

A FREE BOUNDARY PROBLEM FOR A NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION INVOLVING A SMALL PARAMETER

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Abstract. A free boundary problem for a nonlinear second order differential equation involving a small parameter is studied. The problem arises in the research of certain generalized diffusion processes. Its solution is constructed by the aid of the unique solution of a two-point boundary value problem for two nonlinear first order differential equations, in which the right endpoint is singular.

1. Introduction. The present paper is a continuation of the paper [1]. In this paper we study a free boundary problem of the form

$$(1_\varepsilon) \quad [(k(v(y)) + \varepsilon) |v'(y)|^{N-1} v'(y)]' = -[yg(v(y)) + f(v(y))]v'(y) \quad \text{for } y > Y,$$

$$(2_\varepsilon) \quad v|_{y=Y} = 0, \quad -A[k(v) + \varepsilon] |v'|^{N-1} v'|_{y=Y} = Y, \quad v|_{y=+\infty} = B,$$

where ε is a small positive parameter, the constant Y and the function $v(y)$ are to be determined, while the constants A, B, N , and the functions $f(s), g(s), k(s)$ are assumed to satisfy the following hypotheses:

- (I) Both N and B are given positive numbers.
- (II) $G(s) := \int_0^s g(t)dt$ is a strictly increasing, absolutely continuous function defined on $[0, B]$.
- (III) $k(s)$ is a nonnegative measurable function defined on $[0, B]$ such that the function $(G(B) - G(s))k^{1/N}(s)$ is Lebesgue integrable on $[0, B]$.
- (IV) $A > -1/G(B)$ is a given real number.
- (V) $F(s) := \int_0^s f(t)dt$ is an absolutely continuous function defined on $[0, B]$ such that the function $F(B)(1 + AG(s))/(1 + AG(B)) - F(s)$ is non-negative on $[0, B]$.

By a solution of the free boundary problem $(1_\varepsilon) - (2_\varepsilon)$ with $\varepsilon > 0$, we shall mean the pair $(Y_\varepsilon, v_\varepsilon(y))$ satisfying the following conditions:

- (i) Y_ε is a finite real number.
- (ii) $v_\varepsilon(y)$ is an absolutely continuous function defined on $[Y_\varepsilon, +\infty)$.
- (iii) $K_\varepsilon(y) := (k(v_\varepsilon(y)) + \varepsilon) |v'_\varepsilon(y)|^{N-1} v'_\varepsilon(y)$ is (equivalent to) an absolutely continuous function defined on $[Y_\varepsilon, +\infty)$.

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(iv) $v_\varepsilon(Y_\varepsilon) = 0$, $-AK_\varepsilon(Y_\varepsilon) = Y_\varepsilon$, and $v_\varepsilon(+\infty) = B$.

(v) The equality $K'_\varepsilon(y) = -[yg(v_\varepsilon(y)) + f(v_\varepsilon(y))]v'_\varepsilon(y)$ holds almost everywhere in $(Y_\varepsilon, +\infty)$.

If the solution $(Y_\varepsilon, v_\varepsilon(y))$ converges to a limit $(Y_0, v_0(y))$ as ε tends to zero, then the limit is called a solution of the reduced free boundary problem $(1_0) - (2_0)$.

From the form and the definition of solutions of the free boundary problem $(1_0) - (2_0)$, the free boundary problem (2.1) studied in the paper [1] is a particular case of the problem in this paper. In the paper [1] the function $k(s)$ was assumed to be positive a.e. on $[0, B]$, while in this paper $k(s)$ is allowed to have intervals of degeneracy in $[0, B]$ (whose definition will be given in the last section). This is an important difference between these papers.

The free boundary problem $(1_\varepsilon) - (2_\varepsilon)$ is a perturbation problem. When the function $k(s)$ has at least one interval of degeneracy in $[0, B]$, it is a singular perturbation problem. The aim of studying the problem is to determine the solution of the reduced free boundary problem $(1_0) - (2_0)$ and to ascertain properties of the solution. As we shall see later, $v_0(y)$, as a function defined on $[Y_0, +\infty)$, has jump points if and only if the function $k(s)$ possesses intervals of degeneracy in $[0, B]$, and there exists a one-to-one correspondence between the set of all jump points and the collection of all intervals of degeneracy.

The plan of this paper is as follows. In Section 2 we convert the free boundary problem $(1_\varepsilon) - (2_\varepsilon)$ into the two-point boundary value problem, namely,

$$(3_\varepsilon) \quad -\frac{w'(s) + f(s)}{g(s)} = z(s), \quad z'(s) = \left(\frac{k(s) + \varepsilon}{w(s)}\right)^{1/N}, \quad s \in (0, B),$$

$$(4_0) \quad -Aw(0) = z(0), \quad w(B) = 0.$$

Section 3 is devoted to the two-point boundary value problem. We show that for each fixed $\varepsilon \geq 0$ the two-point boundary value problem has a unique solution $(w_\varepsilon(s), z_\varepsilon(s))$ under the hypotheses (I)–(V). In the last section we construct the solution $(Y_\varepsilon, v_\varepsilon(y))$ of the free boundary problem $(1_\varepsilon) - (2_\varepsilon)$ with $\varepsilon \geq 0$, utilizing the unique solution $(w_\varepsilon(s), z_\varepsilon(s))$.

2. Formal reduction. Let $(Y, v(y))$ be a solution of the free boundary problem $(1_\varepsilon) - (2_\varepsilon)$. We claim that $v(y)$ is monotone (increasing) on $[Y, +\infty)$. For if this is not the case, there will be numbers $a, b \in [Y, +\infty)$, $a < b$, and $C \in [0, B]$, such that $v(a) = v(b) = C$ and $v(y) > C$ (say) in (a, b) . Integrating the equation (1_ε) over (a, b) , we get

$$0 \geq K_\varepsilon(b) - K_\varepsilon(a) = \int_a^b (G(v(y)) - G(C)) dy > 0.$$

This contradiction proves our claim. If $v(y)$ is still strictly increasing on $[Y, +\infty)$, then $v'(+\infty) = 0$ and the function $y = z(s)$, inverse to $s = v(y)$, exists. Furthermore, $z(0) = Y$, $s = v(z(s))$ in $[0, B]$, and $v'(z(s)) = 1/z'(s) > 0$ a.e. in $(0, B)$. Substituting $y = z(s)$ into the

equation (1_ε) and then putting $w(s) = (k(s) + \varepsilon)/(z'(s))^N$, we obtain the two-point boundary value problem (3_ε)–(4₀).

3. Two-point boundary value problem. In this section we address ourselves to the two-point boundary value problem (3_ε)–(4₀). As the endpoints $s = B$ is singular for it, we need to consider the two-point boundary value problem without singularity, namely,

$$(3_\varepsilon) \quad -\frac{w'(s) + f(s)}{g(s)} = z(s), \quad z'(s) = \left(\frac{k(s) + \varepsilon}{w(s)}\right)^{1/N} := P_\varepsilon(s, w(s)), \quad s \in (0, B),$$

$$(4_h) \quad -Aw(0) = z(0), \quad w(B) = h > 0.$$

We shall call a pair $(w(s), z(s))$ a solution of the two-point boundary value problem (3_ε)–(4_h) with $h \geq 0$, if it satisfies the following conditions:

(i) $w(s)$ is an absolutely continuous function defined on $[0, B]$ and positive in $[0, B)$, while $z(s)$ is (equivalent to) an increasing, locally absolutely continuous function defined in $[0, B)$.

(ii) $-Aw(0) = z(0)$ and $w(B) = h$.

(iii) The equalities

$$-(w'(s) + f(s)) = z(s)g(s) \quad \text{and} \quad z'(s) = P_\varepsilon(s, w(s))$$

hold a.e. in $(0, B)$.

By the hypotheses (I)–(V), it is readily verified that if a nonnegative continuous function $w(s)$ is a solution of the integral equation

$$(5_B^h) \quad w(s) = \frac{1 + AG(s)}{1 + AG(B)} (F(B) + h) - F(s) + \int_0^B J_b(s, t) P_\varepsilon(t, w(t)) dt := (Mw)(s), \quad s \in [0, B],$$

where

$$J_b(s, t) := \begin{cases} (G(b) - G(s))(1 + AG(t))/(1 + AG(b)) & \text{for } 0 \leq t \leq s, \\ (G(b) - G(t))(1 + AG(s))/(1 + AG(b)) & \text{for } s \leq t \leq b \leq B, \end{cases}$$

then the pair $(w(s), z(s))$ is a solution of the two-point boundary value problem (3_ε)–(4_h) with $\varepsilon \geq 0$ and $h \geq 0$, where the function $z(s)$ is defined by

$$(6) \quad z(s) = \frac{-A(F(B) + h)}{1 + AG(B)} + \int_0^s \frac{1 + AG(t)}{1 + AG(B)} P_\varepsilon(t, w(t)) dt - \int_s^B \frac{A(G(B) - G(t))}{1 + AG(B)} P_\varepsilon(t, w(t)) dt,$$

and vice versa. It must be pointed out that the integral representations (5_B^h) and (6) are both valid when $k(s) + \varepsilon \equiv 0$.

LEMMA 1. For each fixed $h > 0$ and $\varepsilon \geq 0$, the integral equation (5_B^h) has at least one solution, say $w(s; \varepsilon, h)$, which is positive on $[0, B]$.

PROOF. Define the mapping $M : X \rightarrow X$ by the right side of (5_B^h), where

$$X := \{w(t) \in C[0, B]; 0 < u(t) \leq w(t) \leq (Mu)(t)\},$$

$C[0, B]$ is the set of all real-valued continuous functions defined on $[0, B]$, and $u(t) := h(1 + AG(t))/(1 + AG(B)) > 0$.

By the hypotheses (I)–(V), it is easy to check that M is a compactly continuous mapping from X into X . The Schauder fixed point theorem tells us that the mapping M has at least one fixed point, say $w(s; \varepsilon, h)$, in the set X . $w(s; \varepsilon, h)$ is clearly a solution of the equation (5_B^h).

LEMMA 2. *If $h_1 \geq h_2 > 0$, then for all $s \in [0, B]$*

$$0 \leq w(s; \varepsilon, h_1) - w(s; \varepsilon, h_2) \leq (h_1 - h_2)(1 + AG(s))/(1 + AG(B)).$$

Hence $w(s; \varepsilon, h)$ is a unique solution of the equation (5_B^h) with $h > 0$.

PROOF. We denote the solutions $w(s; \varepsilon, h_1)$ and $w(s; \varepsilon, h_2)$ by $w_1(s)$ and $w_2(s)$, respectively, and first prove the inequality on the left hand side, namely, $w_1(s) - w_2(s) \geq 0$ on $[0, B]$. If not, then there will be a point $s = E$ at which $w_1(E) - w_2(E) < 0$. We now distinguish two cases.

Case (i) $w_1(0) - w_2(0) < 0$. In this case we can choose the left endpoint $s = 0$ as the point $s = E$. As $w_1(B) - w_2(B) = h_1 - h_2 \geq 0$, there exists a maximal interval $[0, b]$ such that $w_1(b) - w_2(b) = 0$ and $w_1(s) - w_2(s) < 0$ in $[0, b]$. Note that for almost all $s \in (0, B)$

$$(7) \quad -\left(\frac{w_1'(s) - w_2'(s)}{g(s)}\right)' = P_\varepsilon(s, w_1(s)) - P_\varepsilon(s, w_2(s)).$$

Multiplying the equality (7) by $J_b(0, s)$ and then integrating the result over $(0, b)$, we get

$$0 > w_1(0) - w_2(0) = \int_0^b J_b(0, s)(P_\varepsilon(s, w_1(s)) - P_\varepsilon(s, w_2(s)))ds \geq 0.$$

This is impossible.

Case (ii) $w_1(0) - w_2(0) \geq 0$. In this situation there exists a maximal interval (a, b) , $0 \leq a < E < b \leq B$, such that $w_1(a) - w_2(a) = w_1(b) - w_2(b) = 0$ and $w_1(s) - w_2(s) < 0$ in (a, b) . Multiplying (7) by

$$J(E, s) := \begin{cases} (G(b) - G(E))(G(s) - G(a))/(G(b) - G(a)) & \text{for } a \leq s \leq E, \\ (G(b) - G(s))(G(E) - G(a))/(G(b) - G(a)) & \text{for } E \leq s \leq b \end{cases}$$

and then integrating the result over (a, b) , we get

$$0 > w_1(E) - w_2(E) = \int_a^b J(E, s)(P_\varepsilon(s, w_1(s)) - P_\varepsilon(s, w_2(s)))ds \geq 0.$$

This is also impossible.

The above argument shows that the inequality on the left hand side is true. The

inequality on the right hand side follows from that on the left by (5_B^h) .

LEMMA 3. *The equation (5_B^h) has a solution $w(s; \varepsilon, 0) \geq 0$.*

PROOF. When $k(s) + \varepsilon \equiv 0$, $F(B)(1 + AG(s))/(1 + AG(B)) - F(s)$ is a solution of (5_B^0) . We now consider the case $k(s) + \varepsilon \not\equiv 0$. In view of Lemma 2, the solution $w(s; \varepsilon, h)$ converges to a limit $w(s; \varepsilon, 0)$ uniformly on $[0, B]$, as $h \rightarrow +0$. Inserting $w(s; \varepsilon, h)$ into the equation (5_B^h) and then letting $h \rightarrow +0$, we have the equality (5_B^0) , by the monotone convergence theorem. This shows that the uniform limit $w(s; \varepsilon, 0)$ is a solution of the equation (5_B^0) .

LEMMA 4. *If $\varepsilon_1 \geq \varepsilon_2 \geq 0$, then for all $s \in [0, B]$*

$$0 \leq w(s; \varepsilon_1, h) - w(s; \varepsilon_2, h) \leq Ch^{-1/N}(\varepsilon_1 - \varepsilon_2)^\theta,$$

where $\theta := \min\{1, 1/N\}$ and C is a number independent of $\varepsilon_1, \varepsilon_2 \in [0, 1]$ and $h > 0$.

LEMMA 5. *The equation (5_B^h) with $\varepsilon \geq 0$ and $h \geq 0$ has at most one solution.*

The proofs of the two lemmas above are similar to that of Lemma 2 and hence omitted here.

LEMMA 6. *The solution $w(s; \varepsilon, 0)$ uniformly converges to the solution $w(s; 0, 0)$ as $\varepsilon \rightarrow +0$.*

PROOF. In virtue of Lemmas 2 and 4, we have

$$\begin{aligned} w(s; 0, 0) &\leq w(s; 0, h) \leq w(s; 0, 0) + h && \text{on } [0, B], \\ w(s; 0, h) &\leq w(s; \varepsilon, h) \leq w(s; 0, h) + Ch^{-1/N}\varepsilon^\theta && \text{on } [0, B]. \end{aligned}$$

Hence, setting $\varepsilon^\theta = h^{1+1/N}$, we obtain

$$0 \leq w(s; 0, 0) \leq w(s; \varepsilon, 0) \leq w(s; \varepsilon, h) \leq w(s; 0, 0) + (C + 1)h$$

for all $s \in [0, B]$. This shows that $w(s; \varepsilon, 0)$ uniformly converges to $w(s; 0, 0)$ as $\varepsilon^\theta = h^{1+1/N} \rightarrow +0$.

Let $(w(s; \varepsilon, 0), z(s; \varepsilon, 0))$ be the unique solution of the two-point boundary value problem $(3_\varepsilon) - (4_0)$. By the representation (6), it follows from Lemma 6 that as $\varepsilon \rightarrow +0$, $z(s; \varepsilon, 0)$ converges to $z(s; 0, 0)$ uniformly on $[0, B - \delta]$ whenever $\delta \in (0, B)$.

LEMMA 7. *Let $f(s)/g(s)$ be (equivalent to) an increasing, absolutely continuous function defined on $[0, B]$; namely, let $F(Q(r))$ be a continuous convex function defined on $[0, G(B)]$, where $s = Q(r)$ is the function inverse to $r = G(s)$. Then a necessary and sufficient condition for $z(B; \varepsilon, 0)$ to be finite is that*

$$(8) \quad \int_0^B \left(\frac{k(s) + \varepsilon}{G(B) - G(s)} \right)^{1/N} ds < +\infty.$$

PROOF. When $k(s) + \varepsilon \equiv 0$, $z(s; 0, 0) = -AF(B)/(1 + AG(B))$ is constant. Hence we

consider only the case where $k(s) + \varepsilon \neq 0$. In this situation, $-w(Q(r); \varepsilon, 0)$ is a continuous convex function defined on $[0, G(B)]$ and $w(0; \varepsilon, 0) > 0$, in virtue of the assumption of the lemma. Hence

$$w(Q(r); \varepsilon, 0) \geq \theta(G(B) - r) \quad \text{on } [0, G(B)],$$

where $\theta = w(0; \varepsilon, 0)/G(B)$, namely,

$$w(s; \varepsilon, 0) \geq \theta(G(B) - G(s)) \quad \text{on } [0, B].$$

If the condition (8) holds, then

$$\begin{aligned} z(B; \varepsilon, 0) + \frac{AF(B)}{1 + AG(B)} &= \int_0^B \frac{1 + AG(t)}{1 + AG(B)} P_\varepsilon(t, w(t; \varepsilon, 0)) dt \\ &\leq \int_0^B \frac{1 + AG(t)}{1 + AG(B)} \left(\frac{k(t) + \varepsilon}{\theta(G(B) - G(t))} \right)^{1/N} dt. \end{aligned}$$

This shows that $z(B; \varepsilon, 0)$ is finite.

Assume $z(B; \varepsilon, 0)$ is finite. Then there is a positive number M such that

$$-\frac{d}{dr} w(Q(r); \varepsilon, 0) \Big|_{r=G(B)} = \frac{d}{dr} F(Q(r)) \Big|_{r=G(B)} + z(B; \varepsilon, 0) \leq M.$$

Hence it follows that $w(Q(r); \varepsilon, 0) \leq M(G(B) - r)$ on $[0, G(B)]$, that is,

$$w(s; \varepsilon, 0) \leq M(G(B) - G(s)) \quad \text{on } [0, B].$$

Consequently,

$$\begin{aligned} z(B; \varepsilon, 0) + \frac{AF(B)}{1 + AG(B)} &= \int_0^B \frac{1 + AG(t)}{1 + G(B)A} P_\varepsilon(t, w(t; \varepsilon, 0)) dt \\ &\geq \int_0^B \frac{1 + AG(t)}{1 + AG(B)} \left(\frac{k(t) + \varepsilon}{M(G(B) - G(t))} \right)^{1/N} dt. \end{aligned}$$

This shows that the condition (8) holds.

We can summarize the above results in the following statement.

THEOREM 1. *Under the hypotheses (I)–(V), for each $\varepsilon \geq 0$ the two-point boundary value problem $(3_\varepsilon) - (4_0)$ has a unique solution $(w_\varepsilon(s), z_\varepsilon(s))$. Moreover, as ε tends to zero, $w_\varepsilon(s)$ converges to $w_0(s)$ uniformly on $[0, B]$ and $z_\varepsilon(s)$ converges to $z_0(s)$ uniformly on $[0, B - \delta]$ for any $\delta \in (0, B)$.*

Theorem 1 shows that the two-point boundary value problem $(3_\varepsilon) - (4_0)$ involving a small parameter ε is a regular perturbation.

4. Free boundary problem. In this section we construct the solution $(Y_\varepsilon, v_\varepsilon(y))$,

utilizing the unique solution $(w_\epsilon(s), z_\epsilon(s))$ of the two-point boundary value problem $(3_\epsilon) - (4_0)$.

We first introduce several propositions and definitions.

PROPOSITION 1. *Let $v(s)$ be an increasing function defined on $[a, b]$ and $u(t)$ an absolutely continuous function defined on $[v(a), v(b)]$. Then $u(v(s))$ has a finite derivative a.e. on $[a, b]$ and the chain rule*

$$\frac{d}{ds} u(v(s)) = u'(v(s))v'(s)$$

holds a.e. on $[a, b]$.

PROPOSITION 2. *Let $v(s)$ be an increasing, absolutely continuous function defined on $[a, b]$ and $u(t)$ an integrable function defined on $[v(a), v(b)]$. Then $u(v(s))v'(s)$ is integrable on $[a, b]$ and the change of variables formula*

$$\int_a^b u(v(s))v'(s)ds = U(v(b)) - U(v(a))$$

holds, where $U(t)$ is an indefinite integral of $u(t)$.

PROPOSITION 3. *Let (a, b) be a finite open interval and $z(s)$ a strictly increasing, locally absolutely continuous function defined in (a, b) . Then the function $s=v(y)$, inverse to $y=z(s)$, is a strictly increasing, absolutely continuous function defined in $(z(a+0), z(b-0))$.*

Propositions 1 and 2 are respectively Corollaries 4 and 6 in [2], and Proposition 3 is a direct consequence of them. Hence the proofs are omitted.

DEFINITION 1. Let $k(s)$ be a nonnegative measurable function defined on $[0, B]$ such that $k^{1/N}(s)$ is integrable on $[0, B]$. A closed subinterval $[a, b]$ is said to be an interval of degeneracy possessed by $k(s)$ in $[0, B]$, if, for any $\delta > 0$,

$$\int_a^b k^{1/N}(s)ds = 0, \quad \int_{a-\delta}^a k^{1/N}(s)ds > 0 \quad \text{or} \quad \int_b^{b+\delta} k^{1/N}(s)ds > 0,$$

whenever $a - \delta \geq 0$ or $b + \delta \leq B$.

DEFINITION 2. Let $y = z(s)$ be an increasing, locally absolutely continuous function defined in $[0, B)$. A function $s = v(y)$ is called a generalized inverse to $y = z(s)$, if it is defined on $[Y, +\infty)$, increasing, of bounded variation, $v(+\infty) = B$, and possibly multiple-valued, and its graph in the region $\{(y, s); Y \leq y < z(B-0), 0 \leq s < B\}$ is congruent with the graph of $y = z(s)$. Here $Y := z(0)$.

For example, if $z(s) \equiv Y$ on $[0, B)$, then $v(y) = BH(y - Y)$, where $H(y) = 0$ for $y < 0$, $H(y) = 1$ for $y > 0$, and $H(0) = [0, 1]$.

Let $(w_\varepsilon(s), z_\varepsilon(s))$ be the unique solution of the two-point boundary value problem $(3_\varepsilon) - (4_0)$, $\varepsilon \geq 0$. Then $z_\varepsilon(s)$ is an increasing, locally absolutely continuous function defined in $[0, B)$. Consequently, the function $s = v_\varepsilon(y)$, generalized inverse to $y = z_\varepsilon(s)$, is defined on $[Y_\varepsilon, +\infty)$, where $Y_\varepsilon := z_\varepsilon(0)$; when $z_\varepsilon(B-0)$ is finite, $v_\varepsilon(y) = B$ for $y \geq z_\varepsilon(B-0)$. We now prove that $(Y_\varepsilon, v_\varepsilon(y))$ is a solution of the free boundary problem $(1_\varepsilon) - (2_\varepsilon)$.

LEMMA 8. *As $\varepsilon \rightarrow +0$, $Y_\varepsilon = z_\varepsilon(0)$ converges to $Y_0 = z_0(0)$ and $v_\varepsilon(y)$ converges to $v_0(y)$ pointwise on $[Y_0, +\infty)$.*

This lemma is an immediate consequence of the definition of $(Y_\varepsilon, v_\varepsilon(y))$ and Theorem 1.

LEMMA 9. *If $\varepsilon > 0$ or $\varepsilon = 0$ and $k(s) > 0$ a.e. on $[0, B]$, then $(Y_\varepsilon, v_\varepsilon(y))$ is a solution of the free boundary problem $(1_\varepsilon) - (2_\varepsilon)$.*

PROOF. In virtue of the assumption of the lemma, we have

$$(9) \quad z'_\varepsilon(s) = \left(\frac{k(s) + \varepsilon}{w_\varepsilon(s)} \right)^{1/N}$$

is positive a.e. in $(0, B)$, i.e., $z_\varepsilon(s)$ is strictly increasing in $[0, B)$. Thus, $v_\varepsilon(y)$ is a strictly increasing, absolutely continuous function defined on $[Y_\varepsilon, z_\varepsilon(B-0))$ by Proposition 3. Hence it follows that

$$(10_\varepsilon) \quad w_\varepsilon(s) = (k(s) + \varepsilon) / (z'_\varepsilon(s))^N$$

holds a.e. in $(0, B)$. As the function $w_\varepsilon(s)$ is absolutely continuous on $[0, B]$, $(k(s) + \varepsilon) / (z'_\varepsilon(s))^N$ can be regarded as an absolutely continuous function defined on $[0, B]$. Inserting $s = v_\varepsilon(y)$ into (10_ε) and the first of (3_ε) , we get

$$(11_\varepsilon) \quad w_\varepsilon(v_\varepsilon(y)) = (k(v_\varepsilon(y)) + \varepsilon) |v'_\varepsilon(y)|^{N-1} v'_\varepsilon(y) \quad \text{a.e. in } [Y_\varepsilon, z_\varepsilon(B-0)),$$

$$(12_\varepsilon) \quad W'_\varepsilon(v_\varepsilon(y)) = -[yg(v_\varepsilon(y)) + f(v_\varepsilon(y))] \quad \text{a.e. in } [Y_\varepsilon, z_\varepsilon(B-0)).$$

Here we have used the facts that $z_\varepsilon(v_\varepsilon(y)) = y$ in $[Y_\varepsilon, z_\varepsilon(B-0))$ and that $v'_\varepsilon(y) = 1/z'_\varepsilon(v_\varepsilon(y))$ in $(Y_\varepsilon, z_\varepsilon(B-0))$. When $z_\varepsilon(B-0)$ is finite, the two equalities above read $0=0$ for $y \geq z_\varepsilon(B-0)$. By the chain rule, we obtain

$$[(k(v_\varepsilon(y)) + \varepsilon) |v'_\varepsilon(y)|^{N-1} v'_\varepsilon(y)]' = w'_\varepsilon(v_\varepsilon(y)) v'_\varepsilon(y) = -[yg(v_\varepsilon(y)) + f(v_\varepsilon(y))] v'_\varepsilon(y)$$

holds a.e. in $(Y_\varepsilon, z_\varepsilon(B-0))$; when $z_\varepsilon(B-0)$ is finite, the equality reads $0=0$ for $y \geq z_\varepsilon(B-0)$. From the definition of $v_\varepsilon(y)$, (11_ε) and (4_0) , we conclude that $v_\varepsilon(y)$ satisfies all the boundary conditions in (2_ε) . Thus, the pair $(Y_\varepsilon, v_\varepsilon(y))$ is a solution of the free boundary problem $(1_\varepsilon) - (2_\varepsilon)$, as both $v_\varepsilon(y)$ and $K_\varepsilon(y) := (k(v_\varepsilon(y)) + \varepsilon) |v'_\varepsilon(y)|^{N-1} v'_\varepsilon(y)$ are absolutely continuous functions defined on $[Y_\varepsilon, +\infty)$ regardless of whether $z_\varepsilon(B-0)$ is finite or not.

Lemmas 8 and 9 tell us that $(Y_0, v_0(y))$ must be a solution of the reduced free boundary problem $(1_0) - (2_0)$; in particular, when $k(s) > 0$ a.e. on $[0, B]$, i.e., $k(s)$ has

no interval of degeneracy in $[0, B]$, $v_0(y)$ is absolutely continuous on $[Y_0, +\infty)$ and hence the free boundary problem $(1_\varepsilon) - (2_\varepsilon)$ is a regular perturbation problem. In the ensuing paragraphs we consider the case where $k(s)$ has at least one interval of degeneracy in $[0, B]$.

Let $\{[a_j, b_j]; j = 1, 2, \dots\}$ be the collection of all intervals of degeneracy possessed by the function $k(s)$ in $[0, B]$. Then $(0, B) \setminus D$, where D is the closure of $\bigcup_j [a_j, b_j]$, must be an open set in which $k(s) > 0$ a.e.

It follows from (10_0) that $z'_0(s)$ has the same interval of degeneracy in $[0, B]$ as $k(s)$. Therefore, $z_0(s)$ is strictly increasing in $(0, B) \setminus D$, whilst on each interval of degeneracy $[a_j, b_j], j = 1, 2, \dots, z_0(s) = y_j$ are constant, and

$$z_0(a_j - 0) = z_0(b_j + 0) = y_j.$$

Thus, $y = y_j, j = 1, 2, \dots$, is a jump point of $v_0(y)$, where

$$(13) \quad v_0(y_j - 0) = a_j, \quad v_0(y_j + 0) = b_j, \quad \text{and} \quad v_0(y_j) = [a_j, b_j].$$

In virtue of Proposition 3, from the fact that $z_0(s)$ is strictly increasing and (locally) absolutely continuous in $(0, B) \setminus D$, it follows that $v_0(y)$ is strictly increasing and absolutely continuous in each connected component of the open set $z_0((0, B) \setminus D) = (Y_0, z_0(B - 0)) \setminus \bigcup_j \{y_j\}$. Repeating the proof of Lemma 9, we draw the following conclusion: $v_0(y)$ satisfies the equation (1_0) a.e. in each connected component of the open set $(Y_0, +\infty) \setminus \bigcup_j \{y_j\}$.

Integrating the equality

$$w'_0(s) + z_0(s)g(s) + f(s) = 0 \quad \text{for all } s \in (0, B)$$

over each interval of degeneracy $[a_j, b_j], j = 1, 2, \dots$, we get

$$(14) \quad (k(v_0(y)) | v'_0(y) |^{N-1} v'_0(y) + yG(v_0(y)) + F(v_0(y))) \Big|_{y=y_j-0}^{y=y_j+0} = 0.$$

Here we have used the equalities (11_0) and (13) .

We can summarize the above results in the following statement.

THEOREM 2. *Under the hypotheses (I)–(V), for each fixed $\varepsilon \geq 0$ the free boundary problem $(1_\varepsilon) - (2_\varepsilon)$ has a solution $(Y_\varepsilon, v_\varepsilon(y))$, where $Y_\varepsilon < 0$ if $A > 0$, $Y_\varepsilon = 0$ if $A = 0$, $Y_\varepsilon > 0$ if $A < 0$, and $v_\varepsilon(y)$ is increasing and of bounded variation; when $\varepsilon > 0$ or $\varepsilon = 0$ and $k(s) > 0$ a.e. on $[0, B]$, $v_\varepsilon(y)$ is absolutely continuous on $[Y_\varepsilon, +\infty)$. As ε tends to zero, Y_ε converges to Y_0 and $v_\varepsilon(y)$ pointwise converges to $v_0(y)$. Moreover,*

$$v_0(y) = \sum_j (b_j - a_j) H(y - y_j) + \int_{y_0}^y v'_0(s) ds \quad \text{for all } y \geq Y_0,$$

where $v'_0(y)$, a derivative of $v_0(y)$, is nonnegative and integrable on $[Y_0, +\infty)$, $\{[a_j, b_j]; j = 1, 2, \dots\}$ is the collection of all intervals of degeneracy possessed by the function $k(s)$ in $[0, B]$, and $\{y_j; j = 1, 2, \dots\}$ is the set of all jump points of $v_0(y)$, whilst $H(y) = 0$ for $y < 0$, $H(y) = 1$ for $y > 0$, and $H(0) = [0, 1]$. In each connected component of the open set

$(y_0, z_0(B-0)) \setminus \bigcup_j \{y_j\}$, $v_0(y)$ is strictly increasing and absolutely continuous, and almost everywhere satisfies the equation (1_0) , while at each jump point $y=y_j$, $j=1, 2, \dots$, $v_0(y)$ satisfies the conditions (13) and (14). Here $z_0(s)$ is the second component of the unique solution $(w_0(s), z_0(s))$ to the two-point boundary value problem (3_0) – (4_0) . In addition, $z_\varepsilon(B-0)$ is finite and $v_\varepsilon(y)=B$ for all $y \geq z_\varepsilon(B-0)$, if and only if the condition (8) holds.

Theorem 2 shows that when $k(s)$ is positive a.e. on $[0, B]$, the free boundary problem (1_ε) – (2_ε) is a regular perturbation problem, whilst when $k(s)$ has intervals of degeneracy in $[0, B]$ it is a singular perturbation problem; in the second case, $v_0(y)$ possesses jump points and there exists a one-to-one correspondence between the collection of all intervals of degeneracy and the set of all jump points.

Finally, it should be pointed out that the jump condition (14) is exactly an extension of the Rankine–Hugoniot condition.

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