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ON COMPACT CONFORMALLY FLAT 4-MANIFOLDS

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Abstract. We prove a certain gap theorem concerning the Yamabe invariants for compact conformally flat 4-manifolds with positive Euler numbers.

1. Introduction. Let M be a smooth manifold and C a conformal class of metrics of M. (M, C) is said to be *conformally flat* if, for any $p \in M$, C contains a metric which is flat in some neighborhood of p. A conformal class C is called a *flat conformal structure* on M if (M, C) is conformally flat. A Riemannian manifold $(M\ g)$ is said to be *conformally flat* if (M, C) is conformally flat for the class C containing g. In order to understand conformally flat manifolds from the Riemannian-geometric viewpoint, it is useful to choose reasonable metrics as representatives of conformal classes. In the two-dimensional case, for any conformal class of a connected surface, such metrics are given as complete constant curvature metrics by the uniformization theorem of Riemann surfaces. If M is compact and connected, then the Yamabe problem gives representatives in higher dimension.

Let M be a compact connected manifold with $\dim M = n \geqq 3$. The *Yamabe functional* I on a conformal class C (not necessarily conformally flat) of M is defined as

$$I(g) = \frac{\int_M R_g dV_g}{\left(\int_M dV_g\right)^{(n-2)/n}},$$

for $g \in C$, where R_g and dV_g denote the scalar curvature and the volume element of g, respectively. The infimum of this functional is denoted by $\mu(M, C)$, i.e.,

$$\mu(M, C) = \inf_{g \in C} I(g),$$

and called the *Yamabe invariant*. The Yamabe problem asks the existence of a metric satisfying $I(g) = \mu(M, C)$. By Yamabe [13], Trudinger [12], Aubin [2], and Schoen [11] this question was answered affirmatively as we see in Theorem 2.1. Thus, for any (M, C), we can choose a metric $g \in C$ such that $I(g) = \mu(M, C)$. Moreover, by so normalizing g that $\mathrm{Vol}(M, g) = 1$, the scalar curvature of g is constant and equal to $\mu(M, C)$ (see

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Proposition 2.2). Then it seems that the Yamabe invariant $\mu(M, C)$, which is considered as the scalar curvature of a representative metric of C, gives some information on (M, C). The purpose of this paper is to prove a certain gap theorem concerning the Yamabe invariants for conformally flat 4-manifolds.

We denote by C_0 the conformal class of the standard metric g_0 of S^n and $\boldsymbol{RP}^n$. Then our result is stated as follows:

THEOREM. *Let M be a compact connected 4-manifold with positive Euler number, and C a flat conformal structure on M.*

(1) *If M is orientable and $\mu(M, C) > -\mu(S^4, C_0)$, then (M, C) is conformal to (S^4, C_0).*

(2) *If M is orientable and $\mu(M, C) = -\mu(S^4, C_0)$, then C contains a negative constant curvature metric.*

(3) *If M is non-orientable and $\mu(M, C) > -\mu(\boldsymbol{RP}^4, C_0)$, then (M, C) is conformal to $(\boldsymbol{RP}^4, C_0)$.*

(4) *If M is non-orientable and $\mu(M, C) = -\mu(\boldsymbol{RP}^4, C_0)$, then C contains a negative constant curvature metric.*

REMARK. The conformal classes of flat metrics on T^4 are conformally flat and their Yamabe invariants are equal to zero. For any positive integer k and any positive real number ε, there exists a flat conformal structure C on a connected sum of k-copies of $S^1 \times S^3$ such that its Yamabe invariant is greater than $\mu(S^4, C_0) - \varepsilon$ (this follows from [5, Theorem 2]). Thus, for a manifold M with non-positive Euler number, we cannot determine (M, C) up to conformal equivalence by the Yamabe invariants.

By Lemma 2.3, Corollary 2.4, and Proposition 2.5 in the next section, we have the following corollary.

COROLLARY. *Let (M, g) be a compact, connected, and conformally flat Riemannian 4-manifold with positive Euler number.*

(1) *If M is orientable and* $\min R_g\{\mathrm{Vol}(M, g)\}^{1/2} > -\mu(S^4, C_0)$, *then (M, g) is conformal to (S^4, g_0).*

(2) *If M is orientable and* $R_g\{\mathrm{Vol}(M, g)\}^{1/2} \equiv -\mu(S^4, C_0)$, *then g has negative constant curvature.*

(3) *If M is non-orientable and* $\min R_g\{\mathrm{Vol}(M, g)\}^{1/2} > -\mu(\boldsymbol{RP}^4, C_0)$, *then (M, g) is conformal to $(\boldsymbol{RP}^4, g_0)$.*

(4) *If M is non-orientable and* $R_g\{\mathrm{Vol}(M, g)\}^{1/2} \equiv -\mu(\boldsymbol{RP}^4, C_0)$, *then g has negative constant curvature.*

2. Preliminaries. First, we discuss Yamabe metrics and their known properties which will be needed later. We assume that M is a compact and connected manifold with $\dim M = n \geqq 3$, and that (M, C) is not necessarily conformally flat.

THEOREM 2.1. *For any (M, C), there exists a metric g in C such that $I(g)=\mu(M, C)$. Moreover $\mu(M, C)$ satisfies*

$$\mu(M, C) \leqq \mu(S^n, C_0) = n(n-1)\{\mathrm{Vol}(S^n, g_0)\}^{2/n},$$

and the equality holds if and only if (M, C) is conformal to (S^n, C_0). In particular, $I(g_0)=\mu(S^n, C_0)$.

Since the Yamabe functional I remains invariant under homothetic change of metrics, we can choose a metric g with $\mathrm{Vol}(M, g)=1$, and $I(g)=\mu(M, C)$ for any (M, C). By computing the first variation of I, we see that $g \in C$ is a critical point of I if and only if the scalar curvature R_g of g is constant. Thus we can rewrite Theorem 2.1 as follows:

PROPOSITION 2.2. *For any (M, C), there exists a metric $g \in C$ with* $\mathrm{Vol}(M, g)=1$, *and $R_g \equiv \mu(M, C) \leqq \mu(S^n, C_0)$. The equality $R_g = \mu(S^n, C_0)$ holds if and only if (M, C) is conformal to (S^n, C_0).*

A metric $g \in C$ is called a *Yamabe metric* if $I(g)=\mu(M, C)$. A Yamabe metric with $\mathrm{Vol}(M, g)=1$ is called a *normalized Yamabe metric* in this paper.

By the definition of $\mu(M, C)$, any (M, C) satisfies $\mu(M, C) \leqq (\max R_g)\{\mathrm{Vol}(M, g)\}^{2/n}$ for arbitrary $g \in C$.

LEMMA 2.3 (see [5] or [6]). *If $\mu(M, C) \leqq 0$, then the scalar curvature R_g of any $g \in C$ satisfies*

$$(\min R_g)\{\mathrm{Vol}(M, g)\}^{2/n} \leqq \mu(M, C) \leqq (\max R_g)\{\mathrm{Vol}(M, g)\}^{2/n},$$

and each of the two equalities implies that R_g is constant.

COROLLARY 2.4 (see [6]). *If C contains a metric with positive (resp. zero, resp. negative) scalar curvature, then $\mu(M, C)$ is positive (resp. zero, resp. negative).*

PROOF. If the scalar curvature R_g is negative for some $g \in C$, then $\mu(M, C)<0$ by the second inequality in Lemma 2.3, which always holds. Similarly if $R_g \equiv 0$ then $\mu(M, C) \leqq 0$. Applying the first inequality, we get $\mu(M, C)=0$. For the case $R_g>0$, suppose $\mu(M, C) \leqq 0$. Then by the first inequality, $\min R_g \leqq 0$, a contradiction. q.e.d.

Combining Corollary 2.4 with Lemma 2.3, we see that if the scalar curvature R_g of g is equal to a non-positive constant, then g is a Yamabe metric. On the other hand, it is well-known that if $\mu(M, C) \leqq 0$ (i.e., C contains a metric with non-positive constant scalar curvature), then two metrics with constant scalar curvature in C are proportional to each other (see [1]). Therefore we get the following:

PROPOSITION 2.5. *If (M, g) has non-positive constant scalar curvature and* $\mathrm{Vol}(M, g)=1$, *then g is a uniquely determined normalized Yamabe metric.*

A similar result holds for Einstein manifolds other than (S^n, g_0).

PROPOSITION 2.6 (see [10]). *Suppose that* (M, C) *is not conformal to* (S^n, C_0), *and that* C *contains an Einstein metric* g *with* $\mathrm{Vol}(M, g)=1$. *If* $g'\in C$ *has constant scalar curvature and* $\mathrm{Vol}(M, g')=1$, *then* $g=g'$. *In particular,* g *is a uniquely determined normalized Yamabe metric of* (M, C).

$\{\mathrm{Vol}(S^n, g_0)\}^{-2/n}g_0$ is a normalized Yamabe metric for (S^n, C_0) by Theorem 2.1, but the uniqueness does not hold because the functional I remains invariant under the action of conformal transformations, and (S^n, g_0) admits non-isometric conformal transformations. In this case, Obata's theorem [10] gives the following.

PROPOSITION 2.7. *Every* $g\in C_0$ *with constant scalar curvature and* $\mathrm{Vol}(M, g)=1$ *is obtained as a pull-back of* $\{\mathrm{Vol}(S^n, g_0)\}^{-2/n}g_0$ *by some conformal transformation. In other words,* g *is a normalized Yamabe metric.*

Generally speaking, the uniqueness does not hold. The space of normalized Yamabe metrics is studied, for example, in [4] and [7].

Next, we review the formulae for the characteristic numbers of 4-manifolds. For any Riemannian n-manifold ($n\geqq 4$) (M, g), the Riemannian curvature tesnor R of g has the orthogonal decomposition

$$R=\frac{R_g}{2n(n-1)}g\cdot g+\frac{1}{n-2}(\mathrm{Ric})^\circ\cdot g+W,$$

where $(\mathrm{Ric})^\circ$ denotes the traceless Ricci tensor, i.e., $(\mathrm{Ric})^\circ=\mathrm{Ric}-(R_g/n)g$, and the 4-tensor $h\cdot k$ denotes the Kulkarni-Nomizu product of symmetric 2-tensors h and k, and is defined by

$$\begin{aligned}(h\cdot k)(X, Y, Z, U)&=h(X, Z)k(Y, U)+h(Y, U)k(X, Z)\\&\quad-h(X, U)k(Y, Z)-h(Y, Z)k(X, U)\,.\end{aligned}$$

W is called the Weyl tensor and the theorem of Weyl-Schouten states that (M, g) is conformally flat if and only if the Weyl tensor vanishes. (When $n=3$ the above decomposition is still valid in principle. But the Weyl tensor always vanishes and conformal flatness is characterized in another way. See for example [9]). It is easy to see that $(\mathrm{Ric})^\circ$ vanishes if and only if g is an Einstein metric. Moreover both $(\mathrm{Ric})^\circ$ and W vanish if and only if g has constant curvature. With respect to this decomposition, the Gauss–Bonnet formula for a compact Riemannian 4-manifold (M, g) is written as

$$(2.1)\qquad 32\pi^2\chi(M)=\frac{1}{6}\int_M R_g^2dV_g-2\int_M|(\mathrm{Ric})^\circ|^2dV_g+\int_M|W|^2dV_g\,.$$

If M is oriented, then its orientation and volume form determine the star operator $*$ of (M, g). With respect to $*$, a 2-form α has an orthogonal decomposition $\alpha=\alpha^++\alpha^-$, where $\alpha^+=(\alpha+*\alpha)/2$ and $\alpha^-=(\alpha-*\alpha)/2$. Thus by considering the Weyl tensor W of g as an $\mathrm{End}(TM)$-valued 2-form, W splits as $W=W^++W^-$. Then, the signature $\tau(M)$

of a compact oriented (M, g) is given by

$$\tau(M)=\frac{1}{24\pi^2}\int_M(|W^+|^2-|W^-|^2)dV_g. \tag{2.2}$$

For a detailed exposition on the subject above, see [3].

3. Proof of Theorem.

(1) Fix an orientation of M and take a normalized Yamabe metric g of (M, C). Since (M, C) is conformally flat, the Weyl tensor of g vanishes. Thus by (2.2), the signature $\tau(M)$ of M vanishes. $\tau(M)$ is the signature of the interesection form of M, which is a non-degenerate bilinear form defined on $H^2(M, \boldsymbol{R})$. Therefore $b_2(M)=\dim H^2(M, \boldsymbol{R})$ is even. Hence $\chi(M)$ is even by the Poincaré duality. Since we assume $\chi(M)$ to be positive, we have $\chi(M)\geqq 2$. Thus by (2.1),

$$32\pi^2\chi(M)=\frac{1}{6}\{\mu(M, C)\}^2-2\int_M|(\mathrm{Ric})^\circ|^2dV_g\geqq 64\pi^2$$

holds for g. By Theorem 2.1, the standard metric of S^n is a Yamabe metric of (S^n, C_0). Then,

$$32\pi^2\chi(S^4)=64\pi^2=\frac{1}{6}\{\mu(S^4, C_0)\}^2$$

follows from (2.1). Theorefore g satisfies

$$\frac{1}{6}\{\mu(M, C)\}^2-2\int_M|(\mathrm{Ric})^\circ|^2dV_g\geqq\frac{1}{6}\{\mu(S^4, C_0)\}^2\,. \tag{3.1}$$

In particular, $|\mu(M, C)|\geqq\mu(S^4, C_0)$ holds. Since we assume $\mu(M, C)>-\mu(S^4, C_0)$, we get $\mu(M, C)\geqq\mu(S^4, C_0)$. Hence, by Theorem 2.1, (M, C) is conformal to (S^4, C_0).

(2) Fix an orientation of M and take a normalized Yamabe metric g. By the proof of (1), g satisfies (3.1). Since we assume $\mu(M, C)=-\mu(S^4, C_0)$, $(\mathrm{Ric})^\circ$ must vanish. Thus g has negative constant curvature.

(3) Take a normalized Yamabe metric g. Let $\tilde{M}$ be an orientable double cover of M, $\tilde{g}$ a pull-back of g by the covering map, and $\tilde{C}$ its conformal class. Note that $R_{\tilde{g}}\equiv\mu(M, C)$. If $\mu(M, C)\leqq 0$, then $\tilde{g}$ is a Yamabe metric by Proposition 2.5. Thus,

$$\mu(\tilde{M}, \tilde{C})=R_{\tilde{g}}\{\mathrm{Vol}(\tilde{M}, \tilde{g})\}^{1/2}=2^{1/2}\mu(M, C)\,.$$

If $\mu(M, C)>0$, then $\mu(\tilde{M}, \tilde{C})>0$ by Corollary 2.4. Hence, by our assumption, in both cases

$$\mu(\tilde{M}, \tilde{C})>-2^{1/2}\mu(\boldsymbol{R}P^4, C_0) \tag{3.2}$$

holds. By Proposition 2.6, the Yamabe metric g' of $(\boldsymbol{R}P^4, C_0)$ is a constant curvature metric. Thus, by Theorem 2.1,

$$2^{1/2}\mu(\boldsymbol{RP}^4, C_0)=2^{1/2}I(g')=I(\tilde{g}')=\mu(S^4, C_0)\,,$$

where $\tilde{g}'$ is the pull-back of g' by the covering map $S^4\to\boldsymbol{RP}^4$. Thus $(\tilde{M}, \tilde{C})$ is conformal to (S^4, C_0) by (1) (note that $\chi(\tilde{M})=2\chi(M)>0$). By Proposition 2.7, $\tilde{g}$ has positive constant curvature, hence so does g. Since an even-dimensional Riemannian manifold with positive constant curvature is homothetic to (S^n, g_0) or $(\boldsymbol{RP}^n, g_0)$ (see for example [8]), (M, g) must be homothetic to $(\boldsymbol{RP}^4, g_0)$. That is, (M, C) is conformal to $(\boldsymbol{RP}^4, C_0)$.

(4) We use the same notation as in the proof of (3). By the proof of (3), $\mu(\tilde{M}, \tilde{C})=-\mu(S^4, C_0)$ and $\chi(\tilde{M})>0$. Thus $\tilde{C}$ contains a negative constant curvature metric by (2). Then, by Proposition 2.5, $\tilde{g}$ has negative constant curvature, hence so does g. q.e.d.

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