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RELATIONSHIPS BETWEEN BC-STABILITIES AND ρ-STABILITIES IN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. For functional differential equations on a fading memory space, some relationships between the BC-stabilities and ρ-stabilities are studied. Although the BC-uniform stability is weaker than the ρ-uniform stability, it is shown that the BC-total stability and BC-uniform asymptotic stability are respectively equivalent to the ρ-total stability and ρ-uniform asymptotic stability.

1. Introduction. In the theory of functional differential equations (FDEs) with infinite delay as well as integrodifferential equations, the space BC which consists of all bounded continuous functions on $(-\infty, 0]$ is one of the important classes for the space of initial functions. When one takes the BC as the space of initial functions, there are mainly two ways to provide it with the structure of a metric space in connection with stability problems. One way is to provide it with the supremum norm, and the other is of compact open topology induced by a metric which is called the "ρ-metric". Throughout this paper, the stabilities corresponding to the two metrics are referred to as the BC-stabilities and the ρ-stabilities, respectively. Although there are some similarities between them, some authors have studied them independently; for the BC-stabilities, see [2], [3], [8], [9]; for the ρ-stabilities, see [5]–[7], [11], [13]. Practical phenomena are intimately related to the BC-stabilities. Thus the BC-stabilities would seem to be more usual than the ρ-stabilities. However, the supremum norm never fade the past memory in contrast with the ρ-metric. This fact would produce some difficulties when one tries to discuss the existence of periodic solutions or almost periodic solutions under some BC-stability assumption. It is a remarkable difference between the BC-stabilities and ρ-stabilities. The purpose of this paper is to study the relationships between the above two stabilities. In what follows, we will do this for FDEs considered on a fading memory space. A fading memory space is a considerably flexible (phase) space for FDEs. Indeed, as pointed out in [1], some integrodifferential equations can be set up as FDEs on a fading memory space. Hence our setting is not so restrictive. As will be seen later, the BC-uniform stability does not necessarily imply the ρ-uniform stability. However, the total stability is an equivalent concept in the BC-stabilities and ρ-stabilities (Theorem 1). Therefore, via the ρ-stabilities, one can often overcome some difficulties which would arise in the BC-stabilities. In fact, we

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can build up an existence result of almost periodic solutions under the BC-total stability assumption (Corollary 1). Furthermore, under a mild assumption, we establish (Theorem 2) that the BC-stabilities and ρ-stabilities are equivalent even for the uniform asymptotic stability.

2. Fading memory spaces and some definitions. First of all, we shall explain some notation and convention employed throughout this paper. Let R^n be the n-dimensional real linear space and denote by $|\cdot|$ an appropriate norm in R^n. For any interval $I \subset R:=(-\infty, \infty)$, we denote by BC($I$) the set of all bounded and continuous functions mapping I into R^n, and set $|\varphi|_I = \sup\{|\varphi(s)|: s \in I\}$. In particular, we employ the conventions $\mathrm{BC}(I)=:\mathrm{BC}$ and $|\cdot|_I =: |\cdot|_{\mathrm{BC}}$ when $I=(-\infty, 0]=: R^-$. For any compact set K in R^n, we denote by int K the interior of K, and moreover we employ the notation $\varphi(\cdot) \in K$ to denote that $\varphi \in \mathrm{BC}$ and $\varphi(s) \in K$ for all $s \in R^-$.

Now, for any function $x: (-\infty, a) \to R^n$ and $t<a$, define a function $x_t: R^- \to R^n$ by $x_t(s)=x(t+s)$ for $s \in R^-$. Let $\mathscr{B}$ be a real linear space of functions mapping R^- into R^n with a complete seminorm $|\cdot|_{\mathscr{B}}$. We always assume the following conditions on the space $\mathscr{B}$.

(A1) There exist positive constants J, M and N with the property that if $x: (-\infty, a) \to R^n$ is continuous on $[\sigma, a)$ with $x_\sigma \in \mathscr{B}$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,

(i) $x_t \in \mathscr{B}$,

(ii) x_t is continuous in t (with respect to the seminorm $|\cdot|_{\mathscr{B}}$),

(iii) $J|x(t)| \le |x_t|_{\mathscr{B}} \le M \sup_{\sigma \le s \le t}|x(s)| + N|x_\sigma|_{\mathscr{B}}$.

(A2) If $\{\varphi^k\}$ is a sequence in $\mathscr{B} \cap \mathrm{BC}$ converging to a function φ uniformly on any compact interval in R^- (compactly on R^-, for short) and $\sup_k |\varphi^k|_{\mathrm{BC}} < \infty$, then $\varphi \in \mathscr{B}$ and $|\varphi^k - \varphi|_{\mathscr{B}} \to 0$ as $k \to \infty$.

It is known [12, Proposition 2.1] that the space $\mathscr{B}$ contains BC and that there is a constant $l>0$ such that

$$|\varphi|_{\mathscr{B}} \le l|\varphi|_{\mathrm{BC}} \qquad \text{for all} \quad \varphi \in \mathrm{BC}. \tag{1}$$

The space $\mathscr{B}$ is called a fading memory space, if it satisfies the following fading memory condition together with (A1) and (A2):

(A3) If $x: R \to R^n$ is a function such that $x_0 \in \mathscr{B}$, and $x(t) \equiv 0$ on $R^+ := [0, \infty)$, then $|x_t|_{\mathscr{B}} \to 0$ as $t \to \infty$.

Let g be any function which satisfies the condition

$$g: R^- \to [1, \infty) \text{ is continuous, nonincreasing and } \lim_{\theta \to -\infty} g(\theta) = \infty. \tag{2}$$

Then the Banach space $C_g^0 := (C_g^0, |\cdot|_g)$ defined by

$$C_g^0 = \{\varphi: R^- \to R^n \text{ is continuous and } \lim_{\theta \to -\infty} |\varphi(s)|/g(s) = 0\}$$

$$|\varphi|_g = \sup_{s \le 0} |\varphi(s)|/g(s), \qquad \varphi \in C_g^0,$$

is a (separable) fading memory space. For other examples of fading memory spaces, see [4], [10].

Some integrodifferential equations can be set up as FDEs on a fading memory space. For instance, consider a system of linear Volterra equations

$$\dot{x}(t)=D(t)x(t)+\int_{-\infty}^{t} E(t,s)x(s)ds\,, \tag{V}$$

where $D(t)$ is an $n\times n$ matrix function continuous on R, and $E(t,s)$ is an $n\times n$ matrix function continuous for $-\infty<s\leq t<\infty$ with the property that

(3) for any $\eta>0$ there exists an $S(\eta)>0$ such that

$$\int_{-\infty}^{t-S(\eta)} |E(t,s)|ds<\eta \qquad \text{for all} \quad t\in R\,.$$

One can easily see ([1], [10, Chapter 9]) that there exists a function g which satisfies $\int_{-\infty}^{0}|E(t,t+s)|g(s)ds<\infty$ together with (2). Thus (V) can be considered as an FDE on C_g^0.

Now we consider a system of FDEs

$$\dot{x}(t)=f(t,x_t) \qquad t\in R^+\,, \tag{E}$$

where f: $R^+\times\mathscr{B}\to R^n$. We impose the following conditions on (E):

(H1) $\sup\{|f(t,\varphi)|: t\in R^+, |\varphi|_{\mathscr{B}}\leq H\}=: L(H)<\infty$ for each $H>0$.

(H2) $f(t,\varphi)$ is uniformly continuous in $(t,\varphi)\in R^+\times W$ for any compact set W in $\mathscr{B}$.

In what follows, restricting initial functions to the elements belonging to BC, we give the definitions of BC-stabilities and ρ-stabilities for a bounded solution of (E). To do this, we impose the following condition on (E), too:

(H3) (E) has a solution u defined on R^+ satisfying $\sup_{t\geq 0}|u(t)|<\infty$ and $u_0\in \text{BC}$.

From (H3) and Axiom (A1) it follows that $\sup_{t\geq 0}|u_t|_{\mathscr{B}}<\infty$; hence $\sup_{t\geq 0}|\dot{u}(t)|<\infty$ by (H1). Thus the set

$$\Gamma(u):=\text{the closure of } \{u_t\colon t\in R^+\}$$

is compact in $\mathscr{B}$ (cf. [12, Theorem 4.1]).

a. BC-stabilities. The solution $u(t)$ of (E) is said to be:

(a-1) BC-uniformly stable (BC-US) if for any $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that $\sigma\in R^+$ and $\varphi\in\text{BC}$ with $|\varphi-u_\sigma|_{\text{BC}}<\delta(\varepsilon)$ imply $|x(t;\sigma,\varphi)-u(t)|<\varepsilon$ for $t\geq\sigma$, where $x(\cdot\,;\sigma,\varphi)$ denotes any solution of System (E) through (σ,φ);

(a-2) BC-uniformly asymptotically stable (BC-UAS) if it is BC-US and if there exists a $\delta_0>0$ with the property that for any $\varepsilon>0$ there exists a $T(\varepsilon)>0$ such that $\sigma\in R^+$ and $|\varphi-u_\sigma|_{\text{BC}}<\delta_0$ imply $|x(t;\sigma,\varphi)-u(t)|<\varepsilon$ for $t\geq\sigma+T(\varepsilon)$;

(a-3) BC-totally stable (BC-TS) if for any $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ with the property that $\sigma \in R^+$, $\varphi \in \mathrm{BC}$ with $|\varphi - u_\sigma|_{\mathrm{BC}} < \delta(\varepsilon)$ and $p \in \mathrm{BC}([\sigma, \infty))$ with $|p|_{[\sigma, \infty)} < \delta(\varepsilon)$ imply $|x(t; \sigma, \varphi, p) - u(t)| < \varepsilon$ for $t \geq \sigma$, where $x(\cdot\,; \sigma, \varphi, p)$ denotes any solution of

$$\text{(P)} \qquad \dot{x}(t) = f(t, x_t) + p(t) \qquad t \in R^+$$

through (σ, φ).

b. ρ-stabilities. For any $\varphi, \psi \in \mathrm{BC}$, we set

$$\rho(\varphi, \psi) = \sum_{j=1}^{\infty} 2^{-j} |\varphi - \psi|_j / \{1 + |\varphi - \psi|_j\} ,$$

where $|\cdot|_j = |\cdot|_{[-j, 0]}$. Then (BC, ρ) is a metric space. Furthermore, it is clear that $\rho(\varphi^k, \varphi) \to 0$ as $k \to \infty$ if and only if $\varphi^k \to \varphi$ compactly on R^-. Now, set

$$\mathcal{O}(u) = \text{the closure of } \{u(t) : t \in R\} ,$$

and consider any compace set K in R^n such that int $K \supset \mathcal{O}(u)$.

The solution $u(t)$ of (E) is said to be:

(b-1) ρ-uniformly stable with respect to K (ρ-US with respect to K) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\sigma \in R^+$ and $\varphi(\cdot) \in K$ with $\rho(\varphi, u_\sigma) < \delta(\varepsilon)$ imply $\rho(x_t(\sigma, \varphi), u_t) < \varepsilon$ for $t \geq \sigma$;

(b-2) ρ-uniformly asymptotically stable with respect to K (ρ-UAS with respect to K) if it is ρ-US with respect to K and if there exists a $\delta_0 > 0$ with the property that for any $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that $\sigma \in R^+$ and $\varphi(\cdot) \in K$ with $\rho(\varphi, u_\sigma) < \delta_0$ imply $\rho(x_t(\sigma, \varphi), u_t) < \varepsilon$ for $t \geq \sigma + T(\varepsilon)$;

(b-3) ρ-totally stable with respect to K (ρ-TS with respect to K) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ with the property that $\sigma \in R^+$, $\varphi(\cdot) \in K$ with $\rho(\varphi, u_\sigma) < \delta(\varepsilon)$ and $p \in \mathrm{BC}([\sigma, \infty))$ with $|p|_{[\sigma, \infty)} < \delta(\varepsilon)$ imply $\rho(x_t(\sigma, \varphi, p), u_t) < \varepsilon$ for $t \geq \sigma$.

As will be shown in Theorems 1 and 2, the definitions (b-2) and (b-3) do not depend on the particular choice of a compact set K such that int $K \supset \mathcal{O}(u)$. If the terms $\rho(x_t(\sigma, \varphi), u_t)$ in (b-1), (b-2) and $\rho(x_t(\sigma, \varphi, p), u_t)$ in (b-3) are respectively replaced by $|x(t; \sigma, \varphi) - u(t)|$ and $|x(t; \sigma, \varphi, p) - u(t)|$, then we have another concept of ρ-stability; which will be referred to as the (ρ, R^n)-stability. Later, it will be shown that these two notions of ρ-stability are equivalent. Therefore the ρ-stability implies the BC-stability, because of $\rho(\varphi, \psi) \leq |\varphi - \psi|_{\mathrm{BC}}$ for $\varphi, \psi \in \mathrm{BC}$. In subsequent sections, we discuss the opposite implications.

3. Equivalence of BC-total stability and ρ-total stability.

THEOREM 1. *Let $\mathscr{B}$ be a fading memory space, and assume Conditions* (H1), (H2) *and* (H3). *Then the following statements are equivalent*:

(i) *The solution $u(t)$ of* (E) *is* BC-TS.

(ii) *For some compact set K in R^n such that* int $K \supset \mathcal{O}(u)$, *the solution $u(t)$ of* (E) *is* ρ-TS *with respect to K.*

(iii) *For any compact set K in R^n such that* int $K \supset \mathcal{O}(u)$, *the solution $u(t)$ of* (E) *is* ρ-TS *with respect to K.*

Obviously, (iii) implies (ii). To establish the theorem, we need a lemma, from which it follows that (ii) implies (i).

LEMMA 1. *Assume Condition* (H3), *and let K be a compact set in R^n such that* int $K \supset \mathcal{O}(u)$. *Then the solution $u(t)$ of* (E) *is* ρ-TS *with respect to K if and only if it is* (ρ, R^n)-TS *with respect to K.*

PROOF. The proof of the "only if" part is obvious. We shall establish the "if" part. Take any $\varepsilon > 0$, $(\sigma, \varphi) \in R^+ \times \mathrm{BC}$ and $p \in \mathrm{BC}([\sigma, \infty))$ with $\varphi(\cdot) \in K$, $\rho(\varphi, u_\sigma) < \delta(\varepsilon)$ and $|p|_{[\sigma, \infty)} < \delta(\varepsilon)$, where $\delta(\cdot)$ is the one for (ρ, R^n)-TS of the solution $u(t)$ of (E). Then $x(t) := x(t; \sigma, \varphi, p)$ satisfies

$$|x(t) - u(t)| < \varepsilon \quad \text{for} \quad t \geq \sigma . \tag{4}$$

To estimate $\rho(x_t, u_t)$, we first estimate $|x_t - u_t|_j$. Let $t \geq \sigma$, and denote by k the largest integer which does not exceed $t - \sigma$. If $j \leq k$, then $j \leq t - \sigma$; hence $|x_t - u_t|_j = \sup_{-j \leq s \leq 0} |x(t+s) - u(t+s)| < \varepsilon$ by (4). On the one hand, if $j \geq k+1$, then $j > t - \sigma$; hence

$$\begin{aligned} |x_t - u_t|_j &= \mathrm{Max}\left\{ \sup_{-j \leq s \leq \sigma - t} |x(t+s) - u(t+s)|, \sup_{\sigma - t \leq s \leq 0} |x(t+s) - u(t+s)| \right\} \\ &\leq \mathrm{Max}\left\{ \sup_{-j \leq \theta \leq 0} |\varphi(\theta) - u(\sigma + \theta)|, \sup_{\sigma \leq \theta} |x(\theta) - u(\theta)| \right\} < |\varphi - u_\sigma|_j + \varepsilon \end{aligned}$$

by (4). Then

$$\begin{aligned} \rho(x_t, u_t) &= \left(\sum_{j=1}^{k} + \sum_{j=k+1}^{\infty} \right) 2^{-j} |x_t - u_t|_j / [1 + |x_t - u_t|_j] \\ &< \sum_{j=1}^{k} 2^{-j} \varepsilon / (1 + \varepsilon) + \sum_{j=k+1}^{\infty} 2^{-j} [|\varphi - u_\sigma|_j + \varepsilon] / [1 + |\varphi - u_\sigma|_j + \varepsilon] \\ &\leq \sum_{j=1}^{\infty} 2^{-j} \varepsilon / (1 + \varepsilon) + \sum_{j=k+1}^{\infty} 2^{-j} |\varphi - u_\sigma|_j / [1 + |\varphi - u_\sigma|_j] < \varepsilon + \delta(\varepsilon) \leq 2\varepsilon , \end{aligned}$$

which shows that the solution $u(t)$ of (E) is ρ-TS with $\delta(\cdot/2)$.

Now, in order to complete the proof of Theorem 1, it is sufficient to show that (i) implies (iii). We shall accomplish it by contradiction. By Lemma 1, we assume that the solution $u(t)$ of (E) is BC-TS but not (ρ, R^n)-TS with respect to K; here, $K \subset \{x \in R^n : |x| \leq c\}$ for some $c > 0$. Since the solution $u(t)$ of (E) is not (ρ, R^n)-TS with respect to K, there exist an $\varepsilon \in (0, 1)$, sequences $\{\tau_m\} \subset R^+$, $\{t_m\}$ $(t_m > \tau_m)$, $\{\varphi^m\} \subset \mathrm{BC}$ with $\varphi^m(\cdot) \in K$, $\{p_m\}$ with $p_m \in \mathrm{BC}([\tau_m, \infty))$, and solutions $\{x(\cdot\,; \tau_m, \varphi^m, p_m)\}$ such that

$$\rho(\varphi^m, u_{\tau_m}) < 1/m \quad \text{and} \quad |p_m|_{[\tau_m, \infty)} < 1/m \tag{5}$$

and that

(6) $\quad |x(t_m;\tau_m,\varphi^m,p_m)-u(t_m)|=\varepsilon \quad$ and $\quad |x(t;\tau_m,\varphi^m,p_m)-u(t)|<\varepsilon \quad$ on $\quad [\tau_m,t_m)$

for $m\in N$ (N denotes the set of all positive integers). For each $m\in N$ and $T\in R^+$, we define $\varphi^{m,T}\in \mathrm{BC}$ by

$$\varphi^{m,T}(\theta)=\begin{cases}\varphi^m(\theta) & \text{if} \quad -T\le\theta\le 0\,,\\ \varphi^m(-T)+u(\tau_m+\theta)-u(\tau_m-T) & \text{if} \quad \theta<-T\,.\end{cases}$$

Notice that $|\varphi^{m,T}-u_{\tau_m}|_{\mathrm{BC}}=|\varphi^m-u_{\tau_m}|_{[-T,0]}$.

Claim 1. $\sup\{|\varphi^{m,T}-\varphi^m|_{\mathscr{B}}: m\in N\}\to 0$ as $T\to\infty$.

If this is not the case, there exist an $\varepsilon>0$ and sequences $\{m_k\}\subset N$ and $\{T_k\}$, $T_k\to\infty$ as $k\to\infty$, such that $|\varphi^{m_k,T_k}-\varphi^{m_k}|_{\mathscr{B}}\ge\varepsilon$ for $k=1,2,\ldots$. Put $\psi^k=\varphi^{m_k,T_k}-\varphi^{m_k}$. Clearly, $\{\psi^k\}$ is a sequence in BC which converges to the zero function compactly on R^- and $\sup_k|\psi^k|_{\mathrm{BC}}<\infty$. Then Axiom (A2) yields that $|\psi^k|_{\mathscr{B}}\to 0$ as $k\to\infty$, a contradiction.

Claim 2. The set $\{\varphi^m, \varphi^{m,T}: m\in N,\ T\in R^+\}$ is relatively compact in $\mathscr{B}$.

Indeed, since the set $\Gamma(u)$ is compact in $\mathscr{B}$, (5) and Axiom (A2) yield that any sequence $\{\varphi^{m_j}\}_{j=1}^\infty$ ($m_j\in N$) has a convergent subsequence in $\mathscr{B}$. Therefore, it suffices to show that any sequence $\{\varphi^{m_j,T_j}\}_{j=1}^\infty$ ($m_j\in N$, $T_j\in R^+$) has a convergent subsequence in $\mathscr{B}$. We assert that the sequence of functions $\{\varphi^{m_j,T_j}(\theta)\}_{j=1}^\infty$ contains a subsequence which is equicontinuous on any compact set in R^-. If this is the case, then the sequence $\{\varphi^{m_j,T_j}\}$ would have a convergent subsequence in $\mathscr{B}$ by Ascoli's theorem and Axiom (A2), as required. Now, notice that the sequence of functions $\{u(\tau_{m_j}+\theta)\}$ is equicontinuous on any compact set in R^-. Then the assertion obviously holds true when the sequence $\{m_j\}$ is bounded. Taking a subsequence if necessary, it is thus sufficient to consider the case $m_j\to\infty$ as $j\to\infty$. In this case, from (5) it follows that $\varphi^{m_j}(\theta)-u(\tau_{m_j}+\theta)=:w^j(\theta)\to 0$ compactly on R^-. Consequently, $\{w^j(\theta)\}$ is equicontinuous on any compact set in R^-, and so is $\{\varphi^{m_j}(\theta)\}$. Therefore the assertion immediately follows from this observation.

Now, for any $m\in N$, set $x^m(t)=x(t+\tau_m;\tau_m,\varphi^m,p_m)$ if $t\le t_m-\tau_m$ and $x^m(t)=x^m(t_m-\tau_m)$ if $t>t_m-\tau_m$. Moreover, set $x^{m,T}(t)=\varphi^{m,T}(t)$ if $t\in R^-$ and $x^{m,T}(t)=x^m(t)$ if $t\in R^+$. Since $(x^m)_0=\varphi^m$ and $|x^m(t)|<1+|u|_{[0,\infty)}=:h<\infty$ for $t\in R^+$, we have

$$|(x^m)_t|_{\mathscr{B}}\le Mh+N|\varphi^m|_{\mathscr{B}}\le Mh+Nl|\varphi^m|_{\mathrm{BC}}\le Mh+Nlc$$

by (1) and Axiom (A1); hence, if $0\le t<t_m-\tau_m$, then

$$|(d/dt)x^m(t)|\le|f(t+\tau_m,(x^m)_t)|+|p_m(t+\tau_m)|\le L(Mh+Nlc)+1/m\le\tilde{L}$$

(independent of $m\in N$)

by (5) and (H1). Consequently,

$$(7)\qquad |x^m(s_1)-x^m(s_2)|\le \tilde{L}|s_1-s_2|\,,\quad s_1, s_2\in R^+\,,\quad m\in N.$$

Set

$$W=\text{the closure of } \{(x^{m,T})_t, (x^m)_t\colon m\in N, t\in R^+, T\in R^+\}\,.$$

Combining (7) with Claim 2, we see by [12, Theorem 4.1] that the set W is compact in $\mathscr{B}$; hence $f(t, \varphi)$ is uniformly continuous on $R^+\times W$ by (H2). Define a continuous function $q_{m,T}$ on R^+ by $q_{m,T}(t)=f(t+\tau_m, (x^m)_t)-f(t+\tau_m, (x^{m,T})_t)$ if $0\le t\le t_m-\tau_m$, and $q_{m,T}(t)=q_{m,T}(t_m-\tau_m)$ if $t>t_m-\tau_m$. Since $|(x^{m,T})_t-(x^m)_t|_{\mathscr{B}}\le N|\varphi^{m,T}-\varphi^m|_{\mathscr{B}}$ $(t\in R^+$, $m\in N)$ by Axiom (A1), if follows from Claim 1 that $\sup\{|(x^{m,T})_t-(x^m)_t|_{\mathscr{B}}\colon t\in \mathrm{R}^+$, $m\in N\}\to 0$ as $T\to\infty$; hence one can choose $T=T(\varepsilon)\in N$ in such a way that

$$\sup\{|q_{m,T}(t)|\colon m\in N, t\in R^+\}<\delta(\varepsilon/2)/2\,,$$

where $\delta(\cdot)$ is the one for BC-TS of the solution $u(t)$ of (E). Moreover, for this T, select an $m\in N$ such that $m>2^T(1+\delta(\varepsilon/2))/\delta(\varepsilon/2)$. Then $2^{-T}|\varphi^m-u_{\tau_m}|_T/[1+|\varphi^m-u_{\tau_m}|_T]\le \rho(\varphi^m, u_{\tau_m})<2^{-T}\delta(\varepsilon/2)/[1+\delta(\varepsilon/2)]$ by (5), which implies that

$$|\varphi^m-u_{\tau_m}|_T<\delta(\varepsilon/2)\quad\text{or}\quad |\varphi^{m,T}-u_{\tau_m}|_{\mathrm{BC}}<\delta(\varepsilon/2)\,.$$

The function $x^{m,T}$ satisfies $(x^{m,T})_0=\varphi^{m,T}$ and

$$\begin{aligned}(d/dt)x^{m,T}(t)&=(d/dt)x^m(t)=f(t+\tau_m, (x^m)_t)+p_m(t+\tau_m)\\ &=f(t+\tau_m, (x^{m,T})_t)+q_{m,T}(t)+p_m(t+\tau_m)\end{aligned}$$

for $t\in[0, t_m-\tau_m)$. Since $u^m(t)=u(t+\tau_m)$ is a BC-totally stable solution of $\dot{x}(t)=f(t+\tau_m, x_t)$ with the same $\delta(\cdot)$ as the one for $u(t)$, from the fact that $\sup_{t\ge 0}|q_{m,T}(t)+p_m(\tau_m+t)|<\delta(\varepsilon/2)/2+1/m<\delta(\varepsilon/2)$ it follows that $|(x^{m,T})(t)-u(t+\tau_m)|<\varepsilon/2$ on $[0, t_m-\tau_m)$. In particular, we have $|x^{m,T}(t_m-\tau_m)-u(t_m)|<\varepsilon$ or $|x(t_m; \tau_m, \varphi^m, p_m)-u(t_m)|<\varepsilon$, which contradicts (6).

It would be natural to ask whether the BC-US and ρ-US are equivalent under Conditions (H1), (H2) and (H3). Needless to say, from the proof of Lemma 1 we easily see that the ρ-US implies the BC-US. As the following example shows, however, the opposite implication does not generally hold good.

EXAMPLE. Consider a scalar equation

$$(8)\qquad \dot{x}(t)=-\frac{1}{t+1}x(t)+\sup_{n\ge 0}\frac{|x(-n)|}{1+t+n}\qquad t\in R^+\,,$$

and set $g(s)=1-s$ for $s\in R^-$ and $f(t, \varphi)=-\varphi(0)/(t+1)+\sup_{n\ge 0}\{|\varphi(-t-n)|/g(-t-n)\}$ for $(t, \varphi)\in R^+\times C_g^0$. Then Equation (8) can be considered as an FDE on $R^+\times C_g^0$. In this case, it is not difficult to see that Conditions (H1), (H2) and (H3) (with $u\equiv 0$) hold true. In the following, we shall prove that the zero solution of Equation (8) is BC-US but not ρ-US with respect to K however small K is chosen. Indeed, for any

$(\sigma, \varphi) \in R^+ \times \mathrm{BC}$, the solution $x(t; \sigma, \varphi)$ of (8) is expressed by the variation-of-constants formula as

$$x(t; \sigma, \varphi) = X(t, \sigma)\varphi(0) + \int_\sigma^t X(t, s)\left(\sup_{n \geq 0} \frac{|\varphi(-\sigma-n)|}{1+s+n}\right) ds, \qquad t \geq \sigma,$$

where $X(t, s) = (s+1)/(t+1)$. Then

$$|x(t; \sigma, \varphi)| \leq |X(t, \sigma)|\,|\varphi(0)| + \int_\sigma^t \frac{X(t, s)}{1+s} ds \cdot |\varphi|_{\mathrm{BC}} \leq |\varphi|_{\mathrm{BC}}$$

for $t \geq \sigma$, which shows that the zero solution of Equation (8) is BC-US. Now, let K be any compact set in R^n which contains zero in its interior. Then there exists a constant $c > 0$ such that $K \supset \{x \in R^n : |x| \leq c\}$. For any $k \in N$, define a function $\varphi^k(\cdot) \in K$ by

$$\varphi^k(\theta) = \begin{cases} c & \text{if } \theta \leq -k \\ 0 & \text{if } \theta \geq -k+1 \\ \text{linear} & \text{if } -k \leq \theta \leq -k+1 . \end{cases}$$

Clearly, $\rho(\varphi^k, 0) \to 0$ as $k \to \infty$. Moreover,

$$\begin{aligned} x(k; 0, \varphi^k) &= X(k, 0)\varphi^k(0) + \int_0^k X(k, s)\left(\sup_{n \geq 0} \frac{|\varphi^k(-n)|}{1+s+n}\right) ds \\ &\geq \int_0^k \frac{s+1}{k+1} \frac{c}{1+s+k} ds = \frac{ck}{k+1}\left[1 - \log \frac{1+2k}{1+k}\right]; \end{aligned}$$

hence $\liminf_{k \to \infty} [\sup_{t \geq 0} |x(t; 0, \varphi^k)|] \geq c(1 - \log 2) > 0$. Consequently, the zero solution of Equation (8) is not ρ-US with respect to K.

As an application of Theorem 1, we can establish an existence result of almost periodic solutions for almost periodic systems under the BC-total stability assumption. This is a natural extension of the corresponding result for ODEs to the one for FDEs on a fading memory space. We emphasize that the method for ODEs cannot be applied directly to the case of FDEs on a fading memory space, because the convergence in the ρ-metric does not necessarily imply the convergence in the BC-norm. Therefore the following result would be of interest.

COROLLARY 1. *Let $\mathscr{B}$ be a separable fading memory space, and $f(t, \varphi): R \times \mathscr{B} \to R^n$ be a function almost periodic in $t \in R$ uniformly for $\varphi \in \mathscr{B}$ which satisfies* (H1) *and* (H3). *If the solution $u(t)$ of* (E) *is* BC-TS, *then there exists an almost periodic solution of* (E).

The proof is easy. Indeed, the BC-TS of $u(t)$ yields the ρ-TS of $u(t)$ by Theorem 1. Then the existence of an almost periodic solution for (E) can be assured by the standard argument; see, e.g., [5]–[7], [11], [13]. An example to which our corollary is applicable is found in [5], [7], [11].

4. Equivalence of BC-uniform asymptotic stability and ρ-uniform asymptotic stability. In this section, we discuss the equivalence of BC-UAS and ρ-UAS of a bounded solution of (E) by means of limiting equations. Throughout this section, we suppose that $\mathscr{B}$ is a separable fading memory space.

Now, let $C(R^+ \times \mathscr{B}, R^n)$ be the space of all continuous functions mapping $R^+ \times \mathscr{B}$ into R^n, which is equipped with the compact open topology. For any $\tau \in R^+$ and any $F \in C(R^+ \times \mathscr{B}, R^n)$, we set $F_\tau(t, \varphi) = F(t+\tau, \varphi)$ for $(t, \varphi) \in R^+ \times \mathscr{B}$. Under Conditions (H1) and (H2), it is known (cf. [10, Chapter 8]) that the set $\{f_\tau : \tau \in R^+\}$ is relatively compact in $C(R^+ \times \mathscr{B}, R^n)$ and that its closure in $C(R^+ \times \mathscr{B}, R^n)$ is metrizable. Therefore, for any sequence $\{\tau'_k\} \subset R^+$, there exists a subsequence $\{\tau_k\}$ of $\{\tau'_k\}$ and an element g in $C(R^+ \times \mathscr{B}, R^n)$ such that $f_{\tau_k} \to g$ in $C(R^+ \times \mathscr{B}, R^n)$; in other words, $f(t+\tau_k, \varphi) \to g(t, \varphi)$ uniformly on any compact subset of $R^+ \times \mathscr{B}$ as $k \to \infty$. Moreover, under Condition (H3), we may assume that there exists a $v \in \mathrm{BC}(R)$ such that $|u_{t+\tau_k} - v_t|_{\mathscr{B}} \to 0$ compactly on R^+. For the sake of simplicity, we write as

$$(u_{\tau_k}, f_{\tau_k}) \to (v, g) \quad \text{compactly on } R^+ \times \mathscr{B}$$

whenever the above situation occurs. Denote by $\Omega(u, f)$ the set of all (v, g)'s with the property that $(u_{\tau_k}, f_{\tau_k}) \to (v, g)$ compactly on $R^+ \times \mathscr{B}$ for some sequence $\{\tau_k\}$ such that $\tau_k \to \infty$ as $k \to \infty$. It is easy to check that if $(v, g) \in \Omega(u, f)$, then v is a solution defined on R^+ of the system

$$(\mathrm{E}_g) \qquad \dot{x}(t) = g(t, x_t)\,.$$

The system (E_g) is called a limiting equation of (E). If the solution of each limiting equation of (E) is unique for the initial conditions, then (E) is said to be regular.

THEOREM 2. *Let $\mathscr{B}$ be a separable fading memory space, and assume Conditions* (H1), (H2) *and* (H3). *Moreover, suppose that* (E) *is regular. Then the following statements are equivalent*:

(i) *The solution $u(t)$ of* (E) *is* BC-UAS.

(ii) *For some compact set K in R^n such that* int $K \supset \mathcal{O}(u)$, *the solution $u(t)$ of* (E) *is ρ-*UAS *with respect to K.*

(iii) *For any compact set K in R^n such that* int $K \supset \mathcal{O}(u)$, *the solution $u(t)$ of* (E) *is ρ-*UAS *with respect to K.*

Obviously, (iii) implies (ii). To prove the theorem, we first establish the following lemmas. From Lemma 2 we can easily see that (ii) implies (i).

LEMMA 2. *Assume Condition* (H3), *and let K be a compact set in R^n such that* int $K \supset \mathcal{O}(u)$. *Then the solution $u(t)$ of* (E) *is ρ-*UAS *with respect to K if and only if it is* (ρ, R^n)-UAS *with respect to K.*

LEMMA 3. *Suppose that all the conditions in Theorem* 2 *are satisfied. If the solution $u(t)$ of* (E) *is* BC-UAS, *then it is* BC-TS.

PROOF OF LEMMA 2. The proof of the "only if" part is obvious. We shall establish the "if" part. Suppose that the solution $u(t)$ of (E) is (ρ, R^n)-UAS with respect to K. The same argument as in the proof of Lemma 1 shows that the solution $u(t)$ of (E) is ρ-US with respect to K. Now, let $\rho(\varphi, u_\sigma)<\delta_0$ and $\varphi(\cdot)\in K$, where δ_0 is the one for (ρ, R^n)-UAS with respect to K of the solution $u(t)$ of (E). Then, for any $\varepsilon>0$ there exists a $T(\varepsilon)>0$ such that $|x(t;\sigma,\varphi)-u(t)|<\varepsilon$ for all $t\geq\sigma+T(\varepsilon)$, which implies that $\sup\{|x(t+\theta+\sigma;\sigma,\varphi)-u(t+\theta+\sigma)|: \sigma\in R^+, \theta\in[-L,0], \rho(\varphi,u_\sigma)<\delta_0 \text{ with } \varphi(\cdot)\in K\}\to 0$ as $t\to\infty$ for each $L>0$, or $\sup\{\rho(x_{t+\sigma}(\sigma,\varphi), u_{t+\sigma}): \sigma\in R^+, \rho(\varphi,u_\sigma)<\delta_0 \text{ with } \varphi(\cdot)\in K\}\to 0$ as $t\to\infty$. This shows that the solution $u(t)$ of (E) is ρ-UAS with respect to K.

PROOF OF LEMMA 3. Suppose that the solution $u(t)$ of (E) is BC-UAS, and let $(\delta(\cdot), \delta_0, T(\cdot))$ be the triple for BC-UAS of $u(t)$, where we may assume $\delta_0<\delta(1)$. We first establish that

(9) $(v, g)\in\Omega(u,f)$ and $|\varphi-v_\sigma|_{\mathrm{BC}}<\delta(\eta/2)$ imply $|y(t;\sigma,\varphi,g)-v(t)|<\eta$ for $t\geq\sigma$,

where $y(\cdot;\sigma,\varphi,g)$ denotes the (unique) solution of (E_g) through (σ,φ). Select a sequence $\{\tau_k\}$ with $\tau_k\to\infty$ as $k\to\infty$ such that $(u_{\tau_k}, f_{\tau_k})\to(v,g)$ compactly on $R^+\times\mathscr{B}$, and consider any solution $x(\cdot;\sigma+\tau_k, \varphi-v_\sigma+u_{\sigma+\tau_k})$ of (E). For any $k\in N$, set $x^k(t)=x(t+\tau_k;\sigma+\tau_k, \varphi-v_\sigma+u_{\sigma+\tau_k})$, $t\in R$. Since the solution $u(t)$ of (E) is BC-US, from the fact that $|(x^k)_\sigma-u_{\sigma+\tau_k}|_{\mathrm{BC}}=|\varphi-v_\sigma|_{\mathrm{BC}}<\delta(\eta/2)$ it follows that

(10) $$|x^k(t)-u(t+\tau_k)|<\eta/2 \qquad \text{for all} \quad t\geq\sigma \quad \text{and} \quad k\in N;$$

hence $\sup\{|(x^k)_t|_{\mathscr{B}}: t\geq\sigma, k\in N\}\leq M(\eta/2+|u|_{[0,\infty)})+N|\varphi-v_\sigma+u_{\sigma+\tau_k}|_{\mathscr{B}}$ by Axiom (A1). In virtue of (H1), this implies that $\{x^k(t)\}$ is uniformly equicontinuous on $[\sigma,\infty)$. Thus we may assume that $x^k(t)\to y(t)$ compactly on $[\sigma,\infty)$ for some function $y: [\sigma,\infty)\to R^n$. Since $x^k(\sigma)=\varphi(0)-v(\sigma)+u(\sigma+\tau_k)$, we obtain $y(\sigma)=\varphi(0)$. Hence, if we extend the function y by setting $y_\sigma=\varphi$, then $y\in C(R, R^n)$ and $|(x^k)_t-y_t|_{\mathscr{B}}\to 0$ compactly on $[\sigma,\infty)$. Letting $k\to\infty$ in the equation

$$x^k(t)=\varphi(0)-v(\sigma)+u(\sigma+\tau_k)+\int_\sigma^t f(s+\tau_k, (x^k)_s)ds, \quad t\geq\sigma,$$

we obtain

$$y(t)=\varphi(0)+\int_\sigma^t g(s, y_s)ds, \quad t\geq\sigma,$$

which means that $y(t)\equiv y(t;\sigma,\varphi,g)$ for $t\geq\sigma$ by the regularity assumption. Then (9) follows from (10) by letting $k\to\infty$.

Repeating the above argument with $\eta=2$, one can establish that

(11) $(v,g)\in\Omega(u,f)$ and $|\varphi-v_\sigma|_{\mathrm{BC}}<\delta_0$ imply $|y(t;\sigma,\varphi,g)-v(t)|<\varepsilon$ for $t\geq\sigma+T(\varepsilon/2)$.

Now, we assert that the solution $u(t)$ of (E) is BC-TS. Suppose not. Then there exist an ε, $0<\varepsilon<\delta_0$, sequences $\{\tau_k\}\subset R^+$, $\{r_k\}$, $r_k>0$, $\{\varphi^k\}\subset \mathrm{BC}$, $\{p_k\}$, $p_k\in \mathrm{BC}([\tau_k, \infty))$, and solutions $\{x(\cdot; \tau_k, \varphi^k, p_k)\}$ such that, for all $k\in N$,

(12) $$|\varphi^k-u_{\tau_k}|_{\mathrm{BC}}<1/k \quad \text{and} \quad |p_k|_{[\tau_k, \infty)}<1/k$$

and

(13) $$|z^k(\tau_k+r_k)-u(\tau_k+r_k)|=\varepsilon \quad \text{and} \quad |z^k(t)-u(t)|<\varepsilon \qquad \text{for} \quad t\in(-\infty, \tau_k+r_k)\,;$$

here $z^k(t):=x(t; \tau_k, \varphi^k, p_k)$. We first consider the case that $\{r_k\}$ is unbounded. Without loss of generality, we may assume that $(u_{\tau_k+r_k-T}, f_{\tau_k+r_k-T})\to(v, g)$ compactly on $R^+\times\mathscr{B}$ for some $(v, g)\in\Omega(u,f)$ and that $z^k(\tau_k+r_k-T+t)\to z(t)$ compactly on $(-\infty, T]$ for some function z, where $T=T(\varepsilon/2)$. Repeating almost the same argument as in the proof of the claim (9), we see by (12) that z satisfies (E_g) on $[0, T]$. Let $k\to\infty$ in (13) to obtain $|z(t)-v(t)|\le\varepsilon$ on $(-\infty, T]$ and $|z(T)-v(T)|=\varepsilon$. This is a contradiction, because $|z_0-v_0|_{\mathrm{BC}}\le\varepsilon<\delta_0$ implies $|z(T)-v(T)|<\varepsilon$ by (11). Therefore the sequence $\{r_k\}$ must be bounded. Thus we may assume that $\{r_k\}$ converges to some r, $0\le r<\infty$. Moreover, we may assume that $\{z^k(\tau_k+t)\}$ converges to a function ξ compactly on $(-\infty, r]$. Consider the case where the sequence $\{\tau_k\}$ is unbounded; hence we may assume that $(u_{\tau_k}, f_{\tau_k})\to(w, h)$ compactly on $R^+\times\mathscr{B}$ for some $(w, h)\in\Omega(u,f)$. Then $\xi(t)$ satisfies the limiting equation (E_h) on $[0, r]$, and moreover we have $|\xi_0-w_0|_{\mathrm{BC}}=0$ and $|\xi(r)-w(r)|=\varepsilon$ by letting $k\to\infty$ in (12) and (13). This is a contradiction, because we must have $\xi\equiv w$ on $[0, r]$ by the regularity assumption. Thus the sequence $\{\tau_k\}$ must be bounded, too. Hence we may assume that $\lim_{k\to\infty}\tau_k=\tau$ for some $\tau<\infty$. Then $\xi(t-\tau)$ satisfies (E) on $[\tau, \tau+r]$, and moreover we have $|\xi_0-u_\tau|_{\mathrm{BC}}=0$ and $|\xi(r)-u(\tau+r)|=\varepsilon$ by (12) and (13). This again contradicts the fact that the solution $u(t)$ of (E) is BC-US.

Now, in order to complete the proof of Theorem 2 it is sufficient to show that (i) implies (iii). Suppose that the solution $u(t)$ of (E) is BC-UAS, and let $(\delta(\cdot), \delta_0, T(\cdot))$ be the triple for BC-UAS of $u(t)$, where we may assume $\delta_0<\delta(1)<1$. Let K be any compact set in R^n such that int $K\supset\mathscr{O}(u)$. Combining Lemma 3 with Theorem 1, we see that the solution $u(t)$ of (E) is ρ-TS with respect to K. Thus it suffices to establish the following assertion:

(*) for any $\varepsilon>0$ there exists a $\bar{T}(\varepsilon)>0$ such that $\rho(\varphi, u_\tau)<\delta_1:=\bar{\delta}(\delta_0/4)$ with $\varphi(\cdot)\in K$ implies $\rho(x_t(\tau, \varphi), u_t)<\varepsilon$ for all $t\ge\tau+\bar{T}(\varepsilon)$, where $\bar{\delta}(\cdot)$ is the number for the ρ-total stability of $u(t)$.

If this is not the case, then there exist an $\varepsilon>0$ and sequences $\{\tau_k\}\subset R^+$, $\{t_k\}$, $t_k\ge\tau_k+2k$, $\{\varphi^k\}\subset\mathrm{BC}$, and solutions $\{x(\cdot; \tau_k, \varphi^k)\}$ such that

(14) $$\rho(\varphi^k, u_{\tau_k})<\delta_1\,, \quad \varphi^k(\cdot)\in K$$

and

(15) $$\rho(x_{t_k}(\tau_k, \varphi^k), u_{t_k})\ge\varepsilon$$

for all $k \in N$. Since $\bar{\delta}(\cdot)$ is the one corresponding to the fact that the solution $u(t)$ of (E) is ρ-TS with respect to K, (14) and (15) imply that

$$(16) \qquad \rho(x_t(\tau_k, \varphi^k), u_t) < \delta_0/4 \qquad \text{for all} \quad t \geq \tau_k$$

and $\rho(x_t(\tau_k, \varphi^k), u_t) \geq \bar{\delta}(\varepsilon)$ for all $t \in [\tau_k, \tau_k + 2k]$ or

$$(17) \qquad \rho(x_{t+\tau_k+k}(\tau_k, \varphi^k), u_{t+\tau_k+k}) \geq \bar{\delta}(\varepsilon) \qquad \text{for all} \quad t \in [-k, k] ,$$

respectively. Set $x^k(t) = x(t + \tau_k + k, \tau_k, \varphi^k)$ for $t \in R$. Since $\rho(\varphi, \psi) \geq 2^{-1} |\varphi - \psi|_1 / [1 + |\varphi - \psi|_1] \geq 2^{-1} |\varphi(0) - \psi(0)| / [1 + |\varphi(0) - \psi(0)|]$, we have $|\varphi(0) - \psi(0)| \leq 2\rho(\varphi, \psi) / [1 - 2\rho(\varphi, \psi)]$ whenever $\rho(\varphi, \psi) \leq 1/2$; hence (16) implies that

$$(18) \qquad |x^k(t) - u(t + \tau_k + k)| \leq \delta_0/[2 - \delta_0] \qquad \text{for all} \quad t \in [-k, k] .$$

Since $\{x^k(t)\}$ is uniformly bounded and equicontinuous on any compact set in R, we may assume that $x^k(t) \to \mu(t)$ compactly on R for some bounded continuous function $\mu : R \to R^n$. Also, we may assume that $(u_{\tau_k+k}, f_{\tau_k+k}) \to (v, g)$ compactly on $R^+ + \mathscr{B}$ for some $(v, g) \in \Omega(u, f)$. Then $\mu(t) = y(t; 0, \mu_0, g)$ on R. Letting $k \to \infty$ in (18), we have $|\mu(t) - v(t)| \leq \delta_0/[2 - \delta_0]$ on R. In particular, $|\mu_0 - v_0|_{\mathrm{BC}} \leq \delta_0/[2 - \delta_0] < \delta_0$. Then $|\mu(t) - v(t)| \to 0$ as $t \to \infty$, by (11); hence $\rho(\mu_t, v_t) \to 0$ as $t \to \infty$. On the other hand, letting $k \to \infty$ in (17), we obtain $\rho(\mu_t, v_t) \geq \bar{\delta}(\varepsilon)$ for all $t \in R$, a contradiction. This completes the proof of Theorem 2.

Under the condition (3) and the conditions (19) and (20) (see below), it is known [8, Theorem 3] that the BC-TS and BC-UAS are equivalent for the zero solution of a system of linear Volterra equations

$$(\mathrm{V}) \qquad \dot{x}(t) = D(t)x(t) + \int_{-\infty}^{t} E(t, s)x(s)ds .$$

Since (V) can be set up as an FDE on a separable fading memory space as shown in Section 2, combining [8, Theorem 3] with Theorems 1 and 2, we obtain the following counterpart for the ρ-stabilities.

COROLLARY 2. *Suppose that Condition* (3) *and the following are satisfied*:

(19) *$D(t)$ and $E(t, t+s)$ are bounded and uniformly continuous in $(t, s) \in R \times W$ for any compact set $W \subset R^-$.*

$$(20) \qquad \sup_{t \in R} \{|D(t)| + \int_{-\infty}^{0} |E(t, t+s)| ds\} < \infty .$$

Then, for any compact set K in R^n which contains zero in its interior, the zero solution of (V) *is ρ-TS with respect to K if and only if it is ρ-UAS with respect to K.*

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