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ON CERTAIN QUARTIC FORMS FOR AFFINE SURFACES

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The main purpose of the present note is to introduce certain quartic forms and other related invariants for nondegenerate affine surfaces and characterize a number of classes of affine surfaces by the vanishing of each of such invariants (see Theorems 1–4). Our results are closely related to the work in [B-N-S], [M-R], [N-M], [N-P1] and others.

Actually, our quartic forms can be defined for affine immersions and, more generally, in the projective setting, as in [N-P1], [N-P2], [N-P4], [S]. We hope to come back to applications of this general approach. Here we shall prove yet another result (Theorem 5, and equivalently, Theorem 6) characterizing quadrics among nondegenerate hypersurfaces in the real projective space in a fashion similar to Theorem 12 in [N-P4].

1. Quartic forms for nondegenerate affine hypersurfaces. Let $f: M^n \to \boldsymbol{R}^{n+1}$ be a nondegenerate hypersurface in the affine space $\boldsymbol{R}^{n+1}$ provided with a volume element by the usual determinant function det and the flat affine connection D. Then there exists an affine normal ξ such that

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)\xi \tag{1}$$

$$D_X \xi = -f_*(SX), \tag{2}$$

where ∇ is the induced affine connection on M^n, h the affine metric, and S the shape operator. Moreover, the volume element defined by

$$\theta(X_1, \ldots, X_n) = \det[f_*(X_1), \ldots, f_*(X_n), \xi] \tag{3}$$

coincides with the volume element for the nondegenerate metric h. The structure given by (∇, h, S) is called the Blashke structure on M^n. We have the following fundamental equations (see [N-P1]):

(4) (Gauss) $\qquad R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY$

(5) (Codazzi) $\qquad (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z)$

(6) (Codazzi) $\qquad (\nabla_X S)(Y) = (\nabla_Y S)(X)$

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(7) (Ricci) $$h(SX, Y)=h(X, SY).$$

In view of (5), $C(X, Y, Z)=(\nabla_X h)(Y, Z)$ is totally symmetric in X, Y, Z. We call C the cubic form. We now define

$$Q(X, Y, Z, W)=(\nabla_X C)(Y, Z, W)+\mathscr{S}_{Y,Z,W}[h(SX, Y)h(Z, W)], \tag{8}$$

where $\mathscr{S}_{Y,Z,W}$ denotes the cyclic sum over Y, Z, and W. It is clear that Q is symmetric in Y, Z, W. We shall show in a moment that it is symmetric in all four variables X, Y, Z, W, that is, Q is a quartic form.

LEMMA 1. *We have*

$$(\nabla_X C)(Y, Z, W)-(\nabla_Y C)(X, Z, W)=(R(X, Y)\cdot h)(Z, W). \tag{9}$$

PROOF. From

$$\begin{aligned}(\nabla_X C)(Y, Z, W)&=XC(Y, Z, W)-C(\nabla_X Y, Z, W)-C(Y, \nabla_X Z, W)-C(Y, Z, \nabla_X W)\\ &=X(\nabla_Y h)(Z, W)-(\nabla_{\nabla_X Y}h)(Z, W)-(\nabla_Y h)(\nabla_X Z, W)-(\nabla_Y h)(Z, \nabla_X W)\\ &=(\nabla_X\nabla_Y h)(Z, W)+(\nabla_Y h)(\nabla_X Z, W)+(\nabla_Y h)(Z, \nabla_X W)\\ &\quad-(\nabla_{\nabla_X Y}h)(Z, W)-(\nabla_Y h)(\nabla_X Z, W)-(\nabla_Y h)(Z, \nabla_X W)\\ &=(\nabla_X\nabla_Y h)(Z, W)-(\nabla_{\nabla_X Y}h)(Z, W),\end{aligned}$$

we get

$$(\nabla_X C)(Y, Z, W)=(\nabla_X\nabla_Y h)(Z, W)-(\nabla_{\nabla_X Y}h)(Z, W).$$

Alternating this equation, we obtain (9).

REMARK. We have $(\nabla_X C)(Y, Z, W)=(\nabla_Y C)(X, Z, W)$ for all X, Y, Z, W if and only if ∇C is totally symmetric. By Lemma 1, this is the case if and only if $R(X, Y)\cdot h=0$ for all X, Y. It is known that this last condition is equivalent to the condition that M^n is an affine hypersphere. Hence ∇C is totally symmetric if and only if M^n is an affine hypersphere (as is known in [B-N-S]).

LEMMA 2. *The form Q is totally symmetric.*

PROOF. The symmetry in Y, Z, W is obvious. We now check the symmetry in X and Y. From (8) and (9) we get

$$\begin{aligned}Q(X, Y, Z, W)-Q(Y, X, Z, W)&=(R(X, Y)\cdot h)(Z, W)+h(X, SZ)h(W, Y)\\ &\quad+h(X, SW)h(Y, Z)-h(Y, SZ)h(W, X)-h(SY, W)h(X, Z).\end{aligned}$$

By the Gauss equation, we can check that the right-hand side is 0. Lemma 2 is thus proved.

We write $B(X, Y)=h(SX, Y)$ and observe that $\mathrm{trace}_h B=\mathrm{trace}\, S=nH$, where H is the affine mean curvature. We now introduce

$$B^0(X, Y)=B(X, Y)-nH\cdot h(X, Y) \tag{10}$$

as well as

$$Q^0(X, Y, Z, W)=(\nabla_X C)(Y, Z, W)+\mathscr{S}_{Y,Z,W}[B^0(X, Y)h(Z, W)], \tag{11}$$

where $\mathscr{S}$ denotes the cyclic sum as before.

Since

$$(Q-Q^0)(X, Y, Z, W)=nH\mathscr{S}_{Y,Z,W}[h(X, Y)h(Z, W)],$$

it follows that Q^0 is also symmetric.

2. Other invariants and some formulas. We recall that the affine connection ∇ and the Levi-Civita connection $\hat{\nabla}$ differ by a tensor K of type (1, 2) such that

$$K_X=\nabla_X-\hat{\nabla}_X \tag{12}$$

$$C(X, Y, Z)=-2h(K_X Y, Z), \quad K_X Y=K_Y X. \tag{13}$$

Apolarity can be expressed by $\operatorname{trace} K_X=0$ for all tangent vectors X; or, equivalently, $\operatorname{trace}_h K=0$.

We define a tensor L of type (1, 2) by

$$L(X, Y)=\operatorname{trace}_h[Z\to(\nabla_Z K)(X, Y)]. \tag{14}$$

We also set

$$S^0=S-HI, \tag{15}$$

where I denotes the identity endomorphism. S^0 is the trace 0 component of S.

LEMMA 3. *We have*

$$L(X, Y)=-\frac{n}{2}h(S^0X, Y)-2\operatorname{trace}(K_X K_Y). \tag{16}$$

PROOF. In [N] we defined

$$\hat{L}(X, Y)=\operatorname{trace}[Z\to(\hat{\nabla}_Z K)(X, Y)] \tag{17}$$

and showed

$$\hat{L}(X, Y)=\frac{n}{2}[Hh(X, Y)-h(SX, Y)].$$

Since the right hand side is $-(n/2)h(S^0X, Y)$ and since $\hat{L}(X, Y)-L(X, Y)=2\operatorname{trace}(K_X K_Y)$, we obtain (16).

Taking ∇_X of both sides of (13), we obtain

$$(\nabla_X C)(Y, Z, W) = 4h(K_X K_Y Z, W) - 2h((\nabla_X K)_Y Z, W)\,. \tag{18}$$

Hence we may express the quartic form Q by

$$Q(X, Y, Z, W) = -2h(\hat{Q}_X(Y, Z), W)\,, \tag{19}$$

where

$$\hat{Q}_X(Y, Z) = (\nabla_X K)_Y Z - 2K_X K_Y Z - \frac{1}{2}[h(Y, Z)SX + h(SX, Y)Z + h(SX, Z)Y]. \tag{20}$$

We define

$$U(X, Y) = \mathrm{trace}_h[Z \to \hat{Q}_Z(X, Y)]\,. \tag{21}$$

We find

$$U(X, Y) = L(X, Y) - \frac{1}{2}(\mathrm{trace}\, S)\cdot h(X, Y) - h(SX, Y)\,,$$

from which we get

LEMMA 4. *We have*

$$U(X, Y) = -\frac{(n+2)}{2}h(SX, Y) - 2\,\mathrm{trace}\, K_X K_Y\,.$$

Similarly to (19), we have

$$Q^0(X, Y, Z, W) = -2h(\hat{Q}^0_X(Y, Z), W)\,, \tag{22}$$

where

$$\hat{Q}^0_X(Y, Z) = (\nabla_X K)_Y Z - 2K_X K_Y Z - \frac{1}{2}[h(Y, Z)S^0X + h(S^0X, Y)Z + h(S^0X, Z)Y]\,. \tag{23}$$

We define

$$U^0(X, Y) = \mathrm{trace}_h[Z \to \hat{Q}^0_Z(X, Y)] \tag{24}$$

and obtain

LEMMA 5. *We have*

$$U^0(X, Y) = -\frac{(n+2)}{2}h(S^0X, Y) - 2\,\mathrm{trace}\, K_X K_Y\,.$$

3. Affine surfaces. The following is found in [B, p. 157].

LEMMA 6. *We have*

$$\operatorname{trace} K_X K_Y = J \cdot h(X, Y),$$

where $J = h(K, K)/2$ is the Pick invariant.

For the sake of completeness, we indicate the proof. Take a basis $\{X_1, X_2\}$ in the tangent space such that $h(X_1, X_1) = 1$, $h(X_1, X_2) = 0$, $h(X_2, X_2) = \varepsilon = \pm 1$. Using $h(K_{X_1}X_1, X_2) = h(K_{X_2}X_1, X_1)$, $h(K_{X_1}X_2, X_2) = h(K_{X_2}X_1, X_2)$ together with $\operatorname{trace} K_{X_1} = \operatorname{trace} K_{X_2} = 0$, we see that K_{X_1} and K_{X_2} are represented by matrices of the form

$$\begin{bmatrix} a & b \\ \varepsilon b & -a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b & -\varepsilon a \\ -a & -b \end{bmatrix},$$

respectively. We compute

$$\operatorname{trace} K_{X_1}^2 = 2(a^2 + \varepsilon b^2), \quad \operatorname{trace} K_{X_1} K_{X_2} = 0, \quad \operatorname{trace} K_{X_2}^2 = 2\varepsilon(a^2 + \varepsilon b^2).$$

We have $h(K, K) = 4(a^2 + \varepsilon b^2)$ and $J = 2(a^2 + \varepsilon b^2)$. This proves the formula.

We now obtain the following results.

THEOREM 1. *Let M^2 be a nondegenerate affine surface in $\boldsymbol{R}^3$ with the Blaschke structure. If U is identically* 0, *then the surface is an affine sphere and its affine metric is flat.*

REMARK. Affine spheres with flat affine metrics have been classified in [M-R] as the graphs of the following functions:

(I) $z = x^2 + y^2$;
(II) $z = 1/(xy)$;
(III) $z = 1/(x^2 + y^2)$;
(IV) $z = xy + F(x)$, where F is an arbitrary smooth function of y.

REMARK. Surfaces of type (IV) above are exactly nondegenerate ruled improper affine spheres, as is known in [B, p. 221].

THEOREM 2. *Let M^2 be a nondegenerate affine surface in $\boldsymbol{R}^3$. If Q is identically* 0, *then the surface is one of* (I), (II), (III) *in Theorem* 1 *or Cayley surface*

(IV*) $z = xy - y^3/3$.

REMARK. As is shown in [M-N], a nondegenerate affine surface such that $C \neq 0$ and $\hat{\nabla} C = 0$ is either (II), or (III) or (IV*).

THEOREM 3. *Let M^2 be a nondegenerate affine surface in $\boldsymbol{R}^3$. If U^0 is identically* 0, *then the surface is either a quadric or a surface* (IV) *or a ruled proper affine sphere, which was described by Radon as follows*:

(V) $(u, v), v > 0 \mapsto f(u, v) = vz(u) + z'(u)$,

where $z(u)$ is an $\boldsymbol{R}^3$-valued smooth function such that $\det[z\, z'\, z'']$ *is a nonzero constant.*

REMARK. For (IV) and (V), see [B, p. 220], as well as [M-R] and [Si].

THEOREM 4. *Let M^2 be a nondegenerate affine surface in $\boldsymbol{R}^3$. If Q^0 is identically*

0, *then the surface is a quadric whose affine metric is definite or a Cayley surface.*

We are going to prove these theorems.

PROOF OF THEOREM 1. By Lemmas 4 and 6 we get $h(SX, Y) = -Jh(X, Y)$ for all X, Y. Thus $S = -JI$, where I denotes the identity transformation. Hence $H = -J$. The scalar curvature $\hat{\rho}$ of the affine metric h is equal to $H + J$ (see Proposition 5.3 of [N]), which is 0 in our case. This means that the surface is an affine sphere with flat affine metric. By [M-R] we get the desired classification (I)–(IV) of such surfaces. For these surfaces, it is obvious that the tensor Q^0 is identically 0.

PROOF OF THEOREM 2. The assumption $Q = 0$ implies $U = 0$. By Theorem 1, we get affine surfaces (I)–(IV). For a surface of type (IV), we have $H = -J = 0$. Thus $\nabla C = Q = 0$. Excluding quadrics ($C = 0$) we get a Cayley surface. Among the quadrics, only the surface (I) satisfies $Q = 0$. We have thus shown that $Q = 0$ leads to the affine surfaces (I), (II), (III) and (IV*). Conversely, the surfaces (I) and (IV*) obviously satisfy $Q = 0$. Now we can verify by direct computation that the surfaces (II) and (III) also satisfy $Q = 0$. We shall illustrate this computation for the surface (II) only.

Denote $\{x, y\}$ by $\{x^1, x^2\}$. Then the affine metric has components

$$h_{11} = \frac{2}{\sqrt[4]{3}} x^{-2}, \quad h_{12} = \frac{1}{\sqrt[4]{3}} x^{-1} y^{-1}, \quad h_{22} = \frac{2}{\sqrt[4]{3}} y^{-2}.$$

The Christoffel symbols of the connection ∇ are given by

$$\Gamma^1_{11} = \frac{-2}{3} x^{-1}, \quad \Gamma^2_{11} = \frac{-2}{3} x^{-2} y, \quad \Gamma^1_{12} = \frac{-1}{3} y^{-1},$$

$$\Gamma^2_{12} = \frac{-1}{3} x^{-1}, \quad \Gamma^1_{22} = \frac{-2}{3} x y^{-2}, \quad \Gamma^2_{22} = \frac{-2}{3} y^{-1}.$$

Computing $C = \nabla h$ we find its components to be

$$C_{111} = 0, \quad C_{112} = \frac{2}{\sqrt[4]{3}} x^{-2} y^{-1}, \quad C_{122} = \frac{2}{\sqrt[4]{3}} x^{-1} y^{-2}, \quad C_{222} = 0.$$

We further obtain

$$C_{111;1} = \frac{4}{\sqrt[4]{3}} x^{-4}, \quad C_{111;2} = \frac{2}{\sqrt[4]{3}} x^{-3} y^{-1}, \quad C_{112;2} = \frac{2}{\sqrt[4]{3}} x^{-2} y^{-2},$$
$$C_{122;2} = \frac{2}{\sqrt[4]{3}} x^{-1} y^{-3}, \quad C_{222;2} = \frac{2}{\sqrt[4]{3}} y^{-4}.$$

Using $S = \sqrt[4]{3}\, I/3$ we can now check that all the components of Q are equal to 0. With this we conclude the proof of Theorem 2.

PROOF OF THEOREM 3. By Lemmas 5 and 6 we have $S^0=-JI$. But trace $S^0=0$, thus $J=0$, $S^0=0$, and $S=HI$.

Case (i): $C=0$. The surface is a quadric. Conversely, every quadric has $J=0$ and $S^0=0$, which imply $U^0=0$.

Case (ii): $C\neq 0$. In this case, h must be indefinite. Depending on whether the surface is an improper affine sphere or a proper affine sphere, it belongs to the class of surfaces (IV) or to the class of surfaces (V). Conversely, these classes of surfaces satisfy $J=0$ and $S^0=0$ and hence $U^0=0$ as well.

PROOF OF THEOREM 4. By (22) and (23) $Q^0=0$ implies $U^0=0$ and hence $J=0$, $S^0=0$. By (23) and (18) we get $\nabla C=0$. Now if $C=0$, the surface is a quadric. Otherwise, it is a Cayley surface (IV*) by a result in [N-P3]. Conversely, these surfaces satisfy $Q^0=0$.

4. Characterization of quadrics in $\boldsymbol{RP}^{n+1}$. In this section, we follow Section 5 of [N-P4] in order to consider a nondegenerate immersion f of a differentiable manifold M^n into the real projective space $\boldsymbol{RP}^{n+1}$. The space $\boldsymbol{RP}^{n+1}$ admits a flat projective structure, which we can describe as follows. By using any homogeneous coordinates $(x^1, \ldots, x^{n+2})$ in $\boldsymbol{RP}^{n+1}$ we take $n+1$ open subsets $V_k=\{x^k\neq 0\}$, where $1\leq k\leq n+1$. We may consider each V_k as an affine space $\boldsymbol{R}^{n+1}$ by regarding $\{y^i=x^i/x^k\}$, where $1\leq i\leq n+1$, $i\neq k$, as affine coordinates, thus providing V_k with a flat affine connection D_k. In the intersection $V_k\cap V_m$ of two open subsets the affine connections D_k and D_m are projectively related, that is, there is a 1-form μ such that $D_{kX}Y=D_{mX}Y+\mu(X)Y+\mu(Y)X$ for all vector fields X and Y.

We may actually consider an atlas of all locally defined affine connections which are projectively related to any local affine connection obtained in the manner above. Moreover, we may restrict ourselves to an atlas of such local affine connections which are equiaffine (that is, having a parallel volume element).

We now consider an immersion f of a differentiable manifold M^n into $\boldsymbol{RP}^{n+1}$. For each point p of M^n we may consider the restriction of f to a neighborhood U of p into some open subset V with an affine connection D belonging to the flat projective structure of $\boldsymbol{RP}^{n+1}$. We know what it means to say that f is nondegenerate on U, and this notion does not depend on the choice of (V, D). Thus it makes sense to say that f is nondegenerate at p. If this is the case for every p, we say that f is nondegenerate. We may further say that f is locally an affine hypersphere if the following condition is satisfied: If a neighborhood U of each point p is mapped into an open subset V with an affine connection D belonging to the flat projective structure of $\boldsymbol{RP}^{n+1}$, then $f\colon U\to V$ is an affine hypersphere in the sense of Blaschke. Note that we require this condition no matter how (V, D) is chosen. We may now state

THEOREM 5. *If a nondegenerate immersion $f\colon M^n\to \boldsymbol{RP}^{n+1}$ is locally an affine hypersphere, then $f(M^n)$ lies in a quadric in $\boldsymbol{RP}^{n+1}$.*

We may formulate Theorem 5 as follows. The proof of the equivalence of the two results is left to the reader.

THEOREM 6. *Let $f: M^n \to \boldsymbol{R}^{n+1}$ be a nondegenerate immersion. If for every projective transformation Φ of $\boldsymbol{R}^{n+1}$ the immersion $\Phi \cdot f$ is an affine hypersphere, then $f(M^n)$ lies on a quadric in $\boldsymbol{R}^{n+1}$.*

PROOF OF THEOREM 5. We follow the same kind of computation for a projective change of local connections $D \to D'$ in [N-P4, Section 5]. Suppose the connections and the parallel volume elements are related by $D'_X Y = D_X Y + \mu(X)Y + \mu(Y)X$, $\omega' = \phi\omega$, where $\mu = d(\log\phi)/(n+2)$.

The affine metric h, the affine normal ξ, and the affine shape operator S relative to D are related to h', ξ', and S' relative to D' as follows:

$$h' = \phi^{2/(n+2)} h \,;$$
$$\xi' = Z + \phi^{-2/(n+2)} \xi \,,$$

where Z is determined by $h(X, Z) = \phi^{-2/(n+2)} \mu(X)$;

$$\phi^{2/(n+2)} S'X = SX - \nabla_X U + \mu(X)U - \mu(\xi + U)X \,.$$

We define a vector field U by $h(U, X) = \mu(X)$ for all X. By computation we obtain

$$(25) \qquad (\nabla'_X C')(Y, Z, W)/\phi^{2/(n+2)} = (\nabla_X C)(Y, Z, W) - \frac{1}{6}\,\mathrm{Sym}[\mu(X)C(Y, Z, W)] + \mathscr{S}[h(X, Y)C(U, Z, W)] \,,$$

where Sym denotes the symmetric sum over X, Y, Z, W and $\mathscr{S}$ the cyclic sum over Y, Z, W.

Now by assumption f is an affine hypersphere relative to D and D'. Thus both ∇C and $\nabla' C'$ are totally symmetric (cf. Remark preceding Lemma 2). From (25) it follows that $\mathscr{S}[h(X, Y)C(U, Z, W)]$ is symmetric in X and Y. Hence we have

$$(26) \quad h(X, Z)C(U, W, Y) + h(X, W)C(U, Y, Z) = h(Y, Z)C(U, W, X) + h(Y, W)C(U, X, Z) \,.$$

Taking the trace with respect to X, Z relative to h, we obtain $C(U, Y, W) = 0$. Here Y, W are arbitrary. Now given a point x_0 of M and given any nonzero vector U at x_0, take the corresponding covector μ_0 at x_0 (now identified with $f(x_0)$ in $\boldsymbol{RP}^{n+1}$). We may find a function $\phi > 0$ in a neighborhood V (say, with a flat connection D) such that $\mu = d(\log\phi)/(n+2)$ equals μ_0 at x_0 and such that the Hessian of ϕ relative to the connection D is 0. Then the connection $D'_X Y = D_X Y + \mu(X)Y + \mu(Y)X$ is flat, projectively related to D. Relative to D, D' our previous argument shows that $C(U, Y, W) = 0$ at x_0. We have thus established that $C = 0$ at x_0 and hence everywhere. It is now clear that $f(M)$ lies on a quadric in $\boldsymbol{RP}^{n+1}$, completing the proof of Theorem 5.

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