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ON A CONJECTURE OF FUCHS CONCERNING FACTORIZATION OF ENTIRE FUNCTIONS

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Abstract. In 1982, Fuchs raised the following conjecture: Suppose that F is an entire function of finite order with at least one finite deficient value. Then F is pseudo-prime. In this paper, we prove this conjecture under the additional condition that the number of the limiting directions of Julia directions of F is finite.

1. Introduction. Following Gross [5] we say that a meromorphic function $F(z)$ has a factorization with factors $f(z)$ and $g(z)$, if

$$F(z)=f(g(z)),$$

where $f(z)$ is meromorphic and $g(z)$ is entire (g may be meromorphic when f is rational). $F(z)$ is said to be pseudo-prime if every factorization of the above form implies that $f(z)$ is rational or $g(z)$ is a polynomial.

Let $F(z)$ be a meromorphic function. A ray $J(\theta):=\{z:\arg z=\theta, 0\leqslant\theta<2\pi\}$ is called a Julia direction of $F(z)$ if, in any open sector containing the ray, $F(z)$ takes all finte values, with at most two finite exceptional values, infinitely often.

Goldstein [3] proved the following:

THEOREM A. *Let $F(z)$ be an entire function of finite order such that $\delta(a, F)=1$ for some $a\neq\infty$. Then $F(z)$ is pseudo-prime.*

Recently Fuchs (cf. [11]) conjectured that the conclusion of Theorem A remains true under the weaker assumption $\delta(a, F)>0$.

In this paper, we will prove:

THEOREM. *Suppose that $F(z)$ is an entire function of finite lower order such that $\delta(a, F)>0$ for some $a\neq\infty$. If the number of the limiting directions of Julia directions of $F(z)$ is finite, then $F(z)$ is pseudo-prime.*

REMARK. In the original version of this paper the author used the condition that the number of Julia directions of $F(z)$ is finite. The weakened version is due to the referee.

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2. Notation. Let $\theta_1<\theta_2$ and $0\leqslant r_1<r<r_2\leqslant+\infty$, we define

$$\Gamma(\theta_1,\theta_2;r)=\{z:\theta_1<\arg z<\theta_2,|z|=r\},$$

$$\Omega(\theta_1,\theta_2)=\{z:\theta_1<\arg z<\theta_2\},$$

$$\Omega(\theta_1,\theta_2;r)=\{z:\theta_1<\arg z<\theta_2,|z|\leqslant r\}$$

and

$$\Omega(\theta_1,\theta_2;r_1,r_2)=\{z:\theta_1<\arg z<\theta_2,r_1\leqslant|z|\leqslant r_2\}.$$

Also, we denote by $\bar{E}$ the closure of a set E with respect to the plane $|z|<+\infty$.

Furthermore, we write $n(E,f=a)$ to be the number of roots with due count of multiplicity of the equation $f=a$ in E.

In addition, we assume that the reader is familiar with the standard notation of the Nevanlinna theory (see [6]).

3. Known results.

LEMMA 1 (cf. [2]). *Meromorphic functions with more than one deficient value have a positive lower order.*

The following lemma is the upper half part of Lemma 2 in [12].

LEMMA 2. *Suppose that $F(z)$ is an entire function of finite lower order μ such that $\delta=\delta(a,F)>0$ for some $a\neq\infty$. Set*

$$E(t)=\{\theta:\log|F(te^{i\theta})-a|\leqslant-\frac{\delta}{4}T(t,F),\ 0\leqslant\theta<2\pi\}.$$

Then there exists a sequence $\{t_n\}$ of positive numbers tending to infinity such that

$$\operatorname{mes}E(t_n)\geqslant K>0,$$

where K is an absolute constant depending only on δ and μ.

LEMMA 3 (a modified version of [12, Lemma 3]). *Let $F(z)$ be an entire function of lower order μ such that $0<\mu<+\infty$. Suppose that there exist two finite complex numbers b_1 and b_2 ($b_1\neq b_2$) such that*

$$\limsup_{r\to+\infty}\frac{\log^+\left\{\sum_{i=1}^{2}n(\Omega(\theta',\theta'';r),F=b_i)\right\}}{\log r}=0.$$

If there are postive numbers δ and B, a finite complex value a_0 and a positive and sufficiently large number R_k such that

$$\operatorname{mes}E>B$$

for

$$E=\{\theta:\theta'<\theta<\theta'', \log|F(R_k e^{i\theta})-a_0|\leqslant -(\delta/4)T(R_k, F)\},$$

then for two positive numbers ε *and* Q, $Q>1$, $0<\varepsilon<\min\{B/4, (\theta''-\theta')/4\}$,

$$\log|F(z)-a_0|\leqslant -AT(R_k, F)$$

holds in the region

$$\bar{\Omega}(\theta'+\varepsilon, \theta''-\varepsilon; 10^{-4Q}R_k, 10^{4Q}R_k),$$

where $0<A<+\infty$ *is a constant depending only on* δ, ε, B *and* Q, *i.e.*

$$A=\frac{\delta}{4\left\{4\left(5+4\log\frac{1}{h}\right)\right\}^{2N_1+N_2}},\quad N_1=[10\pi/\varepsilon],\quad N_2=\left[\frac{20(10^{4Q}-1)}{\varepsilon}\right]+4,$$

$$h=B\varepsilon/8(2e+1)(10\pi+\varepsilon).$$

REMARK. In [12] it is assumed that the order of F is finite. The referee showed to the author that this condition is unnecessary. Note that $n_1<R_k^{\lambda+1}$ on p. 591 in [12] is unnecessary.

LEMMA 4 (cf. [1]). *Suppose that* $f(z)$ *is a transcendental meromorphic function and* $g(z)$ *is a transcendental entire function. Then*

$$\lim_{r\to+\infty}\frac{T(r, f(g))}{T(r, g)}=+\infty.$$

By virtue of Lemma 4 (as well as Pólya's lemma). We have the following result.

LEMMA 5 (cf. [9]). *Let* $f(z)$ *be meromorphic with at most one pole and* $g(z)$ *be entire. If the lower order of* $f(g)$ *is finite, then either* f *is of zero lower order or* g *is a polynomial.*

LEMMA 6 (cf. [8]). *Suppose that* $f(z)$ *is a meromorphic function of lower order* μ *with* $0\leqslant\mu<1/2$. *If* $\delta(\infty, f)>1-\cos\pi\mu$, *then*

$$\limsup_{r\to\infty}\frac{\log^+\mu(r, f)}{T(r, f)}\geqslant C(\mu)$$

for some constant $C(\mu)>0$, *where* $\mu(r, f)=\min\{|f(z)|; |z|=r\}$.

LEMMA 7 (cf. [4]). *Let* g *be an entire function,* f, F *be meromorphic functions such that* $F=f(g)$. *Suppose that* L *is a path tending to infinity such that* $F(z)\to 0$ *as* $z\to\infty$ *along* L, *and* $g(L)$ *is bounded. Then* $g(z)\to z_0$ *as* $z\to\infty$ *along* L, *where* z_0 *is a zero of* f.

4. Properties of functions having no Julia directions in angles. Now we shall give a property of meromorphic functions without Julia directions in angles, which essentially belongs to Zhang [13].

LEMMA 8. *If a meromorphic function* $F(z)$ *has no Julia directions in* p *angles* $\Omega(\theta_{k1}, \theta_{k2})$ $(k=1, \dots, p;\ 0 \leqslant \theta_{11} < \theta_{12} \leqslant \theta_{21} < \theta_{22} \leqslant \cdots \leqslant \theta_{p1} < \theta_{p2} < 2\pi)$, *then for any small* ε $(0 < \varepsilon < \min_{1 \leqslant k \leqslant p}(\theta_{k2} - \theta_{k1})/2)$, *there exist three distinct complex numbers* b_1, b_2 *and* b_3 *such that*

$$\limsup_{r\to\infty} \frac{\log^+ \sum_{j=1}^{3} n(\bar{\Omega}, F=b_j)}{\log r} = 0$$

and the mutual spherical distances between b_1, b_2 *and* $b_3 \geqslant d$, $0 < d < 1/2$, *where* $\bar{\Omega} = \bigcup_{k=1}^{p} \bar{\Omega}(\theta_{k1}+\varepsilon, \theta_{k2}-\varepsilon; r)$.

PROOF. Assume the conclusion is false. Then for all values Z, except possibly two values,

$$\limsup_{r\to\infty} \frac{\log^+ n(\bar{\Omega}, F=Z)}{\log r} > 0 .$$

By the finite covering theorem, for any $\eta > 0$, there exists a half line $J(\theta) \in \bigcup_{k=1}^{p} \bar{\Omega}(\theta_{k1}+\varepsilon, \theta_{k2}-\varepsilon)$ such that the line measure of the set

$$\left\{ Z : \limsup_{r\to\infty} \frac{\log^+ n(\Omega(\theta-\eta, \theta+\eta; r), F=Z)}{\log r} > 0 \right\}$$

is positive. On the other hand, since $J(\theta)$ is not a Julia direction of $F(z)$, it follows that there exist a positive number η' and three distinct values α, β and γ such that the series

$$\sum_{n=1}^{\infty} |a_n(\eta', Z)|^{-\sigma}, \qquad Z = \alpha, \beta, \gamma$$

converges for an arbitrary small number $\sigma > 0$, where $a_n(\eta', Z)$ $(n = 1, 2, \dots; |a_1(\eta', Z)| \leqslant |a_2(\eta', Z)| \leqslant \cdots)$ denote all zeros of $F(z)$ in $\Omega(\theta-\eta', \theta+\eta')$. According to a known result [10, p. 31], the series

$$\sum_{n=1}^{\infty} |a_n(\eta'', Z)|^{-\sigma}, \qquad \eta'' < \eta'$$

converges for any value Z, except a set with zero line measure. Note that σ can be arbitrarily small, thus for all Z, except a set with zero line measure,

$$\limsup_{r\to\infty} \frac{\log^+ n(\Omega(\theta-\eta'', \theta+\eta''; r), F=Z)}{\log r} = 0 ,$$

which is a contradiction. This completes the proof of Lemma 8.

5. Proof of Theorem. From Lemma 1 and the assumption we have $0<\mu<+\infty$, where μ is the lower order of $F(z)$. By a well known classical result of Julia, $F(z)$ has at least one Julia direction. Let m be the number of the limiting directions of Julia directions of $F(z)$. Then $0<m<+\infty$. Without loss of generality, we assume that $a=0$ and $m=1$. For convenience, we suppose that the limiting direction is $J(0)$. Next we distinguish the proof into five steps.

Step 1. According to Lemma 2, there eixst a sequence $\{t_n\}$ of positive numbers tending to infinity and a set $E_n \subset [0, 2\pi)$ such that, for $\theta \in E_n$,

$$\log|F(t_n e^{i\theta})| \leqslant -\frac{\delta}{4} T(t_n, F), \qquad t_1 > r_0 \tag{1}$$

and

$$\text{mes } E_n \geqslant K > 0, \tag{2}$$

where $\delta=\delta(0, F)>0$. Now we take a small number η with $0<\eta<k/32$. Then, by $m=1$, the number of Julia directions of $F(z)$ in the complement of $\Omega(-\eta, \eta)$ is finite. Thus there exist q rays $J(\theta_k)$ $(\eta=\theta_1<\cdots<\theta_{q-1}<\theta_q=2\pi-\eta)$ such that $F(z)$ has no Julia directions in the region $\bigcup_{k=1}^{q-1} \Omega(\theta_k, \theta_{k+1})$, where $1<q<+\infty$ and q depends on η. Put $\omega=\min_{1\leqslant k \leqslant -1}(\theta_{k+1}-\theta_k)$ and take a number ε such that $0<\varepsilon<\min\{\omega/32, K/32q, 1/8e\}$ and define a sequence $\{\varepsilon_j\}$ of positive numbers

$$\varepsilon_j = 2^{-(j+1)}\varepsilon \qquad (j=0, 1, \ldots). \tag{3}$$

By Lemma 8 applied to F and ε_0, there exist distinct finite complex numbers b_1 and b_2 such that

$$\limsup_{r\to+\infty} \frac{\log^+\{n(\bar{\Omega}, F=b_1)+n(\bar{\Omega}, F=b_2)\}}{\log r} = 0, \tag{4}$$

where

$$\bar{\Omega} = \bigcup_{k=1}^{q} \bar{\Omega}(\theta_k+\varepsilon_0, \theta_{k+1}-\varepsilon_0; r).$$

Step 2. For any n, by (2) and the choise of ε_0 there exists an integer $k(n)$ $(1\leqslant k(n)\leqslant q-1)$ such that, for $\theta \in E_n^* = E_n \cap (\theta_{k(n)}+\varepsilon_0, \theta_{k(n)+1}-\varepsilon_0)$,

$$\log|F(t_n e^{i\theta})| \leqslant -\frac{\delta}{4} T(t_n, F) \tag{5}$$

and

(6) $$\text{mes}\, E_n^* \geq \frac{K}{2q}.$$

Since $q < +\infty$, there exists a sequence $\{n_j\}$ tending to infinity such that

(7) $$k(n_1) = k(n_2) = \cdots = 1(\text{say}).$$

Thus for

(8) $$\theta\varepsilon_{n_j}^* = E_{n_j}^* \cap (\theta_1 + \varepsilon_0, \theta_2 - \varepsilon_0)$$

we have

(9) $$\log |F(t_{n_j} e^{i\theta})| \leq -\frac{\delta}{4} T(t_{n_j}, F)$$

and

(10) $$\text{mes}\, E_{n_j}^* \geq \frac{K}{2q}.$$

In particular,

(11) $$\theta_2 - \theta_1 > \frac{K}{2q}.$$

Step 3. In this step, we shall deal with the values of $F(z)$ in the region $\bar{\Omega}(\theta_1 + \varepsilon, \theta_2 - \varepsilon)$.

For a chosen positive number Q $(1 < Q)$ and any $j \geq 2$, there exists a non-negative integer m_j such that

(12) $$10^{4Qm_j} t_{n_{j-1}} < t_{n_j} \leq 10^{4Q(m_j+1)} t_{n_{j-1}}.$$

Now put

(13) $$R_{j,s} = 10^{-4Qs} t_{n_j} \quad (s = -1, 0, 1, \ldots, m_j + 1).$$

By applying Lemma 3 with

$$R_k = R_{j,0}, \quad \theta' = \theta_1 + \varepsilon_0, \quad \theta'' = \theta_2 - \varepsilon_0, \quad \varepsilon = \varepsilon_1, \quad B = K/2q, \quad E = E_{n_j}^* \quad \text{and} \quad a_0 = 0,$$

we deduce from (9) and (10) that

(14) $$\log |F(z)| \leq -A_1 T(R_{j,0}, F), \leq -A_1 T(R_{j,1}, F),$$

where $z \in \bar{\Omega}(\theta_1 + \varepsilon_0 + \varepsilon_1, \theta_2 - \varepsilon_0 - \varepsilon_1; R_{j,1}, R_{j,-1})$, $0 < A_1 < +\infty$ is a constant depending only on K, Q, q, δ and ε_1. In particular, we have

(15) $$\log |F(z)| \leq -A_1 T(R_{j,1}, F)$$

for $z \in \Gamma(\theta_1 + \varepsilon_0 + \varepsilon_1, \theta_2 - \varepsilon_0 - \varepsilon_1; R_{j,1})$. Now by the choice of ε, we deduce from (3) and

(11) that, for any $s \geqslant 0$,

$$\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_s=\varepsilon(2^{-1}+2^{-2}+\cdots+2^{-s-1})<\varepsilon$$

and

$$(\theta_2-\varepsilon_0-\varepsilon_1-\cdots-\varepsilon_s)-(\theta_1+\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_s)\geqslant\theta_2-\theta_1-2\varepsilon\geqslant K/4q\,.$$

Thus, by applying Lemma 3 again with

$$B=\theta_2-\theta_1-2(\varepsilon_0+\varepsilon_1)>k/4q\,,\quad \theta'=\theta_1+\varepsilon_0+\varepsilon_1\,,\quad \theta''=\theta_2-\varepsilon_0-\varepsilon_1\,,$$

$$\varepsilon=\varepsilon_2\,,\quad R_k=R_{j,1}\,,\quad E=\Gamma(\theta_1+\varepsilon_0+\varepsilon_1,\theta_2-\varepsilon_0-\varepsilon_1;R_{j,1})\,,\quad \delta=4A_1\,,$$

we derive from (15) that

$$\log|F(z)|\leqslant -A_2T(R_{j,1},F)\leqslant -A_2T(R_{j,2},F)\,, \tag{16}$$

where $z\in\bar{\Omega}(\theta_1+\varepsilon_0+\varepsilon_1+\varepsilon_2,\theta_2-\varepsilon_0-\varepsilon_1-\varepsilon_2;R_{j,2},R_{j,0})$, $0<A_2<+\infty$ is a constant depending only on K, Q, q, δ, ε_1 and ε_2.

By induction we obtain

$$\log|F(z)|\leqslant -A_sT(R_{j,s},F)\,, \tag{17}$$

where $1\leqslant S\leqslant m_j+1$ and $z\in\bar{\Omega}(\theta_1+\varepsilon_0+\varepsilon_1+\varepsilon_s,\theta_2-\varepsilon_0-\cdots-\varepsilon_s;R_{j,s},R_{j,s-2})$ and $0<A_s<+\infty$ is a constant depending only on K, Q, q, $\varepsilon_0,\ldots,\varepsilon_s$, δ. Note that

$$\bar{\Omega}(\theta_1+\varepsilon,\theta_2-\varepsilon;t_{n_{j-1}},t_{n_j})$$

is contained in the set

$$\bigcup_{s=1}^{m_j+1}\bar{\Omega}(\theta_1+\varepsilon_0+\cdots+\varepsilon_s,\theta_2-\varepsilon_0-\cdots-\varepsilon_s;R_{j,s},R_{j,s-2})\,.$$

We conclude that, for $z\in\bar{\Omega}(\theta_1+\varepsilon,\theta_2-\varepsilon;t_{n_{j-1}},t_{n_j})$,

$$\log|F(z)|\leqslant -\min_{1\leqslant s\leqslant m_j+1}\{A_sT(R_{j,s},F)\}\leqslant 0\,,$$

i.e., $|F(z)|\leqslant 1$. Since j is arbitrary, $F(z)$ is bounded on $\bar{\Omega}(\theta_1+\varepsilon,\theta_2-\varepsilon)$. Hence, there exists an absolute constant $M>0$ such that

$$|F(z)|\leqslant M\,,\quad z\in\bar{\Omega}(\theta_1+\varepsilon,\theta_2-\varepsilon)\,. \tag{18}$$

Step 4. In this step, we shall prove that $F(z)$ tends to zero in the set $\bar{\Omega}(\theta_1+4\varepsilon,\theta_2-4\varepsilon)$.

Put

$$G(z)=zF(z)\,.$$

Then we can verify that $\delta(0,G)=\delta(0,F)>0$ and that G and F have the same lower order.

Now for sufficiently large r we determine a positive integer n such that

$$2^{n-1} \leqslant r \leqslant 2^n . \tag{19}$$

We define

$$r_j = 2^j \quad (j = 1, \ldots, n) .$$

For any $j \leqslant n$, we consider the mapping

$$w = w_j(z) = \frac{(e^{-i\theta^*} \cdot z)^\theta - (r_j)^\theta}{(e^{-i\theta^*} \cdot z)^\theta + (r_j)^\theta} , \tag{20}$$

where $\theta^* = (\theta_1 + \theta_2)/2$ and $\theta = \pi/(\theta_2 - \theta_1 - 2\varepsilon)$. Then the image of $\bar{\Omega}(\theta_1 + \varepsilon, \theta_2 - \varepsilon)$ in the w-plane is $|w| \leqslant 1$. Now for each $z = te^{i\varphi} \in \bar{\Omega}(\theta_1 + 2\varepsilon, \theta_2 - 2\varepsilon; r_{j-1}, r_j)$ we have

$$|w| = \left| \frac{t^\theta e^{i\theta(\varphi - \theta^*)} - (r_j)^\theta}{t^\theta e^{i\theta(\varphi - \theta^*)} + (r_j)^\theta} \right| = \left(1 - \frac{4t^\theta (r_j)^\theta \cos \theta(\varphi - \theta^*)}{t^{2\theta} + (r_j)^{2\theta} + 2t^\theta (r_j)^\theta \cos \theta(\varphi - \theta^*)} \right)^{1/2} . \tag{21}$$

Note that $r_{j-1} \leqslant t \leqslant r_j$ and $\varphi \leqslant \theta_2 - 2\varepsilon$. Thus

$$t^{2\theta} + (r_j)^{2\theta} + 2t^\theta (r_j)^\theta \cos \theta(\varphi - \theta^*) \leqslant 4(r_j)^{2\theta} ,$$

$$4t^\theta (r_j)^\theta \cos \theta(\varphi - \theta^*) \geqslant 4(r_{j-1})^\theta (r_j)^\theta \cos \left(\frac{\pi}{2} - \theta\varepsilon \right) \geqslant \frac{8\theta\varepsilon}{\pi} (r_{j-1})^\theta (r_j)^\theta .$$

Substituting these into (21) we obtain

$$|w| \leqslant \left(1 - \frac{2\theta\varepsilon}{\pi} (r_{j-1}/r_j)^\theta \right)^{1/2} = \left(1 - \frac{2\theta\varepsilon}{\pi} \left(\frac{1}{2} \right)^\theta \right)^{1/2} \leqslant 1 - \frac{\theta\varepsilon}{\pi} \left(\frac{1}{2} \right)^\theta .$$

Let

$$R = 1 - \frac{\theta\varepsilon}{\pi} \left(\frac{1}{2} \right)^\theta .$$

Then we see that the image of $\bar{\Omega}(\theta_1 + 2\varepsilon, \theta_2 - 2\varepsilon; r_{j-1}, r_j)$ in the w-plane is contained in the circle $|z| \leqslant R < 1$. Furthermore, we can derive from (20) that the inverse mapping of $w = w_j(z)$ is

$$z = z_j(w) = r_i e^{i\theta^*} \left(\frac{1 + w}{1 - w} \right)^{1/\theta} .$$

Thus for $|w| \leqslant (1 + R)/2$, we have

$$|z| \leqslant r_i \left\{ \frac{1 + (1 + R)/2}{1 - (1 + R)/2} \right\}^{1/\theta} \leqslant r_i \left(\frac{4}{1 - R} \right)^{1/\theta} = 2r_j \left(\frac{4\pi}{\theta\varepsilon} \right)^{1/\theta} \leqslant 4 \left(\frac{4\pi}{\theta\varepsilon} \right)^{1/\theta} r .$$

Hence the inverse image of $|w| \leqslant (1 + R)/2$ is contained in the region

$$\bar{\Omega}(\theta_1+\varepsilon, \theta_2-\varepsilon;\ 4(4\pi/\theta\varepsilon)^{1/\theta}r).$$

Now put

$$H_j(w)=G(z_j(w))=z_j(w)F(z_j(w)).$$

Then $H_j(w)$ is holomorphic in $|w|\leqslant 1$. For two distinct and finite complex numbers x, y we denote by $|x,y|$ the spherical distance between x and y. It is easy to verify that

$$\log^+|x|+\log^+|y|+\log\frac{1}{|x-y|}\leqslant\log\frac{1}{|x,y|}. \tag{22}$$

From the Boutroux–Cartan Theorem [10], we have

$$\prod_{j=1}^{n}|H_j(O),\alpha|\geqslant\varepsilon^n \tag{23}$$

for any complex number α, except a set of α which can be enclosed in a finite number of disks with the sum of total spherical radii not exceeding $2e\varepsilon<1/4$. The union of these disks is denoted by (γ).

Choose $\alpha\notin(\gamma)$ such that α satisfies (23). By the first fundamental theorem we deduce from (22) and (18) that

$$\begin{aligned}
&n(\bar{\Omega}(\theta_1+2\varepsilon, \theta_2-2\varepsilon;\ r_{j-1}, r_j), G(z)=\alpha)\leqslant n(R, H_j(w)=\alpha)\\
&\leqslant\frac{1}{\log(1+R)-\log 2R}\int_R^{(1+R)/2}\frac{n(t, H_j(w)=\alpha)}{t}\,dt\\
&\leqslant\frac{1}{\log(1+R)-\log 2R}N\left(\frac{1+R}{2}, \frac{1}{H_i(w)-\alpha}\right)\\
&\leqslant\frac{1}{\log(1+R)-\log 2R}\left\{T\left(\frac{1+R}{2}, H_j(w)-\alpha\right)+\log\frac{1}{|H_j(O)-\alpha|}\right\}\\
&\leqslant\frac{1}{\log(1+R)-\log 2R}\left\{\log^+M\left(\frac{1+R}{2}, H_j(w)\right)+\log 2+\log^+|\alpha|+\log\frac{1}{|H_j(O)-\alpha|}\right\}\\
&\leqslant\frac{1}{\log(1+R)-\log 2R}\left\{\log^+M\left(\bar{\Omega}\left(\theta_1+\varepsilon, \theta_2-\varepsilon;\ 4\left(\frac{4\pi}{\theta\varepsilon}\right)^{1/\theta}r\right), zF(z)\right)\right.\\
&\qquad\qquad\left.+\log 2+\log\frac{1}{|H_j(O),\alpha|}\right\}\\
&\leqslant D\left\{\log r+\log\frac{1}{|H_j(O),\alpha|}+C\right\},
\end{aligned}$$

where

$$C=\log^+ M+\frac{1}{\theta}\log\frac{4\pi}{\theta\varepsilon}+3\log 2\,,\quad D=\left(\log\frac{1+R}{2R}\right)^{-1}.$$

Hence

$$\begin{aligned} &n\{\bar{\Omega}(\theta_1+2\varepsilon,\theta_2-2\varepsilon;r),G(z)=\alpha\}\\ &\leqslant\sum_{j=1}^{n} n\{\bar{\Omega}(\theta_1+2\varepsilon,\theta_2-2\varepsilon;r_{j-1},r_j),G(z)=\alpha\}+O(1)\\ &\leqslant D\left\{n\log r+\log\left(\prod_{j=1}^{n}|H_j(O),\alpha|\right)^{-1}+nC\right\}+O(1)\,. \end{aligned}$$

Now from (19) we have $n\leqslant(\log 2)^{-1}\log r+1$. Therefore, by (23),

$$\begin{aligned} &n\{\bar{\Omega}(\theta_1+2\varepsilon,\theta_2-2\varepsilon;r),G(z)=\alpha\}\\ &\leqslant\frac{D}{\log 2}\left\{(\log r)^2+\left(\log 2+C+\log\frac{1}{\varepsilon}\right)\log r+\log 2\left(C+\log\frac{1}{\varepsilon}\right)\right\}+O(1)\,, \end{aligned}$$

which results in

$$\lim_{r\to+\infty}\frac{\log^+ n\{\bar{\Omega}(\theta_1+2\varepsilon,\theta_2-2\varepsilon;r),G(z)=\alpha\}}{\log r}=0\,,\tag{24}$$

where $\alpha\notin(\gamma)$ and α satisfies (23). Obviously, there are infinitely many such complex values α.

Now we deduce from (9) and $T(r,G)\leqslant\log r+T(r,F)$ that

$$\log|G(t_{n_j}e^{i\theta})|\leqslant\left(1+\frac{\delta}{4}\right)\log t_{n_j}-\frac{\delta}{4}T(t_{n_j},G)\,,\quad \theta\in E^*_{n_j}\,.$$

Since the lower order μ of G is positive, we have

$$(1+\delta/4)\log t_{n_j}=o(T(t_{n_j},G))\,.$$

Thus, for sufficiently large j,

$$\log|G(t_{n_j}e^{i\theta})|\leqslant-\frac{\delta}{5}T(t_{n_j},G),\ \theta\in E^*_{n_j}\,.$$

By the same reasoning as in Step 3 we conclude that, with Lemma 8 replace by (24), the function $G(z)$ is bounded in the region $\bar{\Omega}(\theta_1+4\varepsilon,\theta_2-4\varepsilon)$. Hence

$$\lim_{\substack{z\to\infty\\ z\in\bar{\Omega}(\theta_1+4\varepsilon,\theta_2-4\varepsilon)}} F(z)=0\,.\tag{25}$$

Step 5. Suppose that $F(z)$ is not pseudo-prime. Then there exist a transcendental meromorphic function $f(z)$ and a transcendental entire function $g(z)$ such that

$$F(z)=f(g(z))\,. \tag{26}$$

Thus by Lemma 5, $f(z)$ is of zero lower order. Also, $f(z)$ has at most one pole, since $F(z)$ is entire. Hence $\delta(\infty, f)=1$. By this and Lemma 6 there exists a sequence $\{u_n\}$ with $u_n\to+\infty$ as $n\to\infty$ such that

$$\min_{|z|=u_n}|f(z)|\to+\infty\,. \tag{27}$$

Now take a connected path L running to infinity and having the following properties:

(i) L contains $\Gamma(\theta_1+4\varepsilon, \theta_2-4\varepsilon; t_{n_j})$ $(j=1, \dots)$;

(ii) $L\subset\bar{\Omega}(\theta_1+4\varepsilon, \theta_2-4\varepsilon)$.

From (8), (9) and (10) we have, for $\theta\in\tilde{E}_{n_j}=(\theta_1+4\varepsilon, \theta_2-4\varepsilon)\cap E_{n_j}$,

$$\log|F(z)|\leqslant-\frac{\delta}{4}T(t_{n_j}, F)\,,\quad z=(t_{n_j})e^{i\theta} \tag{28}$$

and

$$\text{mes}\ \tilde{E}_{n_j}\geqslant\frac{K}{4q}\,. \tag{29}$$

By (25), (26) and (27), $g(L)$ must be bounded. Hence Lemma 7 asserts that $g(z)\to z_0$ as $z\to\infty$ along L, where z_0 is a zero of $f(z)$. Thus there exists an integer $j_0\geqslant1$ such that, for $j\geqslant j_0$,

$$|g(z)-z_0|<\varepsilon\,,\quad \theta\in\tilde{E}_{n_j}\quad\text{and}\quad z=(t_{n_j})e^{i\theta}\,. \tag{30}$$

Now, if $f(z)$ has a zero of order m $(m\geqslant1)$ at z_0, then there is a constant $c>0$ such that

$$|f(z)|\geqslant c|z-z_0|^m\qquad\text{if}\quad|z-z_0|<\varepsilon\,.$$

Combining this with (30) we obtain

$$|F(z)|=|f(g(z))|\geqslant c|g(z)-z_0|^m\,,\quad \theta\in\tilde{E}_{n_j}\ (j\geqslant j_0)\quad\text{and}\quad z=(t_{n_j})e^{i\theta}\,.$$

So, for $\theta\in\tilde{E}_{n_j}(j\geqslant j_0)$ and $z=(t_{n_j})e^{i\theta}$, we have

$$m\cdot\log^+\frac{1}{|g(z)-z_0|}\geqslant-\log|F(z)|+\log c\geqslant\frac{\delta}{4}T(t_{n_j}, F)+\log c\,.$$

It follows from (29) and the first fundamental theorem that

$$m\cdot T(t_{n_j}, g)+O(1)\geqslant m\cdot m\left(t_{n_j}, \frac{1}{g-z_0}\right)\geqslant\frac{m}{2\pi}\int_{\tilde{E}_{n_j}}\log^+\frac{1}{|g(t_{n_j}e^{i\theta})-z_0|}\,d\theta$$

$$\geqslant\frac{\text{mes}\ \tilde{E}_{n_j}}{2\pi}\left(\frac{\delta}{4}T(t_{n_j}, F)+\log c\right)\geqslant\frac{K\delta}{32q\pi}T(t_{n_j}, F)+\frac{K}{8q\pi}\log c.$$

This contradicts Lemma 4. The proof is completed.

FINAL REMARK. Niino [7] proved another kind of result: If an entire function f belongs to some family $\mathscr{E}(\lambda, \mu)$ and entire function g is of order λ and lower order μ, then $\delta(a, f(g))=0$ for any a in $\boldsymbol{C}$.

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