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JULIA SET OF THE FUNCTION $z\exp(z+\mu)$

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Abstract. We are concerned with complex dynamics of iteration of an entire transcendental function as in the title of this paper and discuss the following problem: For what value of the real parameter μ does the Julia set of the function coincide with the whole complex plane?

1. Introduction and preliminaries. The study of iteration of complex analytic maps of the complex plane was begun by Fatou [5], [6] and Julia [8]. Fatou [7] discussed the dynamics of iteration of entire transcendental functions. In these three decades, this subject has been studied from diverse view points by many mathematicians. Baker treated the dynamics of entire transcendental functions, while Misiurewicz [9] solved Fatou's conjecture [7] affirmatively by showing that the Julia set of the exponential function $z\mapsto\exp z$ is the whole complex plane $\boldsymbol{C}$. Devaney and Krych [3] discussed symbolic dynamics and bifurcation for a one-parameter family of the functions $z\mapsto\lambda\exp z$ for a non-zero real parameter λ. For a complex parameter λ, this bifurcation was also studied by Baker and Rippon [2].

In this article, we study bifurcation for a one-parameter family of entire transcendental functions $z\mapsto z\exp(z+\mu)$ of exponential type with a parameter μ.

We recall some definitions and fundamental results in complex analytic dynamics.

Let f be an entire transcendental function. For a non-negative integer n, we denote by f^n the n-th iterate $f^n=f\circ f^{n-1}$ of the function f, where f^0 means the identity mapping of the complex plane.

A point z_0 is a periodic point of f with period n, if $f^n(z_0)=z_0$ and $f^k(z_0)\neq z_0$ for $k<n$. In particular, when the period of a periodic point z_0 is one, we call z_0 a fixed point of f. A periodic point z_0 of f with period n is said to be attractive (resp. repulsive), if $|(f^n)'|<1$ (resp. >1).

The Fatou set $\boldsymbol{F}(f)$ of f is the set of points where the sequence $\{f^n\}_{n=0}^{\infty}$ forms a normal family, that is, for a point $\zeta\in\boldsymbol{F}(f)$, there is a neighborhood of ζ such that $\{f^n\}_{n=0}^{\infty}$ is a normal family in the neighborhood. The complementary set $\boldsymbol{J}(f)=\boldsymbol{C}-\boldsymbol{F}(f)$ of the Fatou set $\boldsymbol{F}(f)$ is called the Julia set of f.

It is well-known that the set $\boldsymbol{J}(f)$ is a non-empty perfect set and is completely invariant under f. It is clear that an attractive periodic point of f is contained in the set $\boldsymbol{F}(f)$. It is also known that the set of all the repulsive periodic points of f is dense

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in $J(f)$.

Hereafter, we denote by f_μ an entire transcendental function $z \mapsto z \exp(z+\mu)$, where μ is a complex parameter.

Clearly, the function f_μ has fixed points $z=0$ and $z=-\mu$. If the real part $\operatorname{Re}\mu$ of the parameter μ is negative, then $0<f'_\mu(0)<1$, that is, the point $z=0$ is an attractive fixed point of f_μ. If $|1-\mu|$ is less than 1, then we see $|f'_\mu(-\mu)|=|1-\mu|<1$ and hence the point $z=-\mu$ is also an attractive fixed point of f_μ. Therefore, we see that, if $\operatorname{Re}\mu<0$ or if $|1-\mu|<1$, then the function f_μ has the non-empty Fatou set $F(f_\mu)$. In the case $\mu=0$, the point $z=0$ is fixed by f_0 and $f'_0(0)=1$. Hence the point $z=0$ is a rationally indifferent fixed point of f_0 and there are the so-called petal domains in the non-empty Fatou set $F(f_0)$. (Cf. Blanchard [3].) In other words, if $\operatorname{Re}\mu<0$, if $|1-\mu|<1$ or if $\mu=0$, the Julia set $J(f_\mu)$ of f_μ does not coincide with the whole complex plane.

On the other hand, Baker [1] proved the following interesting theorem.

THEOREM OF BAKER. *For a certain real value μ, the function f_μ has the whole complex plane for its Julia set.*

Thus, the following question arises: For what value of the parameter μ does the Julia set of f_μ coincide with the whole complex plane? In other words, it becomes a problem to discuss the bifurcation diagram for the family $\{f_\mu\}$ with a parameter μ. In the following, we restrict the parameter μ to real values.

2. Fatou sets and Julia sets of f_μ. We consider the one-parameter family of functions $f_\mu(z)=z\exp(z+\mu)$ for a real parameter μ. First we prove two lemmas.

We suppose $\mu>2$. Consider the function

$$G(\mu)=\mu+\alpha(\mu)+(-\mu+\alpha(\mu))\exp\alpha(\mu)\,,$$

where $\alpha(\mu)=(\mu^2-2\mu+2)^{1/2}$.

LEMMA 1. *There exists a μ_* (>2) such that $G(\mu)>0$ for $2<\mu<\mu_*$, $G(\mu_*)=0$ and $G(\mu)<0$ for $\mu>\mu_*$.*

PROOF. Put

$$g(\mu)=\exp\alpha(\mu)-\frac{\mu+\alpha(\mu)}{\mu-\alpha(\mu)}=\exp\alpha(\mu)-\mu-\frac{1+\mu\alpha(\mu)}{\mu-1}\,.$$

We see easily

$$g'(\mu)=\frac{\mu-1}{\alpha(\mu)}\exp\alpha(\mu)-1-\frac{\mu}{\alpha(\mu)}+\frac{1+\alpha(\mu)}{(\mu-1)^2}>\frac{\mu-1}{\alpha(\mu)}\sum_{k=0}^{2}\frac{(\alpha(\mu))^k}{k!}-1-\frac{\mu}{\alpha(\mu)}$$

$$=\mu-2+\frac{(\mu-1)\alpha(\mu)}{2}-\frac{1}{\alpha(\mu)}>0\,,$$

which follows from $(\mu-1)(\mu^2-2\mu+2)>2$ for $\mu>2$. Hence $g(\mu)$ increases monotonously. Since $g(2)=\exp\sqrt{2}-(3+2\sqrt{2})<0$ and $\lim_{\mu\to\infty} g(\mu)=+\infty$, there exists a μ_* (>2) such that $g(\mu)<0$ for $2<\mu<\mu_*$, $g(\mu_*)=0$ and $g(\mu)>0$ for $\mu>\mu_*$. Evidently $G(\mu)=(-\mu+\alpha(\mu))g(\mu)$ and $-\mu+\alpha(\mu)<0$ for $\mu>2$. Thus, the proof of the lemma is complete.

Now, for an arbitrarily fixed μ (>2), we consider the function $h_\mu(x)=x+f_\mu(x)+2\mu$ of the real variable x, where $f_\mu(x)=x\exp(x+\mu)$. Clearly we have $f_\mu^2(x)=x\exp h_\mu(x)$. Put $\beta(\mu)=(\mu^2-2\mu)^{1/2}$.

LEMMA 2. *There exists an x' in the interval $I=(-\mu+\beta(\mu),\ -\mu+\alpha(\mu))$ such that $h_\mu(x')<0$.*

PROOF. First, we note that

$$-2<-\mu+\beta(\mu)<-1<-\mu+\alpha(\mu)\,.$$

Since, for the second derivative h_μ'' of h_μ, we have $h_\mu''(x)=(x+2)\exp(x+\mu)>0$ for x in I, the first derivative h_μ' of h_μ increases in I monotonously. Consider the function $H(\mu)=h_\mu'(-\mu+\beta(\mu))$ of the variable μ (>2). By elementary computation, we see $H(2)=0$ and $H'(\mu)<0$ for $\mu>2$. Hence $H(\mu)$ decreases and $h_\mu'(-\mu+\beta(\mu))=H(\mu)<0$ for $\mu>2$. From the monotonicity of h_μ' in I and from $h_\mu'(-1)=1$, we have the existence of an x' in the interval $(-\mu+\beta(\mu),\ -1)\subset I$ such that $h_\mu'(x')=1+(x'+1)\exp(x'+\mu)=0$. From this equality, we have

$$h_\mu(x')=x'+2\mu+x'\exp(x'+\mu)=\frac{(x'+\mu)^2-\mu^2+2\mu}{x'+1}=\frac{(x'+\mu)^2-\beta(\mu)^2}{x'+1}\,,$$

which is negative bacause of $\beta(\mu)<x'+\mu$ and $x'+1<0$. Thus, we have a proof of the lemma.

Now we prove the following theorem.

THEOREM 1. *If the real parameter μ belongs to the set $(-\infty,2)\cup(2,\mu_*)$, then the function $f_\mu\colon z\mapsto z\exp(z+\mu)$ has the Julia set which is not the whole complex plane. Here μ_* is the one appeared in Lemma 1.*

PROOF. As was already stated in section 1, the theorem holds for $\mu\in(-\infty,2)$. Therefore, it suffices to prove the theorem for $\mu\in(2,\mu_*)$. The function $G(\mu)$ in Lemma 1 and the function h_μ in Lemma 2 satisfy the relation $G(\mu)=h_\mu(-\mu+\alpha(\mu))$. Lemma 1 implies $h_\mu(-\mu+\alpha(\mu))>0$ for our μ. Furthermore, Lemma 2 shows the existence of $x'\in(-\mu+\beta(\mu),\ -\mu+\alpha(\mu))$ satisfying $h_\mu(x')<0$. Therefore, there exists an $x_0\in(-\mu+\beta(\mu),\ -\mu+\alpha(\mu))$ such that $h_\mu(x_0)=x_0+f_\mu(x_0)+2\mu=0$. By the definition of the function h_μ, $h_\mu(x_0)=0$ implies $f_\mu^2(x_0)=x_0$, that is, x_0 is a periodic point of f_μ with period 2. On the other hand, we have easily

$$(f_\mu^2)'(x) = \frac{f_\mu^2(x)}{x}(f_\mu(x)+1)(x+1)\,,$$

which yields

$$(f_\mu^2)'(x_0) = (f_\mu(x_0)+1)(x_0+1) = (1-x_0-2\mu)(x_0+1) = -(x_0+\mu)^2 + (\mu-1)^2\,.$$

Since $\beta(\mu) < x_0 + \mu < \alpha(\mu)$, we have $|(f_\mu^2)'(x_0)| < 1$. Hence, the point x_0 is an attractive periodic point of f_μ. Therefore, the Fatou set $\boldsymbol{F}(f_\mu)$ of f_μ for our μ is non-empty. Thus, we complete the proof of the theorem.

REMARK. The value μ_* in Theorem 1 is the unique zero of the increasing function $g(\mu)$ (in the proof of Lemma 1) in $(2, \infty)$. Easily, we see $g(2.5) < 0$ and $g(2.54) > 0$. Hence we have $\mu_* = 2.5\cdots$.

Next we show that there exist infinitely many values of the real parameter μ such that the Julia set $\boldsymbol{J}(f_\mu)$ of the function $f_\mu(z) = z\exp(z+\mu)$ is the whole complex plane. By Theorem 1, we may assume $\mu > 0$.

The inverse function of f_μ has only one algebraic branch point $f_\mu(-1)$. By the same reasoning as that of Baker [1], we can easily check that, if $f_\mu^n(-1) = -\mu < -2$, then the Julia set $\boldsymbol{J}(f_\mu)$ of f_μ is the whole complex plant. In fact, by showing the existence of a value of μ satisfying $f_\mu^3(-1) = -\mu < -2$, Baker proved his theorem stated in Section 1 of this article. From this point of view, we put $s_0(\mu) = -1$ and define, inductively, a sequence $\{s_n(\mu)\}_{n=1}^\infty$ by

$$s_n(\mu) = -\exp\left(\sum_{k=0}^{n-1} s_k(\mu) + n\mu\right).$$

Note that $s_{n+1}(\mu) = f_\mu(s_n(\mu)) = f_\mu^{n+1}(-1)$.

LEMMA 3. *For any $n \geq 2$, $s_n(\mu)$ tends to 0 as $\mu \to +\infty$.*

PROOF. For any positive constant k, we have easily $s_1(\mu) + k\mu \to -\infty$ as $\mu \to +\infty$. Hence $s_0(\mu) + s_1(\mu) + 2\mu$ tends to $-\infty$ as $\mu \to +\infty$, which shows $s_2(\mu) = -\exp(s_0(\mu) + s_1(\mu) + 2\mu) \to 0$ as $\mu \to +\infty$.

Suppose that $s_k(\mu)$ tends to 0 as $\mu \to +\infty$ for each integer k satisfying $2 \leq k \leq n-1$. Then, we see

$$\sum_{k=0}^{n-1} s_k(\mu) + n\mu = s_0(\mu) + (s_1(\mu) + n\mu) + s_2(\mu) + \cdots + s_{n-1}(\mu) \to -\infty$$

as $\mu \to +\infty$. Therefore, we have $s_n(\mu) \to 0$ as $\mu \to +\infty$. Thus, we are done by induction.

THEOREM 2. *For each integer n (≥ 3), there exists a value μ_n of the parameter μ such that $s_n(\mu_n) = -\mu_n$, $2 < \mu_n < \mu_{n+1}$ and the Julia set $\boldsymbol{J}(f_{\mu_n})$ of f_{μ_n} is the whole complex plane.*

PROOF. If suffices to prove the existence of μ_n ($n \geq 3$) satisfying $2 < \mu_n < \mu_{n+1}$ and $s_n(\mu_n) = -\mu_n$.

Note that, for any positive μ, we have

$$(*) \qquad \mu \leq \exp(-1+\mu)\,,$$

where the equality holds only for $\mu = 1$. Put $\varphi(\mu) = -1 + s_1(\mu) + 2\mu$. Then, there exists a unique μ' (>2) such that $\varphi(\mu') = 0$. Hence $s_2(\mu') = -\exp\varphi(\mu') = -1$. By using (*), we have

$$s_3(\mu') + \mu' = f_{\mu'}(s_2(\mu')) + \mu' = -\exp(-1+\mu') + \mu' < 0\,.$$

On the other hand, Lemma 3 shows $s_3(\mu) + \mu \to +\infty$ as $\mu \to +\infty$. Therefore, by the continuity of the function $\mu \mapsto s_3(\mu) + \mu$, we see that there exists a μ_3 ($>\mu'$) such that $s_3(\mu_3) + \mu_3 = 0$. This proves the existence of the value μ_3 of μ.

Evidently we have $s_3(\mu_3) = -\mu_3 < -1$. By using Lemma 3, we see $s_3(\mu) \to 0$ as $\mu \to +\infty$. Hence there exists a μ'' ($>\mu_3$) satisfying $s_3(\mu'') = -1$. Therefore, by using (*) again, we see

$$s_4(\mu'') + \mu'' = f_{\mu''}(s_3(\mu'')) + \mu'' = -\exp(-1+\mu'') + \mu'' < 0\,.$$

Since $s_4(\mu) + \mu \to +\infty$ as $\mu \to +\infty$ by Lemma 3, we see from the continuity of the function $\mu \mapsto s_4(\mu) + \mu$ that there exists a μ_4 such that $\mu_3 < \mu'' < \mu_4$ and $s_4(\mu_4) = -\mu_4$. Repeating this procedure, we have the theorem.

REMARK 1. The functions $\varphi_1(\mu) = 2\mu - 1$ and $\varphi_2(\mu) = -s_1(\mu) = \exp(-1+\mu)$ are both increasing and we see $\varphi_1(2.25) = 3.5$, $\varphi_1(2.26) = 3.52$, $\varphi_2(2.25) = 3.490\cdots$ and $\varphi_2(2.26) = 3.525\cdots$. Hence, for μ' satisfying $\varphi(\mu') = \varphi_1(\mu') - \varphi_2(\mu') = 0$, we have $\mu' = 2.25\cdots$.

REMARK 2. There exists only one value μ_1 of the parameter μ such that $s_1(\mu_1) = -\mu_1$. Clearly we see $\mu_1 = 1$ and we see by Theorem 1 that the Julia set of the function $f_{\mu_1}: z \mapsto z\exp(z+1)$ is not the whole complex plane. Moreover, we can easily see that there exists no value of μ satisfying $\mu_1 < \mu$ and $s_2(\mu) = -\mu$.

REMARK 3. It is still open whether the sequence $\{\mu_n\}_{n=3}^{\infty}$ obtained in Theorem 2 is unbounded or not.

3. Uniqueness of μ_3. First we suppose $0 \leq \mu < 1$. By (*), we have $-1 < s_1(\mu) < -\mu$ (≤ 0). Since the function $f_\mu(x) = x\exp(x+\mu)$ increases monotonously in the interval $-1 \leq x \leq 0$, we have

$$(-1<)\, s_1(\mu) < s_2(\mu) < -\mu\,.$$

Repeating this, we see

$$-1 < s_1(\mu) < \cdots < s_n(\mu) < s_{n-1}(\mu) < \cdots < -\mu\,.$$

Next, suppose $1<\mu<\mu'$, where μ' (>2) is the value of μ which appeared in the proof of Theorem 2. This μ' satisfies $\varphi(\mu')=-1+s_1(\mu')+2\mu'=0$. Since $\varphi(\mu)$ is positive for $1<\mu<\mu'$, we have $s_2(\mu)<-1$. The inequality $(*)$ shows $s_1(\mu)<-\mu<-1$. Since the function f_μ is decreasing monotonously in the interval $-\infty<x<-1$, we have $f_\mu(-1)<f_\mu(-\mu)<f_\mu(s_1(\mu))$, or

$$s_1(\mu)<-\mu<s_2(\mu)<-1\,.$$

Repeating this procedure, we see

$$s_1(\mu)<s_3(\mu)<\cdots<s_{2n+1}(\mu)<\cdots<-\mu<\cdots<s_{2n}(\mu)<\cdots<s_2(\mu)\,.$$

In particular, for $0\le\mu<1$ and for $1<\mu<\mu'$, we have $s_3(\mu)<-\mu$ and have also $s_3(1)=-1$.

On the other hand, we easily have

$$s_2'(\mu)=s_2(\mu)(s_1(\mu)+2)$$

and

$$s_3'(\mu)=\frac{s_2'(\mu)}{s_2(\mu)}s_3(\mu)\left(s_2(\mu)+1+\frac{1}{s_1(\mu)+2}\right).$$

Hence, by putting $\mu^*=1+\log 2$, we see $s_2'(\mu)<0$ in $0<\mu<\mu^*$, $s_2'(\mu^*)=0$ and $s_2'(\mu)>0$ in $\mu^*<\mu$. Since μ^* is less than μ' by Remark 1 of Theorem 2, we have $s_2'(\mu)(s_2(\mu))^{-1}s_3(\mu)>0$ in $\mu'\le\mu$. Furthermore, the function $-1-(s_1(\mu)+2)^{-1}$ decreases in $\mu'<\mu$ and greater than -1 at $\mu=\mu'$ and tends to -1 as $\mu\to+\infty$. The function $s_2(\mu)$ increases in $\mu'<\mu$ and is equal to -1 at $\mu=\mu'$ and tends to zero as $\mu\to+\infty$. Therefore, the continuity of these two functions shows that there exists a unique μ^{**} $(>\mu')$ such that $s_3'(\mu)<0$ in $\mu'<\mu<\mu^{**}$, $s_3'(\mu^{**})=0$ and $s_3'(\mu)>0$ in $\mu^{**}<\mu$.

By easy computation, we see $s_3'(1+\log 5)>0$, which shows $\mu^{**}<1+\log 5$. Moreover, we also have $s_3(\mu')<-3<-(1+\log 5)$ by Remark 1 of Theorem 2. Hence, we see that there exists no root of the equation $s_3(\mu)=-\mu$ in the interval $\mu'\le\mu\le\mu^{**}$. Therefore, the equation $s_3(\mu)=-\mu$ has only one root μ_3 which is greater than 2. Thus, as a conclusion of the above discussion, we have the following proposition.

PROPOSITION. *The value μ_3 of the parameter μ in Theorem 2 is determined uniquely.*

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