Tôhoku Math. J.
44 (1992), 251–258

YAMABE METRICS AND CONFORMAL TRANSFORMATIONS

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(Received April 11, 1991)

Abstract. We derive higher order variational formulas for the Yamabe functional, and give an example of infinitesimal deformation of a solution of the Yamabe problem which does not come from conformal vector field.

The Yamabe theorem, which was proved by Schoen [7], states that for any conformal class on a compact cnnected manifold there exists a metric of constant scalar curvature which minimizes the Yamabe functional (see § 1) defined on the conformal class. In this paper, we are interested in the space of solutions of the Yamabe problem, that is, the space of minimizers for the Yamabe functional. The conformal transformation group acts naturally on this space, and a naïve question will be whether this action is transitive (up to homothety) or not. We shall show new necessary conditions for a vector field to be conformal, and give examples which negatively answer the question at the infinitesimal level.

1. The space of Yamabe metrics. Let M be a compact connected n-manifold, and C a conformal class of Riemannian metrics of M, i.e., $C=\{e^{2u}g;\ u\in C^{\infty}(M)\}$ for any fixed metric $g\in C$. Throughout this paper, we assume that the dimension n is at least 3. The *Yamabe functional* I: $C\to \boldsymbol{R}$ is defined as

$$I(g)=\int_M R_g dv_g \Big/ \left(\int_M dv_g\right)^{(n-2)/n} \qquad \text{for} \quad g\in C\,,$$

where R_g is the scalar curvature function of a metric $g\in C$. We set

$$S(M, C)=\{g\in C;\ I(g)=\mu(M, C)\}\,,$$

where

$$\mu(M, C)=\inf\{I(g);\ g\in C\}\,.$$

We call a metric in $S(M, C)$ a solution of the Yamabe problem, or simply a *Yamabe metric*. Since a Yamabe metric is a minimizer of I: $C\to \boldsymbol{R}$, variational formulas show the following properties for $g\in S(M, C)$:

$$R_g=\text{const.} \tag{1.1}$$

1991 *Mathematics Subject Classification*. Primary 53C20.

$$\lambda_1(-\Delta_g) \geqq R_g/(n-1)\,, \tag{1.2}$$

where $\lambda_1(-\Delta_g)$ is the first nonzero (positive) eigenvalue of the Laplacian. Moreover, it is also known that for $g \in S(M, C)$,

$$\mu(M, C) = R_g \,\mathrm{Vol}(M, g)^{2/n} \leqq n(n-1)\mathrm{Vol}(S^n(1))^{2/n}\,, \tag{1.3}$$

where $S^n(1)$ is the Euclidean n-sphere of radius 1 (cf. [1]).

Since $S(M, C)$ is closed under multiplication by positive constants, it is convenient to consider

$$S_1(M, C) = \{g \in S(M, C);\ \mathrm{Vol}(M, g) = 1\}$$

instead of $S(M, C)$. $S_1(M, C)$ is not empty because of the Yamabe theorem.

Let $\mathrm{Conf}(M, C)$ denote the conformal transformation group of (M, C). It is obvious that $\varphi_* g \in S_1(M, C)$ if $\varphi \in \mathrm{Conf}(M, C)$ and $g \in S_1(M, C)$. In this way, $\mathrm{Conf}(M, C)$ acts on $S_1(M, C)$. The stabilizer of this action at $g \in S_1(M, C)$ is $\mathrm{Isom}(M, g)$, the isometry group of (M, g). Hence for each $g \in S_1(M, C)$ we have an inclusion map

$$i_g\colon \mathrm{Conf}(M, C)/\mathrm{Isom}(M, g) \to S_1(M, C)\,.$$

This trivial observation gives us examples of (M, C) for which a solution of the Yamabe problem is not unique.

PROPOSITION 1.1 (cf. [6]). *Let (M_i, g_i), $i = 1, 2$, be compact connected Riemannian manifolds with constant scalar curvature. Assume that* $\dim M_1 \geqq 1$, $R_1 \geqq 0$, $R_2 > 0$ *and that* $\mathrm{Isom}(M_i, g_i)$ *acts transitively on M_i for $i = 1, 2$. Let C_r be the conformal class on $M = M_1 \times M_2$ that contains the metric $r^2 g_1 + g_2$. Then for sufficiently large r,* $\mathrm{Conf}(M, C_r)$ *is strictly larger than* $\mathrm{Isom}(M, g)$, *where $g \in S_1(M, C_r)$.*

PROOF. Suppose on the contrary that $\mathrm{Conf}(M, C_r) = \mathrm{Isom}(M, g)$. Then

$$\mathrm{Isom}(M, g) = \mathrm{Conf}(M, C_r) \supset \mathrm{Isom}(M, r^2 g_1 + g_2) \supset \mathrm{Isom}(M_1, g_1) \times \mathrm{Isom}(M_2, g_2)\,.$$

Therefore g is $\mathrm{Isom}(M_i, g_i)$-invariant, $i = 1, 2$. In view of the transitivity of $\mathrm{Isom}(M_i, g_i)$-actions, this implies that g is homothetic to $r^2 g_1 + g_2$. Hence the metric $r^2 g_1 + g_2$ must be a Yamabe metric. On the other hand, it is easy to see that the metric $r^2 g_1 + g_2$ violates the conditions (1.2) and/or (1.3) for sufficiently large r, though its scalar curvature is constant, a contradiction.

REMARK. This result is an extension of [2]. See also [4].

We formulate our question as follows:

Q.1. Is i_g bijective?

Since a Yamabe metric has constant scalar curvature, we may pose the following more general question:

Q.2. For $g_1, g_2 \in C$ such that $R_{g_1} = R_{g_2} = \text{const}$ and $\mathrm{Vol}(M, g_1) = \mathrm{Vol}(M, g_2)$, is there a conformal transformation $\varphi \in \mathrm{Conf}(M, C)$ such that $\varphi^* g_1 = g_2$?

For each $g \in C$, we have a bijection

$$C^\infty(M) \to C; \quad u \mapsto e^{2u} g,$$

and can regard $S_1(M, C)$ as a subset of $C^\infty(M)$:

$$S_1(M, C) \cong \{u \in C^\infty(M);\ R_{e^{2u}g} = \mu(M, C),\ \mathrm{Vol}(M, e^{2u}g) = 1\}.$$

Differentiating the equations, we formally compute the tangent space, denoted by $s_1(M, C)_g$, to $S_1(M, C)$ at $g \in S_1(M, C)$ as

$$s_1(M, C)_g \cong \left\{ u \in C^\infty(M);\ -\Delta_g u = \frac{1}{n-1} R_g u,\ \int_M u\, dv_g = 0 \right\}.$$

As we shall see later, this formal tangent space can differ from the actual tangent space. Let $\mathrm{conf}(M, C)$ and $\mathrm{isom}(M, g)$ denote the Lie algebras of $\mathrm{Conf}(M, C)$ and $\mathrm{Isom}(M, g)$, respectively. We have the following identification:

$$\mathrm{conf}(M, C)/\mathrm{isom}(M, g) = \left\{ -\frac{1}{n} \mathrm{div}_g X;\ X \in \mathrm{conf}(M, C) \right\} \subset C^\infty(M).$$

With these identifications we see that the differential $(i_g)_*$ of i_g is the inclusion map:

$$(i_g)_* : \mathrm{conf}(M, C)/\mathrm{isom}(M, g) \subset s_1(M, C)_g,$$

where $g \in S_1(M, C)$. This inclusion is also a consequence of the well-known formula $-\Delta_g \mathrm{div}_g X = (R_g/(n-1))\, \mathrm{div}_g X$ for a conformal vector field X and $g \in C$ with constant scalar curvature.

In this setting, the following correspond to Q.1 and Q.2, respectively:

Q.1′. Is $(i_g)_*$ bijective for $g \in S_1(M, C)$?

Q.2′. If g has constant scalar curvature and $u \in C^\infty(M)$ satisfies

$$-\Delta_g u = \frac{1}{n-1} R_g u,$$

then is there a conformal vector field whose divergence is equal to u?

In §3 we shall answer these two questions negatively.

2. Conformal vector fields and higher order variations of the Yamabe functional.

THEOREM 2.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ with constant scalar curvature R_g. Let X be a conformal vector field and $u = \mathrm{div}_g X$. Then,*

(i) $$\int_M u^3 dv_g = 0 \quad \text{and} \quad \left(\Delta_g + \frac{1}{n-1} R_g\right) v = -\frac{n+2}{(n-1)(n-2)} R_g u^2$$

is solvable for v;

(ii)
$$3\int_M u^2 v\,dv_g=\frac{n-6}{n-2}\int_M u^4 dv_g\,,$$

where v *is as in* (i).

PROOF. First we note that all are trivial when $R_g \leqq 0$, because $\mathrm{div}_g X=0$ if $R_g \leqq 0$. Secondly, if some solution v of the equation in (i) satisfies the equality in (ii), then any other solution, say v', satisfies the equality, because then

$$\begin{aligned}\int_M u^2(v-v')dv_g&=-\frac{(n-1)(n-2)}{(n+2)R_g}\int_M\left(\left(\Delta_g+\frac{1}{n-1}R_g\right)v\right)(v-v')dv_g\\&=-\frac{(n-1)(n-2)}{(n+2)R_g}\int_M v\left(\Delta_g+\frac{1}{n-1}R_g\right)(v-v')dv_g=0\,.\end{aligned}$$

Let $\{\varphi_t\}$ be the one-parameter transformation group generated by X. Since X is a conformal vector field, $g_t:=\varphi_t^* g$ is conformal to g. Define $w_t \in C^\infty(M)$ by

$$g_t=w_t^{(n-2)/4}g\,,\qquad w_t>0\,. \tag{2.1}$$

Then $u=\mathrm{div}_g X=(2n/(n-2))\dot{w}_0$, where $\cdot$ stands for d/dt. The scalar curvature R_t of g_t is written as

$$R_t=w_t^{-q}L_g w_t\,, \tag{2.2}$$

where $q=(n+2)/(n-2)$ and $L_g=-4((n-1)/(n-2))\Delta_g+R_g$. Hence we have

$$\dot{R}_t=w_t^{-q-1}(w_t L_g-q(L_g w_t))\dot{w}_t\,. \tag{2.3}$$

Differentiating this repeatedly, we get

$$(w_t^{q+1}\dot{R}_t)^{(m-1)}=(w_t L_g-q(L_g w_t))w_t^{(m)}+\sum_{k=1}^{m-1}\left\{\binom{m-1}{m-k}-\binom{m-1}{k}q\right\}w_t^{(m-k)}L_g w_t^{(k)}\,. \tag{2.4}$$

Since R_g is constant, $R_t=\varphi_t^* R_g$ is a constant independent of t. Thus the left hand side of (2.4) is identically equal to 0. So we expand (2.4) explicitly at $t=0$ for $m=1$, 2 and 3, respectively as follows:

$$P_g\dot{w}_0=0\,, \tag{2.5}$$

$$P_g\ddot{w}_0=q(q-1)R_g\dot{w}_0^2\,, \tag{2.6}$$

$$P_g\dddot{w}_0=q(q-1)R_g(3\dot{w}_0\ddot{w}_0+(q-2)\dot{w}_0^3)\,, \tag{2.7}$$

where $P_g=L_g-qR_g=-4((n-1)/(n-2))(\varDelta_g+R_g/(n-1))$. Thus we have

$$q(q-1)R_g\int_M \dot{w}_0^3 dv_g=\int_M \dot{w}_0 P_g\ddot{w}_0 dv_g=\int_M \ddot{w}_0 P_g\dot{w}_0 dv_g=0\,, \tag{2.8}$$

and

$$q(q-1)R_g\int_M(3\dot{w}_0\ddot{w}_0+(q-2)\dot{w}_0^3)dv_g=\int_M\dot{w}_0P_g\dddot{w}_0dv_g=\int_M\dddot{w}_0P_g\dot{w}_0dv_g=0\,. \tag{2.9}$$

Recall that $u=(2n/(n-2))\dot{w}_0$, and we see that our assertions follow from (2.6), (2.8) and (2.9) by putting $v=(2n/(n-2))^2\ddot{w}_0$.

The above result is related to higher order variational formulas for the Yamabe functional. If the Yamabe functional I: $C\to\boldsymbol{R}$ has a relative minimum at g, then the first and the second variational formulas say that the metric g has the properties (1.1) and (1.2). As for the third and the fourth variational formulas we have the following:

THEOREM 2.2. *Suppose g has positive constant scalar curvature and that the Yamabe functional I: $C\to\boldsymbol{R}$ has a relative minimum at g. Then,*

(i) *If u_1, $u_2\in\mathrm{Ker}(\Delta_g+R_g/(n-1))$, then $\int_M u_1^2u_2dv_g=0$. In particular, for any $u\in\mathrm{Ker}(\Delta_g+R_g/(n-1))$,*

$$\left(\Delta_g+\frac{1}{n-1}R_g\right)v=-\frac{n+2}{(n-1)(n-2)}R_gu^2$$

is solvable for v;

(ii) *For u, v as above, the inequality*

$$3\int_M u^2vdv_g\leqq\frac{n-6}{n-2}\int_M u^4dv_g$$

holds.

PROOF. Let u be an arbitrary function satisfying

$$\left(\Delta_g+\frac{1}{n-1}R_g\right)u=0\,. \tag{2.10}$$

We set

$$g_t=\left(1+tu+\frac{1}{2}t^2v\right)^{4/(n-2)}g\,, \tag{2.11}$$

where v is any function such that

$$\int_M vdv_g=-q\int_M u^2dv_g\,, \tag{2.12}$$

where $q=(n+2)/(n-2)$. Then it is straightforward to see that

$$\frac{d}{dt}\mathrm{Vol}(M,g_t)\Big|_{t=0}=\left(\frac{d}{dt}\right)^2\mathrm{Vol}(M,g_t)\Big|_{t=0}=\frac{d}{dt}\int_M R_tdv_t\Big|_{t=0}=\left(\frac{d}{dt}\right)^2\int_M R_tdv_t\Big|_{t=0}=0\,,$$

where R_t and dv_t are, respectively, the scalar curvature and the volume element of the metric g_t. Then it is easy to see that

$$(2.13)\quad \left(\frac{d}{dt}\right)^3 I(g_t)\Big|_{t=0} = \left(\frac{d}{dt}\right)^3\left(\left(\int_M R_t dv_t\right)\mathrm{Vol}(M, g_t)^{-(n-2)/n}\right)\Big|_{t=0}$$

$$= \left(\frac{d}{dt}\right)^3\left(\int_M R_t dv_t\right)\Big|_{t=0} \mathrm{Vol}(M, g)^{-(n-2)/n} + \int_M R_g dv_g \left(\frac{d}{dt}\right)^3 (\mathrm{Vol}(M, g_t)^{-(n-2)/n})\Big|_{t=0}$$

$$= -2R_g q(q-1)\mathrm{Vol}(M, g)^{-(n-2)/n}\int_M u^3 dv_g\,.$$

Since I takes a relative minimum at g, we have

$$(2.14)\qquad \int_M u^3 dv_g = 0\,.$$

This holds for any $u \in \mathrm{Ker}(\Delta_g + R_g/(n-1))$. Hence for any $u_1, u_2 \in \mathrm{Ker}(\Delta_g + R_g/(n-1))$, we have

$$(2.15)\qquad \int_M u_1^2 u_2 dv_g = \frac{1}{6}\int_M ((u_1+u_2)^3 - (u_1-u_2)^3 - 2u_2^3)dv_g = 0\,,$$

which implies $u^2 \in \mathrm{Im}(\Delta_g + R_g/(n-1))$ for any $u \in \mathrm{Ker}(\Delta_g + R_g/(n-1))$. Hence the equation

$$(2.16)\qquad \left(\Delta_g + \frac{1}{n-1}R_g\right)v = -\frac{n+2}{(n-1)(n-2)}R_g u^2$$

is solvable for v. It is easy to see that this v also satisfies the condition (2.12). So we assume that the v in (2.11) satisfies the equation (2.16). Then we easily get

$$(2.17)\quad \left(\frac{d}{dt}\right)^4 I(g_t)\Big|_{t=0} = -4R_g q(q-1)\mathrm{Vol}(M, g)^{-(n-2)/n}\int_M (3u^2 v + (q-2)u^4)dv_g\,.$$

$(d/dt)^4 I(g_t)|_{t=0}$ is nonnegative by our assumption, and we get the desired inequality.

3. Examples. By $S^n(r)$ we denote the n-dimensional Euclidean sphere of radius r. We suppose $(M, g) = S^p(\sqrt{p}) \times S^{n-p}(\sqrt{n-p-1})$. Let

$$M = S^p(\sqrt{p}) \times S^{n-p}(\sqrt{n-p-1}) \subsetneq \boldsymbol{R}^{p+1} \times \boldsymbol{R}^{n-p+1}$$

be the canonical isometric embedding, and $u \in C^\infty(M)$ be any one of the first $p+1$ coordinate functions of $\boldsymbol{R}^{p+1} \times \boldsymbol{R}^{n-p+1}$ restricted to M. Then,

$$\left(\Delta_g + \frac{1}{n-1}R_g\right)u = (\Delta_g + 1)u = 0\,.$$

Moreover, u satisfies the equation

$$u^2+p|du|^2=p\,.$$

Hence putting

$$v=\frac{p(n+2)}{(p+2)(n-2)}(u^2-2)\,,$$

we have

$$\left(\Delta_g+\frac{1}{n-1}R_g\right)v=-\frac{n+2}{(n-1)(n-2)}R_g u^2$$

and

$$\int_M u^2 dv_g=\frac{p}{p+1}\mathrm{Vol}(M,g)\,.$$

It is also easy to see that

$$\int_M u^4 dv_g=\frac{3p}{p+3}\int_M u^2 dv_g=\frac{3p^2}{(p+1)(p+3)}\mathrm{Vol}(M,g)\,.$$

Consequently, we get

$$\int_M\left(3u^2 v-\frac{n-6}{n-2}u^4\right)dv_g=-\frac{24p^2(n-p)}{(p+1)(p+2)(p+3)(n-2)}\mathrm{Vol}(M,g)\,.$$

This is negative if $n\geqq 3$ and $0<p<n$. Therefore it follows from Theorem 2.1 that the function u cannot be the divergence of any conformal vector field. Thus the answer to Q.2′ is negative.

If $n\geqq 3$ and $p=1$, then it can be shown, by using a theorem of Gidas, Ni and Nirenberg [3], that the metric g is a solution of the Yamabe problem (cf. [5], [8]). Hence in this case (M,g) is a counterexample to Q.1′. In this case, however, i_g is bijective (cf. [5], [8]), and the question Q.1 and Q.2 remain open.

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