even.

We have shown the following:

THEOREM 1.4. *For any even canonical surface S with* $c_1^2=3p_g-7$, *there exists a positive integer k satisfying* $p_g=16k-3$. *Furthermore, S is the minimal resolution of a surface S′ with only rational double points which is defined in* $\boldsymbol{P}(\mathcal{O}(2k-2)\oplus\mathcal{O}(4k-2)\oplus\mathcal{O}(10k-2))$ *by Equation* (1.7).

It would be worth stating here the following:

THEOREM 1.5. *The moduli space of even canonical surfaces with* $p_g=13$ *and* $c_1^2=32$ *is non-reduced.*

PROOF. Let S be an even canonical surface with the above numerical invariants. By Lemma 1.3, S has a (-2)-curve G. In order to show the assertion, it is sufficient to show that its Kuranishi space M is singular at S. Note that, since S is canonical and even, every S_t, $t\in M$, enjoys the same properties. Then, as we have seen in Lemma 1.3, S_t contains a (-2)-curve G_t. However, a result of Burns and Wahl [2] tells us that a general vector in $H^1(S, \Theta_S)$ kills every (-2)-curve on S, where Θ_S denotes the tangent sheaf of S. Since $H^1(\Theta_S)$ is nothing but the Zariski tangent space of M, we see that $\dim M$ is strictly smaller than $h^1(\Theta_S)$. Therefore, M cannot be nonsingular at S. q.e.d.

2. The case $c_1^2=3p_g-6$. In this section, we denote by S an even canonical surface with $c_1^2=3p_g-6$. Put $K=2L$ as before. Then we can find a positive integer n satisfying

(2.1) $$L^2=6n\,, \qquad p_g=8n+2\,.$$

If $n=1$, such surfaces are numerical sextic surfaces which are completely classified in [9]. Therefore, we assume $n\geq 2$ in the following.

By the Riemann-Roch theorem, we have

(2.2) $$2h^0(L)-h^1(L)=5n+3\,.$$

In particular, we get

(2.3) $$h^0(L)\geq\frac{1}{2}(5n+3)\,.$$

The following can be shown in the same way as in Lemma 1.1.

LEMMA 2.1. *If $n\geq 2$, then Φ_L is composed of a pencil of nonhyperelliptic curves of genus* 3.

Put $|L|=|mD|+Z$, where $|D|$ is a pencil of nonhyperelliptic curves of genus 3 and Z is the fixed part of $|L|$. Then we have $LD=2$, $D^2=0$. As in §1, let $f\colon S\to \boldsymbol{P}^1$ denote the holomorphic map induced by $|D|$, and put $f_*\mathcal{O}(K)=\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c)$. The integers a, b and c satisfy (1.4). By [10, Lemma 9.3], these further satisfy

(2.4) $$a+c\leq 3b+3\,, \qquad b\leq 2a+2\,.$$

The following can be shown in the same way as in Lemma 1.2 by using (2.4) instead of (1.5).

LEMMA 2.2. *$a=n-1$, $b=2n-1$ and $c=2m=5n+1$. In particular, $h^0(L)=(5n+3)/2$ and $h^1(L)=0$.*

Let k be an integer with $n=2k-1$, $k\geq 2$. Then $L=[(5k-2)D+Z]$ and we have

(2.5) $$LZ=2k-2\,, \quad DZ=2\,, \quad Z^2=-8k+2\,.$$

Though the proof of the following is quite similar to that of Lemma 1.3, we shall need some results in [10].

LEMMA 2.3. *$Z=2G_0+G_1$, where the G_i are nonsingular rational curves satisfying*

$$DG_0=1,\ LG_0=k-1,\ G_0^2=-2k;\quad DG_1=LG_1=0,\ G_1^2=-2\,.$$

PROOF. As in §1, let T and F respectively denote a tautological divisor and a fiber of

$$W=\boldsymbol{P}(\mathcal{O}(2k-2)\oplus\mathcal{O}(4k-3)\oplus\mathcal{O}(10k-4))\to\boldsymbol{P}^1 .$$

We choose sections X_0, X_1 and X_2 of $[T-(2k-2)F]$, $[T-(4k-3)F]$ and $[T-(10k-4)F]$, respectively, so that they form a system of homogeneous coordinates on each fiber of $W\to\boldsymbol{P}^1$. We let (z_0, z_1) denote a system of homogeneous coordinates on $\boldsymbol{P}^1$. Since $|K|$ is free from base points, we get a natural holomorphic map $g: S\to W$ over $\boldsymbol{P}^1$ which satisfies $K=f^*T$. Put $S'=g(S)$. Then S' is linearly equivalent to $4T-(p_g-6)F$ (see [10, §6]). Using X_i's, the equation of any member of $|4T-(p_g-6)F|$ can be written as

$$\begin{aligned}\phi_0X_1^4+X_2(&\phi_2X_0^3+\phi_{2k+1}X_0^2X_1+\phi_{4k}X_0X_1^2+\phi_{6k-1}X_1^3+\phi_{8k}X_0^2X_2\\&+\phi_{10k-1}X_0X_1X_2+\phi_{12k-2}X_1^2X_2+\phi_{16k-6}X_0X_2^2+\phi_{18k-3}X_1X_2^2\\&+\phi_{24k-4}X_2^3)=0\,,\end{aligned}$$

where ϕ_i is a homogeneous form of degree i in z_0, z_1. If it defines S', then it follows from [10, Lemma 9.3] that ϕ_0 and ϕ_2 are not identically zero. Furthermore, S' has only rational double points except for a unique fiber which is a double conic curve (cf. [8] and [10, §9]).

Let G_0 denote the proper inverse image of the rational curve B defined in W by $X_1=X_2=0$. Note that, on S', B is defined by $X_1=0$ in a neighbourhood of its generic point. We have $2Z=g^*(X_2)$. Therefore, the above equation shows that Z is of the form $Z=2G_0+Z'$. It is clear that we have $DG_0=1$. Then we get $DZ'=0$ by (2.5). We have $LG_0\leq k-1$ by $2k-2=LZ=2LG_0+LZ'$. Combining this with $LG_0=((5k-2)D+Z)G_0=5k-2+2G_0^2+G_0Z'$, we get $G_0^2\leq -2k$. Since G_0 is a nonsingular rational curve, we have $KG_0+G_0^2=-2$. From this, we get $G_0^2=-2-2LG_0\geq -2k$. In sum, we get $G_0^2=-2k$, $LG_0=k-1$, $G_0Z'=1$ and $LZ'=0$.

Since $KZ'=2LZ'=0$, Z' consists of (-2)-curves. Let G_1 denote the unique irreducible component of Z' with $G_0G_1=1$. We have $0=LZ'=2G_0Z'+(Z')^2$, that is, $(Z')^2=-2$. Since $0=LG_1=2G_0G_1+G_1^2+G_1(Z'-G_1)$, it follows that $G_1(Z'-G_1)=0$. Hence, we get $(Z'-G_1)^2=0$. Then, Hodge's index theorem shows $Z'=G_1$. q.e.d.

In order to write down the equation of S' explicitly, we follow an idea in [9] to study the bi-graded ring $\bigoplus H^0(\alpha D+\beta Z)$. Though the computation is essentially the same as in [9], we collect it for the sake of completeness.

Let G_i be defined by $\zeta_i\in H^0(G_i)$, $0\leq i\leq 1$, and put $\zeta=\zeta_0^2\zeta_1$. By the choice of $b=4k-3$, we can find a section $\xi\in H^0((6k-1)D+2Z)$ which is linearly independent of $z_0^iz_1^{6k-1-i}\zeta^2$, $0\leq i\leq 6k-1$, where we regard z_0, z_1 as a basis for $H^0(D)$. Since

$$((6k-1)D+2Z)G_0=-2k+1<0\,,\quad ((6k-1)D+3G_0+2G_1)G_1=-1\,,$$

we can write $\xi=\xi_0\zeta_0\zeta_1$ with some $\xi_0\in H^0((6k-1)D+3G_0+G_1)$. Note that ξ_0 is a nonzero constant on G_0. Similarly, by the choice of $a=2k-2$, we can find a section $\eta\in H^0((8k-2)D+2Z)$ which is linearly independent of $z_0^iz_1^{8k-2-i}\zeta^2 (0\leq i\leq 8k-2)$ and

$z_0^j z_1^{2k-1-j}\xi (0 \leq j \leq 2k-1)$. Since $H^0(K-aD) \to H^0(K_D)$ is surjective, we see that η is a nonzero constant on $G_0 \cup G_1$. Note that ζ^2, ξ and η induces a basis for $H^0(K_D)$.

By the Riemann-Roch theorem, we have $\chi((12k-3)D+3Z)=22k-2$. Since $(12k-3)D+3Z=K+(2k+1)D+Z$ and $|(2k+1)D+Z|$ contains a connected member, we have $H^i((12k-3)D+3Z)=0$ for $i \geq 1$. Therefore, we get $h^0((12k-3)D+3Z)=22k-2$. In $H^0((12k-3)D+3Z)$, we have the following $22k-3$ elements:

$$\begin{cases} z_0^i z_1^{12k-3-i}\zeta^3 & (0 \leq i \leq 12k-3), \\ z_0^i z_1^{6k-2-i}\zeta\xi & (0 \leq i \leq 6k-2), \\ z_0^i z_1^{4k-1-i}\zeta\eta & (0 \leq i \leq 4k-1). \end{cases}$$

Therefore, there exists a new element ψ. If ψ were zero on G_0, it is also zero on G_1. Since one can show $h^0((12k-3)D+5G_0+2G_1)=22k-3$, this is impossible.

We next consider the cohomology long exact sequence for

$$0 \to \mathcal{O}((12k-3)D+3Z) \to \mathcal{O}((12k-2)D+3Z) \to \mathcal{O}_D(3Z|_D) \to 0 .$$

Since $Z|_D$ is of degree 2, we have $h^0(\mathcal{O}_D(3Z))=4$. This and $H^1((12k-3)D+3Z)=0$ show $h^0((12k-2)D+3Z)=22k+2$. In $H^0((12k-2)D+3Z)$, however, we have the following $22k+3$ elements:

$$\begin{cases} z_0^i z_1^{12k-2-i}\zeta^3 & (0 \leq i \leq 12k-2), \\ z_0^i z_1^{6k-1-i}\zeta\xi & (0 \leq i \leq 6k-1), \\ z_0^i z_1^{4k-i}\zeta\eta & (0 \leq i \leq 4k), \\ z_0\psi,\ z_1\psi , & \\ \zeta_1\xi_0^2 . & \end{cases}$$

Therefore, there exists a relation of the form

$$A_1\psi = A_0\zeta_1\xi_0^2 + A_{4k}\zeta\eta + A_{6k-1}\zeta\xi + A_{12k-2}\zeta^3 , \tag{2.6}$$

where the A_i are homogeneous forms of degree i in z_0, z_1. We remark that A_1 cannot be zero as a linear form. By restricting (2.6) to G_1, we find that A_1 vanishes on G_1. Geometrically, this implies that G_1 is contained in the fiber defined by $A_1=0$. Similarly, by restricting (2.6) to G_0, we see that $A_0 \neq 0$. Therefore, we may put $A_0=1$ and $A_1=z_0$ by a linear change among z_0 and z_1. Multiplying ζ to (2.6), we get

$$z_0\zeta\psi = \xi^2 + A_{4k}\zeta^2\eta + A_{6k-1}\zeta^2\xi + A_{12k-2}\zeta^2 . \tag{2.7}$$

We write the right hand side of (2.7) as $Q(z_0, z_1, \eta, \xi, \zeta^2)$ for simplicity.

We finally look at $H^0((24k-6)D+6Z)$ which is of dimension $100k-17$. Here, we have the following $100k-16$ elements modulo (2.7):

$$\begin{cases} z_0^i z_1^{24k-6-i}\zeta^6 & (0 \le i \le 24k-6)\,, \\ z_0^i z_1^{18k-5-i}\zeta^4\xi & (0 \le i \le 18k-5)\,, \\ z_0^i z_1^{16k-4-i}\zeta^4\eta & (0 \le i \le 16k-4)\,, \\ z_0^i z_1^{12k-4-i}\zeta^2\xi^2 & (0 \le i \le 12k-4)\,, \\ z_0^i z_1^{10k-3-i}\zeta^2\xi\eta & (0 \le i \le 10k-3)\,, \\ z_0^i z_1^{8k-2-i}\zeta^2\eta^2 & (0 \le i \le 8k-2)\,, \\ z_0^i z_1^{6k-3-i}\zeta^2\xi^3 & (0 \le i \le 6k-3)\,, \\ z_0^i z_1^{4k-2-i}\zeta^2\xi^2\eta & (0 \le i \le 4k-2)\,, \\ z_0^i z_1^{2k-1-i}\zeta^2\xi\eta^2 & (0 \le i \le 2k-1)\,,\,. \\ \eta^3\,, & \\ z_1^{12k-3}\zeta^3\psi\,, & \\ z_1^{6k-2}\zeta\xi\psi\,, & \\ z_1^{4k-1}\zeta\eta\psi\,, & \\ \psi^2\,. & \end{cases}$$

It follows that we have a relation among these. In this relation, the coefficient of ψ^2 cannot be zero. To see this, suppose that we have a relation which does not involve ψ^2. Then, by eliminating ψ from this using (2.7), we would get a cubic relation among ξ, η and ζ^2 with coefficients homogeneous forms in z_0, z_1. Since ξ, η and ζ^2 induce a basis for $H^0(K_D)$, and since D is a nonhyperelliptic curve of genus 3, this leads us to a contradiction. Therefore, by a suitable change of ψ if necessary, we get a relation of the form

$$(2.8)\qquad \begin{aligned} \psi^2 = {} & B_0\eta^3 + B_{2k-1}\xi\eta^2 + B_{4k-2}\xi^2\eta + B_{6k-3}\xi^3 + B_{8k-2}\zeta^2\eta^2 + B_{10k-3}\zeta^2\xi\eta \\ & + B_{12k-4}\zeta^2\xi^2 + B_{16k-4}\zeta^4\eta + B_{18k-5}\zeta^4\xi + B_{24k-6}\zeta^6\,, \end{aligned}$$

where the B_i are homogeneous forms of degree i in z_0, z_1. Since ψ is not zero on G_0, B_0 is a nonzero constant. We write the right hand side of (2.8) as $P(z_0, z_1, \eta, \xi, \zeta^2)$ for simplicity.

Now, eliminating ψ from (2.7) and (2.8), we get

$$(2.9)\qquad Q(z_0, z_1, \eta, \xi, \zeta^2)^2 - z_0^2\zeta^2 P(z_0, z_1, \eta, \xi, \zeta^2) = 0\,.$$

Since the holomorphic map $g: S \to W$ is obtained by putting $X_0 = \eta$, $X_1 = \xi$, $X_2 = \zeta^2$, we see that S' is defined by

$$(2.10)\qquad Q(z_0, z_1, X_0, X_1, X_2)^2 - z_0^2 X_2 P(z_0, z_1, X_0, X_1, X_2) = 0\,.$$

It follows that S' has a double curve along a conic defined by $z_0 = Q = 0$. Let $\sigma: S^* \to S'$ be the blowing up of the conic. In order to describe S^*, we introduce a new variable $w = Q/z_0$ which can be regarded as a fiber coordinate of $[2T-(8k-7)F]$. Then S^* is defined in the total space of $[2T-(8k-7)F]$ by

$$(2.11) \qquad \begin{cases} z_0 w - Q = 0\,, \\ w^2 - X_2 P = 0\,. \end{cases}$$

Since $w = \zeta\psi$, we can lift $g\colon S \to S'$ to $h\colon S \to S^*$. It is easy to see that (2.11) defines a surface which is singular only at $z_0 = w = X_1 = X_2 = 0$ provided that P and Q are sufficiently general. This singularity is given locally by

$$z_0 w - (X_1^2 + \alpha w^2 + \cdots) = 0\,.$$

Therefore, it is a rational double point of type A_1 from which G_1 arises. It may be clear that $X_2 = 0$ induces on S the divisor $2Z = 4G_0 + 2G_1$.

We have shown the following:

THEOREM 2.4. *For any even canonical surface S with $c_1^2 = 3p_g - 6$, there exists a positive integer k satisfying $p_g = 16k - 6$. If $k \geq 2$, then S is the minimal resolution of a surface defined by Equation* (2.10) *in* $\boldsymbol{P}(\mathcal{O}(2k-2) \oplus \mathcal{O}(4k-3) \oplus \mathcal{O}(10k-4))$.

Noting that S contains a (-2)-curve G_1, we can show the following in the same way as in Theorem 1.5.

THEOREM 2.5. *The moduli space of even canonical surfaces with $c_1^2 = 3p_g - 6$, $p_g \neq 10$, is non-reduced.*

REMARK 2.6. When $p_g = 10$ and $K^2 = 24$, an even canonical surface is one of the following (see, [9]):

(1) a sextic surface.

(2) a triple covering of a quadric surface in $\boldsymbol{P}^3$.

(3) a surface with a pencil of nonhyperelliptic curves of genus 3.

See also [10, 4.3, 4.4 and §9]. In [9], it is shown that these together with non-canonical ones form an irreducible family. In particular, (-2)-curves on a surface S of type (3) disappear as S deforms to a sextic surface.

REFERENCES

[1] T. ASHIKAGA AND K. KONNO, Algebraic surfaces of general type with $c_1^2 = 3p_g - 7$, Tôhoku Math. J. 42 (1990), 517–536.

[2] A. BEAUVILLE, L'application canonique pour les surfaces de type général, Invent. Math. 55 (1979), 121–140.

[3] D. BURNS AND J. WAHL, Local contributions to global deformations of surfaces, Invent. Math. 26 (1974), 67–88.

[4] G. CASTELNUOVO, Osservazioni intorno alla geometria sopra una superficie, Nota II, Rendiconti del R. Instituto Lombardo, s. II, vol. 24 (1891).

[5] F. CATANESE, Everywhere non reduced moduli spaces, Invent. Math. 98 (1989), 293–310.

[6] O. DEBARRE, Inégalités numériques pour les surfaces de type général, Bull. Soc. Math. France 110 (1982), 319–346.

[7] E. HORIKAWA, Algebraic surfaces of general type with small c_1^2 III, Invent. Math. 47 (1978), 209–248.

[8] E. Horikawa, Notes on canonical surfaces, Tôhoku Math. J. 43 (1991), 141–148.
[9] E. Horikawa, Deformations of sextic surfaces, preprint (1990).
[10] K. Konno, Algebraic surfaces of general type with $c_1^2=3p_g-6$, Math. Ann. 290 (1991), 77–107.
[11] R. Miranda, On canonical surfaces of general type with $K^2=3\chi-10$, Math. Z. 198 (1988), 83–93.

Department of Mathematics
College of General Education
Kyushu University
Ropponmatsu, Chuo-ku, Fukuoka 810
Japan