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# SYNTOMIC COHOMOLOGY AND $p$-ADIC ÉTALE COHOMOLOGY

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This article is a complement to the paper "$p$-adic periods and $p$-adic étale cohomology" [FM] concerning the $p$-adic étale cohomology of varieties over $p$-adic fields. In [FM], the absolute ramification index of the base $p$-adic field was assumed to be one in the main results. We are interested in composing the method in [FM] and the study of $p$-adic vanishing cycles in the paper [BK]. We show that the composition gives, for a smooth proper variety with good reduction over a $p$-adic field and whose dimension not too big, fairly short proofs of the Hodge-Tate decomposition and of the crystalline conjecture [$\mathrm{Fo}_1$] without the assumption on the absolute ramification index. The Hodge-Tate decomposition and the crystalline conjecture were proved by Faltings without any assumption (cf. [$\mathrm{Fa}_1$], [$\mathrm{Fa}_2$]). The aim of this paper is to show the existence of a different method. This method has been, with a suitable modification and combined with ideas of Fontaine, recently found useful in the semi-stable reduction case as will be discussed elsewhere.

The method of composing the results of [FM] and [BK] as in this paper was found independently by L. Illusie.

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In this paper, $A$ denotes a complete discrete valuation ring with field of fractions $K$ and with residue field $k$ such that $\mathrm{char}(K)=0$, $\mathrm{char}(k)=p>0$, and $k$ is perfect. We denote by $\bar{K}$ (resp. $\bar{k}$) an algebraic closure of $K$ (resp. $k$ corresponding to $\bar{K}$), by $\bar{A}$ the integral closure of $A$ in $\bar{K}$, and by $\boldsymbol{C}_p$ the completion of $\bar{K}$.

For a scheme $X$, let $X_n = X \otimes \boldsymbol{Z}/p^n\boldsymbol{Z}$. For a scheme $X$ over $A$, let $\bar{X} = X \otimes_A \bar{A}$.

**1. Comments on crystalline cohomology and de Rham cohomology.** In this section, we state some results on crystalline cohomology and de Rham cohomology, whose proofs will be given in §4.

(1.1) Let $B_{\mathrm{crys}}$ and $B_{\mathrm{DR}}$ be the rings of Fontaine. We adopt here the following definitions of them given in [FM] using the crystalline cohomology theory, which are slightly different from the original definitions in [Fo]. Let $B_n$ be the crystalline cohomology of degree 0 of $\mathrm{Spec}(\bar{A}/p\bar{A})$ over $W_n = W_n(k)$, and let

$$B^+_{\mathrm{crys}} = \boldsymbol{Q} \otimes \varprojlim_n B_n\,, \qquad B_{\mathrm{crys}} = B^+_{\mathrm{crys}}[t^{-1}]$$

where $t$ is any non-zero element of $\boldsymbol{Q}_p(1)$ which is canonically embedded in $B^+_{\mathrm{crys}}$ ([FM]).

Let $J_{B_n}$ be the kernel of $B_n \to \bar{A}/p^n\bar{A}$, $J_{B_n}^{[r]}$ the $r$-th divided power of the ideal $J_{B_n}$, and let

$$B_{\mathrm{DR}}^{+} = \varprojlim_r (\boldsymbol{Q} \otimes \varprojlim_n B_n/J_{B_n}^{[r]}) , \qquad B_{\mathrm{DR}} = B_{\mathrm{DR}}^{+}[t^{-1}]$$

with $t$ as above. Then $B_{\mathrm{DR}}^{+}$ is a complete discrete valuation ring with field of fractions $B_{\mathrm{DR}}$, with residue field $\boldsymbol{C}_p$ and with a prime element $t$. Let $\mathrm{fil}^{\cdot}\, B_{\mathrm{DR}}$ be the filtration defined by this discrete valuation. We have $B_{\mathrm{DR}}^{+}/\mathrm{fil}^{r}\, B_{\mathrm{DR}} = \boldsymbol{Q} \otimes \varprojlim_n B_n/J_{B_n}^{[r]}$ for $r \geq 0$.

PROPOSITION (1.2). *Let $X$ be a proper smooth scheme over $A$ and let $Y = X \otimes_A k$. Then there exists a canonical isomorphism*

$$\boldsymbol{Q} \otimes \varprojlim_n H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/W_n}) \cong B_{\mathrm{crys}}^{+} \otimes_W H^m((Y/W)_{\mathrm{crys}}, \mathcal{O}_{Y/W}) .$$

(The case $A = W(k)$ is treated in [FM].)

As in [FM], a scheme $X$ over a scheme $Y$ is said to be syntomic over $Y$ if $X$ is flat and locally of complete intersection over $Y$.

PROPOSITION (1.3). *Let $X$ be a proper syntomic scheme over $A$ such that the generic fiber $X_K = X \otimes_A K$ is smooth over $K$. Then, for any $m$ and $r$, we have*:

(1)
$$\begin{aligned} &\boldsymbol{Q} \otimes \varprojlim_n H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/W_n}/J_{\bar{X}_n/W_n}^{[r]}) \\ &\cong (B_{\mathrm{DR}}^{+} \otimes_K H_{\mathrm{DR}}^m(X_K/K))/\mathrm{fil}^r(B_{\mathrm{DR}}^{+} \otimes_K H_{\mathrm{DR}}^m(X_K/K)) , \end{aligned}$$

*where $J_{\bar{X}_n/W_n}^{[r]}$ denotes the $r$-th divided power of the ideal $J_{\bar{X}_n/W_n} = \mathrm{Ker}(\mathcal{O}_{\bar{X}_n/W_n} \to \mathcal{O}_{\bar{X}_n})$.*

(2)
$$\boldsymbol{Q} \otimes \varprojlim_n H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, J_{\bar{X}_n/W_n}^{[r]}/J_{\bar{X}_n/W_n}^{[r+1]}) \cong \bigoplus_{0 \leq i \leq r} \boldsymbol{C}_p(r-i) \otimes_K H^{m-i}(X_K, \Omega_{X_K/K}^i)$$

*where $(r-i)$ means the Tate twist.*

COROLLARY (1.4). *Let $X$ be as in* (1.3).

(1) $\varprojlim_r (\boldsymbol{Q} \otimes \varprojlim_n H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/W_n}/J_{\bar{X}_n/W_n}^{[r]})) \cong B_{\mathrm{DR}}^{+} \otimes_K H_{\mathrm{DR}}^m(X_K/K)$ *for any $m$.*

(2)
$$\boldsymbol{Q} \otimes \varprojlim_n H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, J_{\bar{X}_n/W_n}^{[r]}/J_{\bar{X}_n/W_n}^{[r+1]})(-r) \cong \bigoplus_{i \in \boldsymbol{Z}} \boldsymbol{C}_p(-i) \otimes_K H^{m-i}(X_K, \Omega_{X_K/K}^i)$$

*if $r, m \in \boldsymbol{Z}$ and $r \geq m$.*

**2. $S_n^r$ and $p$-adic vanishing cycles.** We review a relationship between the sheaf $S_n^r$ of [FM] and $p$-adic vanishing cycles, which plays an essential role in this paper.

(2.1) Let $X$ be a smooth scheme over $A$. Let

$$Y = X \otimes_A k , \qquad \bar{Y} = \bar{X} \otimes_{\bar{A}} \bar{k} .$$

Let the sheaf $S_n^r$ on the syntomic site $(\bar{X}_{n+r})_{\mathrm{syn}}$ be as in [FM, III, 3.1]. Let

$$(\bar{X}_{n+r})_{\mathrm{syn}} \xrightarrow{\varepsilon} (\bar{X}_{n+r})_{\mathrm{et}} = \bar{Y}_{\mathrm{et}} \xrightarrow{i} \bar{X}_{\mathrm{et}} \xleftarrow{j} (X_{\bar{K}})_{\mathrm{et}}$$

be the canonical morphisms of sites. The following result (2.2), which relates $S_n^r$ to $p$-adic vanishing cycles, was obtained by Kurihara [Ku] (cf. also [Ka, I, 4.3]) by using the results on the sheaf $i^*R^qj_*(\boldsymbol{Z}/p^n\boldsymbol{Z})$ of $p$-adic vanishing cycles in [BK].

THEOREM (2.2). *Let the notation be as in* (2.1). *Assume* $0\leq r<p-1$. *Then, there exists a canonical isomorphism*

$$\beta\colon R\varepsilon_*(S_n^r)\xrightarrow{\cong}\tau_{\leq r}i^*Rj_*(\boldsymbol{Z}/p^n\boldsymbol{Z}(r))\,.$$

*In particular, if $X$ is furthermore proper over $A$ and if either $m\leq r$ or $r\geq\dim(X_K)$, then* (*by the proper base change theorem*) *we have*

$$H^m(\bar{X},S_n^r)\xrightarrow{\cong}H^m_{\mathrm{et}}(X_{\bar{K}},\boldsymbol{Z}/p^n\boldsymbol{Z}(r))\,.$$

Here, as in [FM], we write by $H^m(\bar{X},S_n^r)$ the group $H^m((\bar{X}_{n+r})_{\mathrm{syn}},S_n^r)$, which coincides with $H^m((\bar{X}_i)_{\mathrm{syn}},S_n^r)$ for any $i\geq n+r$. We denote by $\tau_{\leq r}$ the canonical truncation.

REMARK (2.3). If fact the paper [Ku] of Kurihara includes, not only results over $\bar{A}$ as above but also results over $A$ relating $S_n^r$ on $X_{n+r}$ to $p$-adic vanishing cycles for $Y\to X\leftarrow X_K$.

(2.4) In the rest of this section, we explain how the map $\beta$ in (2.2) is defined by using the theory in [FM] (III §5) (this point does not seem to be explained in detail in [Ku]).

Let $\hat{\bar{X}}$ be the $p$-adic formal completion $\varinjlim\bar{X}_n$ of $\bar{X}$, and let $\bar{X}_{\text{syn-et}}$ and $\hat{\bar{X}}_{\text{syn-et}}$ be the syntomic-étale sites on $\bar{X}$ and on $\hat{\bar{X}}$, respectively. We have a commutative diagram of sites

$$\begin{array}{ccccc}\hat{\bar{X}}_{\text{syn-et}} & \xrightarrow{i_{\mathrm{se}}} & \bar{X}_{\text{syn-et}} & \xleftarrow{j_{\mathrm{se}}} & (X_{\bar{K}})_{\mathrm{et}}\\ {\scriptstyle\hat{\varepsilon}_{\mathrm{se}}}\downarrow & & {\scriptstyle\varepsilon_{\mathrm{se}}}\downarrow & & \Big| \\ \bar{Y}_{\mathrm{et}}=\hat{\bar{X}}_{\mathrm{et}} & \xrightarrow{i_{\mathrm{et}}} & \bar{X}_{\mathrm{et}} & \xleftarrow{j_{\mathrm{et}}} & (X_{\bar{K}})_{\mathrm{et}}\,.\end{array}$$

In (2.1) (2.2), we denoted $i_{\mathrm{et}}$ and $j_{\mathrm{et}}$ by $i$ and $j$, respectively, though $i_{\mathrm{se}}$ and $j_{\mathrm{se}}$ are denoted by $i$ and $j$ in [FM], respectively. Let $S^r_{n,\hat{\bar{X}}}$ be the direct image of $S_n^r$ under the canonical morphism $i_{n+r}\colon(\bar{X}_{n+r})_{\mathrm{syn}}\to\hat{\bar{X}}_{\text{syn-et}}$. Since $(i_{n+r})_*$ is exact [FM, III, 4.1], we have $R\varepsilon_*(S_n^r)=R\hat{\varepsilon}_{\mathrm{se}*}(S^r_{n,\hat{\bar{X}}})$. By [FM, III §5], we have a canonical homomorphism

$$\alpha\colon S^r_{n,\hat{\bar{X}}}\to i^*_{\mathrm{se}}j_{\mathrm{se}*}(\boldsymbol{Z}/p^n\boldsymbol{Z}(r))\,.$$

By applying the functor $R\hat{\varepsilon}_{\mathrm{se}*}$ to the induced map $S^r_{n,\hat{\bar{X}}}\to i^*_{\mathrm{se}}Rj_{\mathrm{se}*}(\boldsymbol{Z}/p^n\boldsymbol{Z}(r))$, we obtain a map

$$R\varepsilon_*(S_n^r)=R\hat{\varepsilon}_{\mathrm{se}*}(S^r_{n,\hat{\bar{X}}})\to R\hat{\varepsilon}_{\mathrm{se}*}i^*_{\mathrm{se}}Rj_{\mathrm{se}*}(\boldsymbol{Z}/p^n\boldsymbol{Z}(r))\,.$$

LEMMA (2.5). $R\hat{\varepsilon}_{se*}i_{se}^* = i_{et}^* R\varepsilon_{se*}$.

This is reduced to $\hat{\varepsilon}_{se*}i_{se}^* = i_{et}^*\varepsilon_{se*}$ which follows from the explicit description of $i_{se}^*$ in [FM, III, 4.4].

(2.6) By (2.5), we have

$$R\hat{\varepsilon}_{se*}i_{se}^* Rj_{se*}(\boldsymbol{Z}/p^n\boldsymbol{Z}(r)) = i_{et}^* R\varepsilon_{se*}Rj_{se*}(\boldsymbol{Z}/p^n\boldsymbol{Z}(r)) = i_{et}^* Rj_{et*}(\boldsymbol{Z}/p^n\boldsymbol{Z}(r)) .$$

By [Ka, I, 3.6],

$$R^q\varepsilon_*(S_n^r) = 0 \qquad \text{for} \quad q > r .$$

Hence our map $R\varepsilon_*(S_n^r) \to i_{et}^* Rj_{et*}(\boldsymbol{Z}/p^n\boldsymbol{Z}(r))$ factors through $\tau_{\leq r} i_{et}^* Rj_{et*}(\boldsymbol{Z}/p^n\boldsymbol{Z}(r))$.

**3. The Hodge-Tate decomposition and the crystalline conjecture.** In this section, let $X$ be a smooth proper scheme over $A$. We explain how the results (1.2), (1.4), (2.2) can be used to prove the Hodge-Tate decomposition and the crystalline conjecture for $X$ in the case $p > 2\dim(X_K) + 1$.

(The Hodge-Tate decomposition for $X$ follows from the crystalline conjecture, but we treat the Hodge-Tate decomposition separately, for one can give a separate easy approach to it.)

(3.1) Recall that the Hodge-Tate decomposition for $X$ is a $\boldsymbol{C}_p$-linear isomorphism

$$\boldsymbol{C}_p \otimes_{\boldsymbol{Q}_p} H^m((X_{\bar{K}})_{et}, \boldsymbol{Q}_p) \cong \bigoplus_{i \in \boldsymbol{Z}} \boldsymbol{C}_p(-i) \otimes_K H^{m-i}(X_K, \Omega^i_{X_K/K}) \tag{3.1.1}$$

preserving the actions of $\mathrm{Gal}(\bar{K}/K)$. Here an element $\sigma$ of $\mathrm{Gal}(\bar{K}/K)$ acts on the left hand side by $\sigma \otimes \sigma$ and on the right hand side by $\sigma \otimes (\mathrm{id.})$.

Consider the canonical map

$$H^m(\bar{X}, S_n^r) \to H^m((\bar{X}_n/W_n)_{crys}, J^{[r]}_{\bar{X}_n/W_n}/J^{[r+1]}_{\bar{X}_n/W_n}) .$$

If $m < p-1$, by taking $\boldsymbol{Q} \otimes \varprojlim_n$ and using (1.4) (2) and (2.2), we obtain the desired $\boldsymbol{C}_p$-linear map from the left hand side of (3.1.1) to the right hand side of (3.1.1) preserving the actions of $\mathrm{Gal}(\bar{K}/K)$. If $p > 2\dim(X_K) + 1$, Poincaré duality shows that this map is an isomorphism (cf. [FM, III, 6.3] for this argument).

(3.2) Recall that the crystalline conjecture says that there exists a $B_{crys}$-linear isomorphism

$$B_{crys} \otimes_{\boldsymbol{Q}_p} H^m((X_{\bar{K}})_{et}, \boldsymbol{Q}_p) \cong B_{crys} \otimes_W H^m_{crys}(Y/W) \tag{3.2.1}$$

preserving the Frobenius and the actions of $\mathrm{Gal}(\bar{K}/K)$ such that the composite map induced by (3.2.1)

$$B_{DR} \otimes_{\boldsymbol{Q}_p} H^m((X_{\bar{K}})_{et}, \boldsymbol{Q}_p) \cong B_{DR} \otimes_W H^m_{crys}(Y/W) \cong B_{DR} \otimes_K H^m_{DR}(X_K/K) \tag{3.2.2}$$

gives an isomorphism of filtrations. Here the Frobenius on the left hand side of (3.2.1) is $\varphi \otimes (\mathrm{id.})$ and that on the right hand side is $\varphi \otimes \varphi$ ($\varphi$ denotes the Frobenius of $B_{\mathrm{crys}}$ and also that of $H^m_{\mathrm{crys}}(Y/W)$), an element $\sigma$ of $\mathrm{Gal}(\bar{K}/K)$ acts on the left hand side of (3.2.1) by $\sigma \otimes \sigma$ and on the right hand side by $\sigma \otimes (\mathrm{id.})$, the filtration on the first group in (3.2.2) is defined by $\mathrm{fil}^i = \mathrm{fil}^i B_{\mathrm{DR}} \otimes H^m((X_{\bar{K}})_{\mathrm{et}}, \boldsymbol{Q}_p)$, and the filtration on the last group in (3.2.2) is the tensor product of the filtration on $B_{\mathrm{DR}}$ and the Hodge filtration on $H^m_{\mathrm{DR}}(X_K/K)$.

By the canonical map $H^m(\bar{X}, S^r_n) \to H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, J^{[r]}_{\bar{X}_n/W_n})$ and the inclusion map $J^{[r]}_{\bar{X}_n/W_n} \to \mathcal{O}_{\bar{X}_n/W_n}$, we obtain a canonical map

$$H^m(\bar{X}, S^r_n) \to H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/W_n})$$

on whose image, the Frobenius acts as the multilication by $p^r$. If $m < p-1$, by taking $\boldsymbol{Q} \otimes \varprojlim_n$ and using (1.2) and (2.2), we have the desired map from the left hand side of (3.2.1) to the right hand side of (3.2.1) which preserves the Frobenius and the actions of $\mathrm{Gal}(\bar{K}/K)$. If $p > 2\dim(X_K)+1$, Poincaré duality shows that this map is an isomorphism. We check that the induced composite isomorphism (3.2.2) preserves the filtrations. Since this map is induced from

$$\begin{aligned}\varprojlim_n H^m(\bar{X}, S^r_n)(-r) &\to \varprojlim_N (\boldsymbol{Q} \otimes \varprojlim_n H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, J^{[r]}_{\bar{X}_n/W_n}/J^{[N]}_{\bar{X}_n/W_n})(-r)) \\ &\cong \mathrm{fil}^r(B^+_{\mathrm{DR}} \otimes H^m_{\mathrm{DR}}(X_K/K))(-r) \qquad (1.4)\,(1)\,,\end{aligned}$$

$\mathrm{fil}^i$ of the first group in (3.2.2) is sent into $\mathrm{fil}^i$ of the last group in (3.2.2). Hence, that (3.2.2) gives an isomorphism of filtrations is, by taking gr. of the filtrations, reduced to the Hodge-Tate decomposition (3.1.1).

Remark (3.3). Here we compare the method in [FM] and that in this paper. Consider the diagram

$$\begin{array}{ccc} B_{\mathrm{crys}} \otimes_{\boldsymbol{Z}_p} \varprojlim_n H^m(\bar{X}, S^r_n)(-r) & \xrightarrow{(1)} & B_{\mathrm{crys}} \otimes_{\boldsymbol{Q}_p} H^m((X_{\bar{K}})_{\mathrm{et}}, \boldsymbol{Q}_p) \\ {\scriptstyle (2)}\downarrow & & \\ B_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(Y/W)\,. & & \end{array}$$

In [FM], it is proved first that if $p > \dim(X_K)$, $m \leq r$, and $A = W(k)$, then the map (2) is bijective. Then, (1) is proved to be bijective (under the same assumption) by using Poincaré duality (cf. [FM, III, 6.1].) In the method in this paper, it is proved first that if $m \leq r < p-1$, the map (1) is bijective (§2). Then, if $m \leq r < p-1$ and $p > 2\dim(X_K)+1$, (2) is proved to be bijective by using Poincaré duality.

(If $A$ ramifies over $W(k)$, it becomes very difficult to prove the bijectivity of (2) first directly, for the necessary theory of filtered modules becomes complicated in this case.)

## 4. Proofs of the Propositions (1.2) and (1.3).

DEFINITION (4.1). (1) If we are given categories $\mathscr{C}_n$ $(n\geq 0)$ and functors $\theta_n$: $\mathscr{C}_{n+1}\to\mathscr{C}_n$ $(n\geq 0)$, we denote by $\varprojlim_n \mathscr{C}_n$ the category of systems $\{A_n, \alpha_n\}_n$ where $A_n$ is an object of $\mathscr{C}_n$ and $\alpha_n$ is a morphism $\theta_n(A_{n+1})\to A_n$ for each $n$. (We sometimes abbreviate $\{A_n, \alpha_n\}_n$ as $\{A_n\}_n$ in the following.)

(2) For an additive category $\mathscr{C}$, let $\boldsymbol{Q}\otimes\mathscr{C}$ be the category whose objects are the same as $\mathscr{C}$ but $\mathrm{Hom}_{\boldsymbol{Q}\otimes\mathscr{C}}=\boldsymbol{Q}\otimes\mathrm{Hom}_{\mathscr{C}}$. For an object $P$ of $\mathscr{C}$, we denote by $\boldsymbol{Q}\otimes P$ the object $P$ regarded as an object of $Q\otimes\mathscr{C}$.

(4.2) Proof of (1.2). We denote $\mathrm{Spec}(A)$ by $S$. Let $f_n$ (resp. $\bar{f}_n$, $g_n$) be the morphism $X_n\to S_n$ (resp. $\bar{X}_n\to\bar{S}_n$, resp. $Y\to W_n$) and let $(f_n)_{\mathrm{crys}}$, $(\bar{f}_n)_{\mathrm{crys}}$, $(g_n)_{\mathrm{crys}}$ be the induced morphisms between the crystalline sites over $W_n$, respectively. By the base change theorem for crystalline cohomology ([B, V, 3.5]), we have

$$R(\bar{f}_n)_{\mathrm{crys}*}(\mathcal{O}_{\bar{X}_n/W_n})\cong\mathcal{O}_{\bar{S}_n/W_n}\otimes^L_{\mathcal{O}_{S_n/W_n}}R(f_n)_{\mathrm{crys}*}(\mathcal{O}_{X_n/W_n}).$$

By Berthelot-Ogus [BO$_2$], we have an isomorphism in the category $\boldsymbol{Q}\otimes\varprojlim_n D((S_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{S_n/W_n}))$;

$$\boldsymbol{Q}\otimes\{R(f_n)_{\mathrm{crys}*}(\mathcal{O}_{X_n/W_n})\}_n\cong\boldsymbol{Q}\otimes\{\mathcal{O}_{S_n/W_n}\otimes^L_{W_n}R(g_n)_{\mathrm{crys}*}(\mathcal{O}_{Y/W_n})\}_n .$$

(Here $D((S_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{S_n/W_n})$ denotes the derived category of the category of $\mathcal{O}_{S_n/W_n}$-modules on $(S_n/W_n)_{\mathrm{crys}}$.) Hence we have an isomorphism in $\boldsymbol{Q}\otimes\varprojlim_n D((\bar{S}_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{\bar{S}_n/W_n})$;

$$\boldsymbol{Q}\otimes\{R(\bar{f}_n)_{\mathrm{crys}*}(\mathcal{O}_{\bar{X}_n/W_n})\}_n\cong\boldsymbol{Q}\otimes\{\mathcal{O}_{\bar{S}_n/W_n}\otimes^L_{W_n}R(g_n)_{\mathrm{crys}*}(\mathcal{O}_{Y/W_n})\}_n .$$

By taking $R\Gamma((\bar{S}_n/W_n)_{\mathrm{crys}},)$ and by using

$$B_n=R\Gamma((\bar{S}_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{\bar{S}_n/W_n})\qquad(\text{cf. [Fo}_2\text{]}),$$

we obtain in $\varprojlim_n D$ ($W_n$-modules);

$$\boldsymbol{Q}\otimes\{R\Gamma((\bar{X}_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/W_n})\}_n\cong\boldsymbol{Q}\otimes\{B_n\otimes_{W_n}R\Gamma((Y/W_n)_{\mathrm{crys}}, \mathcal{O}_{Y/W_n})\}_n .$$

This proves (1.2).

(4.3) We give a preliminary needed for the proof of (1.3).

Generally, let $X$, $Y$, $Z$ be schemes, let $f$: $X\to Y$, $g$: $Y\to Z$ be syntomic morphisms, and assume we are given a quasi-coherent ideal $\mathfrak{A}$ of $\mathcal{O}_Z$ endowed with a $PD$-structure. Endow $\mathfrak{A}\mathcal{O}_Y$ with the unique $PD$-structure compatible with that on $\mathfrak{A}$. In this (4.3), we consider the relationship between

$$Ru_{X/Z*}(J^{[r]}_{X/Z}/J^{[r+1]}_{X/Z}),\qquad Ru_{X/Y*}(J^{[s]}_{X/Y}/J^{[s+1]}_{X/Y}),\qquad Ru_{Y/Z*}(J^{[t]}_{Y/Z}/J^{[t+1]}_{Y/Z})$$

$r, s, t\geq 0$, where $u_{X/Z}$ is the canonical morphism $(X/Z)_{\mathrm{crys}}\to X_{\mathrm{zar}}$, $J^{[r]}_{X/Z}$ is the $r$-th divided power of $J_{X/Z}=\mathrm{Ker}(\mathcal{O}_{X/Z}\to\mathcal{O}_X)$, ..., etc. For $r, i\geq 0$, we construct objects $F^i_r$ in the derived category $D(X, \mathcal{O}_X)$ such that $F_r^0 = Ru_{X/S*}(J_{X/Z}^{[r]}/J_{X/Z}^{[r+1]})$ and $F_r^i = 0$ for $i>r$ with distinguished triangles

$$F_r^{i+1} \to F_r^i \to Ru_{X/Y*}(J_{X/Y}^{[r-i]}/J_{X/Y}^{[r-i+1]}) \otimes^L_{\mathcal{O}_X} Lf^* Ru_{Y/Z}(J_{Y/Z}^{[i]}/J_{Y/Z}^{[i+1]}) \to .$$

If there exists a commutative diagram of schemes

$$\text{(4.3.1)} \qquad \begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \\ E & \longrightarrow & F & & \end{array}$$

such that the vertical arrows are closed immersions and that $E$ is smooth over $F$ and $F$ is smooth over $Z$, then the objects $F_r^i$ are defined as follows. Let $I$ (resp. $I'$, resp. $I''$) be the ideal of $X$ in $E$ (resp. $X$ in $E \times_F Y$, resp. $Y$ in $F$). Then, $Ru_{X/Z*}(J_{X/Z}^{[r]}/J_{X/Z}^{[r+1]})$ is represented by the complex

$$\text{(4.3.2)} \qquad J^{[r]}/J^{[r+1]} \xrightarrow{d} J^{[r-1]}/J^{[r]} \otimes_{\mathcal{O}_E} \Omega^1_{E/Z} \xrightarrow{d} J^{[r-2]}/J^{[r-1]} \otimes_{\mathcal{O}_E} \Omega^2_{E/Z} \xrightarrow{d} \cdots ,$$

where $J^{[r]}$ is the $r$-th divided power of $\mathrm{Ker}(\mathcal{O}_D \to \mathcal{O}_X)$ with $D$ the $PD$-envelope of $X$ in $E$. Since $X$ is syntomic over $Z$, $\bigoplus_{r \in \mathbf{Z}} J^{[r]}/J^{[r+1]}$ is isomorphic to the divided power polynomial ring on the locally free $\mathcal{O}_X$-module $I/I^2$. Denote the degree $r$ part of this ring by $(I/I^2)^{[r]}$. Then (4.3.2) is isomorphic with

$$\text{(4.3.3)} \qquad (I/I^2)^{[r]} \xrightarrow{d} (I/I^2)^{[r-1]} \otimes_{\mathcal{O}_E} \Omega^1_{E/Z} \xrightarrow{d} (I/I^2)^{[r-2]} \otimes_{\mathcal{O}_E} \Omega^2_{E/Z} \xrightarrow{d} \cdots .$$

Let $F_r^i$ $(i \geq 0)$ be the subcomplex of (4.3.3) whose degree $q$ part is the image of

$$\bigoplus_{j \in \mathbf{Z}} (I''/I''^2)^{[j]} \otimes (I/I^2)^{[r-q-i]} \otimes \Omega^{i-j}_{F/Z} \otimes \Omega^{q-i+j}_{E/Z} .$$

Then, $F_r^0$ is the complex (4.3.3) itself, $F_r^i = 0$ for $i > r$, and the complex $F_r^i/F_r^{i+1}$ is isomorphic to the tensor product of the two complex

$$(I''/I''^2)^{[i]} \xrightarrow{d} (I''/I''^2)^{[i-1]} \otimes_{\mathcal{O}_F} \Omega^1_{F/Z} \xrightarrow{d} \cdots ,$$

$$(I'/I'^2)^{[r-i]} \xrightarrow{d} (I'/I'^2)^{[r-i-1]} \otimes_{\mathcal{O}_E} \Omega^1_{E/F} \xrightarrow{d} \cdots$$

where the former (resp. the latter) is isomorphic in the derived category to

$$Ru_{Y/Z*}(J_{Y/Z}^{[i]}/J_{Y/Z}^{[i+1]}) \qquad (\text{resp. } Ru_{X/Y*}(J_{X/Y}^{[r-i]}/J_{X/Y}^{[r-i+1]})) .$$

In general, the diagram (4.3.1) may not exist, but it exists locally on $X$ and $Y$. In the general case, the method of the cohomological descent as in [BO$_1$] shows that $F_r^i$ is defined globally in $D(X_{\mathrm{zar}}, \mathcal{O}_X)$.

(4.4) We prove (1.3) (2). We apply (4.3) to the case $X$, $Y$, $Z$ are $\bar{X}_n$, $\bar{S}_n$, $W_n$, respectively ($S = \mathrm{Spec}(A)$). In fact, $\bar{S}_n$ is not syntomic over $W_n$ but it is a filtered projective limit of syntomic schemes over $W_n$. By taking inductive limit, we obtain objects $F^i_{r,n}$ in $D(\bar{X}_n, \mathcal{O}_{\bar{X}_n})$ ($i \geq 0$) such that $F^0_{r,n} = Ru_{\bar{X}_n/W_n *}(J^{[r]}_{\bar{X}_n/W_n}/J^{[r+1]}_{\bar{X}_n/W_n})$, $F^i_{r,n} = 0$ for $i > r$, and that we have distinguished triangles

$$F^{i+1}_{r,n} \to F^i_{r,n} \to Ru_{\bar{S}_n/W_n *}(J^{[i]}_{\bar{S}_n/W_n}/J^{[i+1]}_{\bar{S}_n/W_n}) \otimes^L_{\mathcal{O}_{\bar{S}_n}} Ru_{\bar{X}_n/\bar{S}_n}(J^{[r-i]}_{\bar{X}_n/\bar{S}_n}/J^{[r-i+1]}_{\bar{X}_n/\bar{S}_n}) \to . \tag{4.4.1}$$

Note

$$Ru_{\bar{S}_n/W_n *}(J^{[r]}_{\bar{S}_n/W_n}/J^{[r+1]}_{\bar{S}_n W_n}) \cong J^{[r]}_{B_n}/J^{[r+1]}_{B_n} \qquad \text{(put in degree 0)}, \tag{4.4.2}$$

$$\boldsymbol{Q} \otimes \varprojlim_n J^{[r]}_{B_n}/J^{[r+1]}_{B_n} \cong C_p(r) ,$$

$$Ru_{\bar{X}_n/\bar{S}_n *}(J^{[r]}_{\bar{X}_n/\bar{S}_n}/J^{[r+1]}_{\bar{X}_n/\bar{S}_n}) \cong Ru_{X_n/S_n *}(J^{[r]}_{X_n/S_n}/J^{[r+1]}_{X_n/S_n}) \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{\bar{X}_n} . \tag{4.4.3}$$

By (4.5) (2) below, we have isomorphisms in $\boldsymbol{Q} \otimes \varprojlim_n D(X_n, \mathcal{O}_{X_n})$;

$$\boldsymbol{Q} \otimes \{Ru_{X_n/W_n *}(J^{[r]}_{X_n/W_n}/J^{[r+1]}_{X_n/W_n})\}_n \xrightarrow{\cong} \boldsymbol{Q} \otimes \{Ru_{X_n/S_n *}(J^{[r]}_{X_n/S_n}/J^{[r+1]}_{X_n/S_n})\}_n \tag{4.4.4}$$

$$\xrightarrow{\cong} \boldsymbol{Q} \otimes \{\Omega^r_{X_n/S_n}[-r]\}_n .$$

Now (4.4.1)–(4.4.4) show that the canonical homomorphism

$$\bigoplus_{0 \leq i \leq r} \boldsymbol{Q} \otimes \{J^{[r-i]}_{B_n}/J^{[r-i+1]}_{B_n} \otimes_{W_n} Ru_{X_n/W_n *}(J^{[i]}_{X_n/W_n}/J^{[i+1]}_{X_n/W_n})\}_n$$

$$\to \boldsymbol{Q} \otimes \{Ru_{\bar{X}_n/W_n *}(J^{[r]}_{\bar{X}_n/W_n}/J^{[r+1]}_{\bar{X}_n/W_n})\}_n$$

is an isomorphism in $\boldsymbol{Q} \otimes \varprojlim_n D(\bar{X}_n, \mathcal{O}_{\bar{X}_n})$ and induces the isomorphism (1.3) (2).

LEMMA (4.5). *Let $X$ be as in* (1.3).

(1) *The canonical morphism in $\varprojlim_n D(X_n, \boldsymbol{Z}/p^n\boldsymbol{Z})$*

$$\{Ru_{X_n/W_n *}(\mathcal{O}_{X_n/W_n}/J^{[r]}_{X_n/W_n})\}_n \to \{\Omega^{<r}_{X_n/S_n}\}_n$$

($\Omega^{<r}$ *denotes the complex* $\Omega^0 \to \Omega^1 \to \cdots \to \Omega^{r-1} \to 0 \to 0 \to \cdots$) *induces an isomorphism in* $\boldsymbol{Q} \otimes \varprojlim_n D(X_n, \boldsymbol{Z}/p^n\boldsymbol{Z})$.

(2) *The canonical morphism in $\varprojlim_n D(X_n, \mathcal{O}_{X_n})$*

$$\{Ru_{X_n/W_n}(J^{[r]}_{X_n/W_n}/J^{[r+1]}_{X_n/W_n})\}_n \to \{\Omega^r_{X_n/A_n}[-r]\}_n$$

*induces an isomorphism in* $\boldsymbol{Q} \otimes \varprojlim_n D(X_n, \mathcal{O}_{X_n})$.

PROOF. Since $X_K$ is smooth over the field of fractions $K'$ of $W$, we have

$$Ru_{X_K/K' *}(J^{[r]}_{X_K/K'}/J^{[r+1]}_{X_K/K'}) = \Omega^r_{X_K/K'}[-r] \cong \Omega^r_{X_K/K}[-r] .$$

Hence the canonical homomorphism

$$Ru_{X/W *}(J^{[r]}_{X/W}/J^{[r+1]}_{X/W}) \to \Omega^r_{X/A}[-r]$$

becomes an isomorphism after $\otimes \boldsymbol{Q}$. This proves (2), and (1) follows from (2).

(4.6) Finally we prove (1.3) (1). By (4.5) (1), we have a canonical homomorphism from $B_{\mathrm{DR}}^{+} \otimes_K H_{\mathrm{DR}}^m(X_K/K)$ to $\boldsymbol{Q} \otimes \varprojlim_n H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/W_n}/J_{\bar{X}_n/W_n}^{[r]})$ which annihilates $\mathrm{fil}^r(B_{\mathrm{DR}}^{+} \otimes_K H_{\mathrm{DR}}^m(X_K/K))$. That the induced map

$$\begin{aligned}(B_{\mathrm{DR}}^{+} \otimes_K H_{\mathrm{DR}}^m(X_K/K))/\mathrm{fil}^r\,(B_{\mathrm{DR}}^{+} \otimes_K H^m(X_K/K))\\ \to \boldsymbol{Q} \otimes \varprojlim_n H^m((\bar{X}_n/W_n)_{\mathrm{crys}}, \mathcal{O}_{\bar{X}_n/W_n}/J_{\bar{X}_n/W_n}^{[r]})\end{aligned}$$

is an isomorphism is reduced to (1.3) (2).

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