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A GENERALIZED IGUSA LOCAL ZETA FUNCTION AND LOCAL DENSITIES OF QUADRATIC FORMS

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Abstract. A certain formal power series attached to local densities of quadratic forms is defined. It is shown that this series can be realized as a coefficient of the Laurent expansion of a generalized Igusa local zeta function.

1. Introduction. Böcherer and Sato [BS] found a relation between the p-adic integrals defined by Denef [D1] and certain formal power series attached to local densities of quadratic forms. In this paper we consider another type of relation between the p-adic integrals defined by Igusa [I] and Deshommes [D2] and similar formal power series.

To be more precise, let A and B be non-degenerate symmetric matrices of degrees m and n, respectively, with entries in the ring $\boldsymbol{Z}_p$ of p-adic integers. Let $R=\boldsymbol{Z}_p[x_{ij}\ (1\le i\le m,\ 1\le j\le n),\ x_i\ (1\le i\le n)]$ be the polynomial ring over $\boldsymbol{Z}_p$. We simply write $X=(x_{ij})_{1\le i\le m, 1\le j\le n}$, and $x=(x_1,\dots,x_n)$. Let (g_{ij}) be the symmetric matrix of degree n with entries in R defined by

$$(g_{ij})=A[X]-B[\mathrm{diag}(x_1,\dots,x_n)]\,,$$

where for an (m, m)-matrix U and an (m, n)-matrix V we write $U[V]={}^tVUV$, and for square matrices $A_1,\dots,A_r$ we often simply write

$$\mathrm{diag}(A_1,\dots,A_r)=\begin{pmatrix} A_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & A_r\end{pmatrix}.$$

For a set S and non-negative integers n and s, put $\langle n\rangle=n(n+1)/2$, and

$$S^{\langle n\rangle+s}=\{((a_{ij})_{1\le i\le j\le n}, a_1,\dots,a_s);\ a_{ij}, a_k\in S\}\,.$$

We often write an element $((a_{ij})_{1\le i\le j\le n}, (a_i)_{1\le i\le s})$ of $S^{\langle n\rangle+s}$ as $((a_{ij}), (a_i))$ if no confusion arises. Further for an element (a_{ij}) of $S^{\langle n\rangle}$ with S a commutative ring, we often write $\sum_{1\le i\le j\le n}a_{ij}$, and $\prod_{1\le i\le j\le n}a_{ij}$ as $\sum a_{ij}$, and $\prod a_{ij}$, respectively. Let $\boldsymbol{C}$ be the field of complex numbers, and $\boldsymbol{Z}$ the ring of rational integers. Further let ord_p be the normalized additive valuation of the field $\boldsymbol{Q}_p$ of p-adic numbers, and put $|v|_p=p^{-\mathrm{ord}_p(v)}$ for $v\in\boldsymbol{Q}_p$.

We then define a function $Z(B, A; (s_{ij}), s_1,\dots,s_n)$ on the set

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$$C_{+}^{\langle n\rangle+n}=\{((s_{ij}),s_1,\dots,s_n)\in C^{\langle n\rangle+n};\ \mathrm{Re}\, s_{ij}>0,\ \mathrm{Re}\, s_i>0\}$$

by

$$Z(B,A;(s_{ij}),s_1,\dots,s_n)=\int_{Z_p^n\times M_{mn}(Z_p)}\prod|g_{ij}|_p^{s_{ij}}\prod_{i=1}^{n}|x_i|_p^{s_i}dxdX\,,$$

where $M_{mn}(Z_p)$ denotes the ring of (m,n)-matrices with entries in Z_p (for the precise definition, see Section 2). We call this function a *generalized Igusa local zeta function* attached to A and B. Put $z_{ij}=p^{-s_{ij}}$ $(1\le i\le j\le n)$, $z_i=p^{-s_i}$ $(1\le i\le n)$. We often write $Z=(z_{ij})$, $w=(z_i)$. Then $Z(B,A;(s_{ij}),s_1,\dots,s_n)$ can be regarded as a function of Z and w. Thus we write $\zeta(B,A;Z,w)=Z(B,A;(s_{ij}),s_1,\dots,s_n)$.

On the other hand, define a *local density* $\alpha_p(B,A)$ by

$$\alpha_p(B,A)=\lim_{e\to\infty}p^{(-mn+\langle n\rangle)e}\#\{\bar{X}\in M_{mn}(Z_p)/p^eM_{mn}(Z_p);\ A[X]\equiv B \bmod p^e\}\,.$$

Then we define a formal power series $P(B,A;x_1,\dots,x_n)$ by

$$P(B,A;x_1,\dots,x_n)=\sum_{r_1,\dots,r_n=0}^{\infty}\alpha_p(B[\mathrm{diag}(p^{r_1},\dots,p^{r_n})],A)x_1^{r_1}\cdots x_n^{r_n}\,.$$

See Section 2 for the relation between our formal power series and the one defined by Böcherer and Sato [BS].

The main purpose of the present paper is to show that the series $P(B,A;x_1,\dots,x_n)$ can be realized as a coefficient of the Laurent expansion of $\zeta(B,A;Z,px_1\prod(p^{-1}z_{ij})^{-4},\dots,px_n\prod(p^{-1}z_{ij})^{-4})$ with respect to (z_{ij}):

THEOREM 1.1. *Put*

$$N(B)=\{(k_{ij})\in Z^{\langle n\rangle};\ \min k_{ij}\ge 2\,\mathrm{ord}_p(2\det B)+1\}\,.$$

Then in the region

$$E=\{((z_{ij}),x_1,\dots,x_n)\in C^{\langle n\rangle+n};\ 0<|z_{ij}|<1,\ 0<|px_i\prod(p^{-1}z_{ij})^{-4}|<1\}\,,$$

we have

$$\begin{aligned}\zeta(B,A;Z,px_1\prod(p^{-1}z_{ij})^{-4},\dots,px_n\prod(p^{-1}z_{ij})^{-4})\\ =\sum_{(k_{ij})\in Z^{\langle n\rangle}\setminus N(B)}P((k_{ij});x_1,\dots,x_n)\prod(p^{-1}z_{ij})^{k_{ij}}\\ +(1-p^{-1})^{\langle n\rangle+n}\sum_{(k_{ij})\in N(B)}P(B,A;x_1,\dots,x_n)\prod(p^{-1}z_{ij})^{k_{ij}}\,,\end{aligned}$$

where $P((k_{ij});x_1,\dots,x_n)$ is a convergent power series of $x_1,\dots,x_n$ for each (k_{ij}) in $Z^{\langle n\rangle}\setminus N(B)$.

In Section 2, we treat a more general case (cf. Theorem 2.4). Using our arguments we can prove the rationality of $P(B, A; x_1, \ldots, x_n)$ and calculate its denominator explicitly. The details will be published in a subsequent paper [K].

2. Generalized Igusa local zeta functions and the proof of the main result. Let $R = \boldsymbol{Z}_p[[x_1, \ldots, x_s]]$ be a formal power series ring, and

$$R_c = \{f(x_1, \ldots, x_s) \in R;\ f(a_1, \ldots, a_s) \text{ converges for any } (a_1, \ldots, a_s) \in \boldsymbol{Z}_p^s\}.$$

For two sets S and Λ, we put

$$S^\Lambda = \prod_{\lambda \in \Lambda} S_\lambda$$

with $S_\lambda = S$. Let Λ be a finite set. Put

$$\boldsymbol{C}_+^\Lambda = \{(s_\lambda) \in \boldsymbol{C}^\Lambda;\ \operatorname{Re} s_\lambda > 0\}.$$

For a subset $\{f_\lambda\}_{\lambda \in \Lambda}$ of R_c, and $(s_\lambda) \in \boldsymbol{C}_+^\Lambda$ define $Z(\{f_\lambda\}; (s_\lambda))$ by

$$Z(\{f_\lambda\}; (s_\lambda)) = \int_{\boldsymbol{Z}_p^s} \prod_{\lambda \in \Lambda} |f_\lambda(x_1, \ldots, x_s)|_p^{s_\lambda} dx,$$

where dx is the Haar measure of $\boldsymbol{Q}_p^s$ so normalized that

$$\int_{\boldsymbol{Z}_p^s} dx = 1.$$

This function was studied by Igusa [I] when $\#\Lambda = 1$, and was generalized by Deshommes to the case where $\#\Lambda$ is arbitrary. So we call this function a *generalized Igusa local zeta function* attached to $\{f_\lambda\}$. The function $Z(\{f_\lambda\}; (s_\lambda))$ is holomorphic on $\boldsymbol{C}_+^\Lambda$. Put $z_\lambda = p^{-s_\lambda}$. Then $Z(\{f_\lambda\}; (s_\lambda))$ can be regarded as a function of (z_λ). So we put $\zeta(\{f_\lambda\}; (z_\lambda)) = Z(\{f_\lambda\}; (s_\lambda))$.

Let m and n be non-negative integers such that $m \geq n \geq 1$. Let A and B be non-degenerate symmetric matrices of degrees m and n, respectively, with entries in $\boldsymbol{Z}_p$. For a subset I of $I_{mn} = \{(i, j);\ 1 \leq i \leq m,\ 1 \leq j \leq n\}$, put

$$\alpha_p(B, A, I) = p^{\#I} \lim_{e \to \infty} p^{(-mn + \langle n \rangle)e} \#\{\overline{(x_{ij})} \in M_{mn}(\boldsymbol{Z}_p)/p^e M_{mn}(\boldsymbol{Z}_p);$$

$$A[(x_{ij})] \equiv B \bmod p^e, \text{ and } x_{ij} \equiv 0 \bmod p \text{ for any } (i, j) \in I\}.$$

We note that $\alpha_p(B, A, I) = \alpha_p(B, A)$ if $I = \varnothing$. Now let m, n, l, and $n_1, \ldots, n_s, n_{s+1}, \ldots, n_{s+t}$ be non-negative integers such that $m \geq n \geq 1$, and $m \geq l$, $n_1, \ldots, n_{s+t} \geq 1$, $n_1 + \cdots + n_s = n$, and $n_{s+1} + \cdots + n_{s+t} = l$. Let A and B be non-degenerate symmetric matrices of degrees m and n, respectively, with entries in $\boldsymbol{Z}_p$, and I be a subset of $I_{m,n}$. Define a formal power series $P(B, A, I; l; n_1, \ldots, n_{s+t}; x_1, \ldots, x_{s+t})$ by

$$P(B, A, I; l; n_1, \dots, n_{s+t}; x_1, \dots, x_{s+t})$$
$$= \sum_{r_1, \cdots, r_{s+t}=0}^{\infty} \alpha_p(B[\mathrm{diag}(p^{r_1}E_{n_1}, \dots, p^{r_s}E_{n_s})], A[\mathrm{diag}(E_{m-l}, p^{r_{s+1}}E_{n_{s+1}}, \dots, p^{r_{s+t}}E_{n_{s+t}})], I)$$
$$\times x_1^{r_1} \cdots x_{s+t}^{r_{s+t}},$$

where E_k denotes the unit matrix of degree k. We note that the formal power series $P(B, A, I; l; n_1, \dots, n_{s+t}; x_1, \dots, x_{s+t})$ coincides with $P(B, A; x_1, \dots, x_n)$ in Section 1 if $l=0$, $s=n$, and $I=\varnothing$. Further if $l=0$, $I=\varnothing$, and $B=\mathrm{diag}(b_1E_{n_1}, \dots, b_sE_{n_s})$, the formal power series

$$\sum_{e_1, \cdots, e_s=0,1} x_1^{e_1} \dots x_s^{e_s} P(\mathrm{diag}(p^{e_1}b_1E_{n_1}, \dots, p^{e_s}b_sE_{n_s}), A, I; 0; n_1, \dots, n_s; x_1^2, \dots, x_s^2)$$

coincides with the $P(B, A, x_1, \dots, x_s)$ defined in 1.2 of [BS]. In this section we show that $P(B, A, I; l; n_1, \dots, n_{s+t}; x_1, \dots, x_{s+t})$ can be realized as a coefficient of the Laurent expansion of a certain generalized Igusa local zeta function.

For this, we give some preliminaries. For a commutative ring R, let $\mathrm{Sym}(k; R)$ denote the set of symmetric matrices of degree k with entries in R. Let $U=(u_{ij}) \in \mathrm{Sym}(m, \boldsymbol{Z}_p)$, and $V=(v_{ij}) \in \mathrm{Sym}(n, \boldsymbol{Z}_p)$, and I be a subset of $\{(i, j); 1 \le i \le m, 1 \le j \le n\}$. For each $(e_{ij}) \in \boldsymbol{Z}^{\langle n \rangle}$, put $M((e_{ij})) = \max_{ij}(e_{ij})$, and

$$A((e_{ij}); V, U, I) = \{\overline{(x_{\alpha i})} \in M_{mn}(\boldsymbol{Z}_p)/p^{M((e_{ij}))}M_{mn}(\boldsymbol{Z}_p)\,;$$
$$\sum_{1 \le \alpha, \beta \le m} u_{\alpha\beta}x_{\alpha i}x_{\beta j} \equiv v_{ij} \bmod p^{e_{ij}} \text{ for any } i, j \text{ and } x_{\alpha i} \equiv 0 \bmod p \text{ for any } (\alpha, i) \in I\},$$

and

$$a((e_{ij}); V, U, I) = \# A((e_{ij}); V, U, I).$$

If $e_{ij}=e$ for all i, j, we simply write $A((e_{ij}); V, U, I) = A(e; V, U, I)$. The following lemma is well known:

LEMMA 2.1 (cf. Siegel [S, Hilfssatz 13]). *In addition to the above notation and assumptions, assume that U and V are non-degenerate, and put $e_0 = 2\,\mathrm{ord}_p(2 \det V) + 1$. Then for any integer $e \ge e_0$, we have*

$$a(e+1; V, U, I) = p^{mn - \langle n \rangle} a(e; V, U, I).$$

The following is essential to proving Theorem 2.4.

PROPOSITION 2.2. *Let the assumptions and notation be as above. Then for any $(e_{ij}) \in \boldsymbol{Z}^{\langle n \rangle}$ such that $\min_{ij} e_{ij} \ge e_0$, we have*

$$p^{-M((e_{ij}))mn + \sum e_{ij}} a((e_{ij}); V, U, I) = p^{e_0(-mn + \langle n \rangle)} a(e_0; V, U, I).$$

PROOF. Put $e = M((e_{ij}))$. For an element $(u_{ij})_{1 \le i \le j \le n}$ of $\boldsymbol{Z}^{\langle n \rangle}$, we define an element

$(u_{ij}^*)_{1\le i,j\le n}$ of Sym$(n, \boldsymbol{Z}_p)$ by $u_{ij}^* = u_{ij}$ or $= u_{ji}$ according as $i \le j$ or not. Then we have

$$a((e_{ij}); V, U, I) = \sum_{(c_{ij})} a(e; V + ((p^{e_{ij}} c_{ij})^*), U, I), \tag{2.1}$$

where (c_{ij}) runs through all representatives of the direct product $\prod \boldsymbol{Z}_p/p^{e-e_{ij}}\boldsymbol{Z}_p$ of $\{\boldsymbol{Z}_p/p^{e-e_{ij}}\boldsymbol{Z}_p\}_{1\le i\le j\le n}$. On the other hand, for any (c_{ij}), we have

$$a(e; V + ((p^{e_{ij}} c_{ij})^*), U, I) = p^{(e-e_0)(mn-\langle n\rangle)} a(e_0; V, U, I).$$

Thus the right-hand side of (2.1) is equal to

$$p^{(e-e_0)(mn-\langle n\rangle)} \# \prod \boldsymbol{Z}_p/p^{e-e_{ij}}\boldsymbol{Z}_p a(e_0; V, U, I) = p^{(e-e_0)(mn-\langle n\rangle)} \prod p^{e-e_{ij}} a(e_0; V, U, I)$$
$$= p^{emn-\sum e_{ij}} p^{e_0(-mn+\langle n\rangle)} a(e_0; V, U, I).$$

Now let A, B and I be as above. Define an element $(g_{ij}) = (g_{ij}(X))$ of Sym$(n, \boldsymbol{Z}_p[x_{\alpha k} (1\le\alpha\le m, 1\le k\le n)]$ by

$$(g_{ij}) = A[(p^{e(\alpha,k;I)} x_{\alpha k})] - B,$$

where $e(\alpha, k; I) = 1$ or $= 0$ according as $(\alpha, k) \in I$ or not. Further for each $(e_{ij}) \in \boldsymbol{Z}^{\langle n\rangle}$ put

$$E((e_{ij}; B, A, I) = \{X \in M_{mn}(\boldsymbol{Z}_p);\ g_{ij}(X) \in p^{e_{ij}}\boldsymbol{Z}_p \text{ for any } 1\le i\le j\le n\},$$

and let $\tilde{v}((e_{ij}); B, A, I)$ be the volume of $E((e_{ij}); B, A, I)$. Then we have:

PROPOSITION 2.3. *For each* $(e_{ij}) \in \boldsymbol{Z}^{\langle n\rangle}$ *such that* $\min_{ij} e_{ij} \ge e_0$, *we have*

$$\tilde{v}((e_{ij}), B, A, I) p^{\sum e_{ij}} = \alpha_p(B, A; I).$$

PROOF. We have

$$\tilde{v}((e_{ij}), B, A, I) = p^{-M((e_{ij}))mn} p^{\# I} a((e_{ij}); B, A, I).$$

Thus the assertion follows from Proposition 2.2.

COROLLARY. *Let* $v((e_{ij}), B, A, I)$ *be the volume of the set*

$$\{(X) \in M_{mn}((\boldsymbol{Z}_p);\ g_{ij}(X) \in p^{e_{ij}}\boldsymbol{Z}_p^* \textit{ for any } 1\le i\le j\le n\},$$

where $\boldsymbol{Z}_p^*$ *denotes the unit group of* $\boldsymbol{Z}_p$. *Then for each* $(e_{ij}) \in \boldsymbol{Z}^{\langle n\rangle}$ *such that* $\min_{ij} e_{ij} \ge e_0$, *we have*

$$v((e_{ij}); B, A, I) p^{\sum e_{ij}} = (1 - p^{-1})^{\langle n\rangle} \alpha_p(B, A; I).$$

PROOF. We simply write $v((e_{ij})) = v((e_{ij}); B, A, I)$. We arrange the quantities $\{e_{ij}\}$ indexed by the set $\{(i,j); 1\le i\le j\le n\}$ in the lexicographic order, and put $e_1 = e_{11}, e_2 = e_{12}, \dots, e_n = e_{1n}, \dots, e_{\langle n\rangle} = e_{nn}$, and $\sum e_i = \sum_{1\le i\le\langle n\rangle} e_i$. Then we have

$$v((e_i)) = \sum_{j=0}^{\langle n\rangle} (-1)^j \sum_{1\le i_1<\cdots<i_j\le\langle n\rangle} \tilde{v}(e_1, \dots, e_{i_1}+1, \dots, e_{i_j}+1, \dots, e_{\langle n\rangle}).$$

Then by Proposition 2.3, for any (e_i) such that $\min e_i \geq e_0$ we have

$$\tilde{v}(e_1, \ldots, e_{i_1}+1, \ldots, e_{i_j}+1, \ldots, e_{\langle n\rangle})p^{\Sigma e_i} = p^{-j}\alpha_p(B, A, I).$$

Thus the assertion holds.

Now let $x_{ij}(1 \leq i \leq m, 1 \leq j \leq n), x_1, \ldots, x_{s+t}$ be variables over $\boldsymbol{Z}_p$, and put $R = \boldsymbol{Z}_p[x_{ij}(1 \leq i \leq m, 1 \leq j \leq n), x_1, \ldots, x_{s+t}]$. Let A, B, I, l, and the others be as above. Define elements $y_{n+1}, \ldots, y_{m+n}$ of R by

$$\mathrm{diag}(y_{n+1}, \ldots, y_{n+m}) = \mathrm{diag}(E_{m-l}, x_{s+1}E_{n_{s+1}}, \ldots, x_{s+t}E_{n_{s+t}}).$$

Define an element $(h_{ij}(x, X))_{1 \leq i,j \leq n}$ of $\mathrm{Sym}(n, R)$ by

$$(h_{ij}(x, X)) = A[(y_{n+\alpha}p^{e(\alpha,k;I)}x_{\alpha k})_{1 \leq \alpha \leq m, 1 \leq k \leq n}] - B[\mathrm{diag}(x_1E_{n_1}, \ldots, x_sE_{n_s})].$$

Now let $\Lambda = \{(i, j); 1 \leq i \leq j \leq n\} \cup \{i; 1 \leq i \leq s+t\}$, and define a subset $\{h_\lambda\}_{\lambda \in \Lambda}$ of R indexed by Λ by

$$h_\lambda = \begin{cases} h_{ij} & \text{if } \lambda = (i, j) \\ x_i & \text{if } \lambda = i, \end{cases}$$

and $\zeta(B, A, I; l, n_1, \ldots, n_{s+t}; (z_{ij})_{1 \leq i \leq j \leq n}, z_1, \ldots, z_{s+t})$ by

$$\zeta(B, A, I; l; n_1, \ldots, n_{s+t}; (z_{ij})_{1 \leq i \leq j \leq n}, z_1, \ldots, z_{s+t}) = \zeta(\{h_\lambda\}; (z_\lambda)).$$

We write $\zeta(B, A, I; l; n_1, \ldots, n_{s+t}; (z_{ij})_{1 \leq i \leq j \leq n}, z_1, \ldots, z_{s+t}) = \zeta(B, A, I; l; n_1, \ldots, n_{s+t}; Z, w)$ as in Section 1. Then $\zeta(B, A, I; l; n_1, \ldots, n_{s+t}; Z, w)$ can be expressed as

$$\zeta(B, A, I; l; n_1, \ldots, n_{s+t}; Z, w) = \int_{\boldsymbol{Z}_p^{s+t} \times M_{mn}(\boldsymbol{Z}_p)} \prod |h_{ij}|_p^{s_{ij}} \prod_{k=1}^{s+t} |x_k|_p^{s_k} dxdX,$$

where dx (resp. dX) denotes the Haar measure of $\boldsymbol{Q}_p^{s+t}$ (resp. $M_{mn}(\boldsymbol{Q}_p)$) so normalized that

$$\int_{\boldsymbol{Z}_p^{s+t}} dx = 1 \quad \left(\text{resp. } \int_{M_{mn}(\boldsymbol{Z}_p)} dX = 1\right).$$

We note that $\zeta(B, A, I; l; n_1, \ldots, n_{s+t}; Z, w)$ coincides with $\zeta(B, A; Z, w)$ if $l = 0$, $s = n$, and $I = \varnothing$. Thus Theorem 1 is a special case of the following:

THEOREM 2.4. *In the region*

$$E = \{((z_{ij}), x_1, \ldots, x_{s+t}) \in \boldsymbol{C}^{\langle n\rangle+s+t}; 0 < |z_{ij}| < 1, 0 < |px_i \prod (p^{-1}z_{ij})^{-4n_i}| < 1\},$$

we have

$$\zeta(B, A, I; l; n_1, \ldots, n_{s+t}; Z, px_1 \prod (p^{-1}z_{ij})^{-4n_1}, \ldots, px_{s+t} \prod (p^{-1}z_{ij})^{-4n_{s+t}})$$

$$= \sum_{(k_{ij}) \in \boldsymbol{Z}^{\langle n\rangle} \setminus N(B)} P((k_{ij}); x_1, \ldots, x_{s+t}) \prod (p^{-1}z_{ij})^{k_{ij}}$$

$$+ (1-p^{-1})^{\langle n\rangle+s+t} \sum_{(k_{ij}) \in N(B)} P(B, A, I; l; n_1, \ldots, n_{s+t}; x_1, \ldots, x_{s+t}) \prod (p^{-1}z_{ij})^{k_{ij}}$$

where $P((k_{ij}); x_1, \dots, x_{s+t})$ *is a convergent power series of* $x_1, \dots, x_{s+t}$ *for each* (k_{ij}) in $\boldsymbol{Z}^{\langle n\rangle} \setminus N(B)$.

Proof. Put $\zeta(Z, w) = \zeta(B, A, I; l; n_1, \dots, n_{s+t}, Z, w)$. Then we have

$$\zeta(Z, w) = \int_{\boldsymbol{Z}_p^{s+t} \times M_{mn}(\boldsymbol{Z}_p)} \prod |h_{ij}(x, X)|_p^{s_{ij}} \prod_{k=1}^{s+t} |x_k|_p^{s_k} dx dX$$

$$= \sum_{r_1, \cdots, r_{s+t}=0}^{\infty} \int_{X'_0(r_1, \cdots, r_{s+t})} \prod |h_{ij}(x, X)|_p^{s_{ij}} \prod_{i=1}^{s+t} |x_i|_p^{s_i} dx dX,$$

where $X'_0(r_1, \dots, r_{s+t}) = \{(x, X) \in \boldsymbol{Z}_p^{s+t} \times M_{mn}(\boldsymbol{Z}_p); |x_i|_p = p^{-r_i}\}$. Thus we have

$$\zeta(Z, w)$$

$$= \sum_{r_1, \cdots, r_{s+t}=0}^{\infty} \prod_{i=1}^{s+t} (p^{-1} p^{-s_i})^{r_i} \int_{\boldsymbol{Z}_p^{*s+1} \times M_{mn}(\boldsymbol{Z}_p)} \prod |h_{ij}(p^{r_1} x_1, \dots, p^{r_{s+t}} x_{s+t}, X)|_p^{s_{ij}} dx dX.$$

Define elements $y_1, \dots, y_{n+m}$ of R by

$$\mathrm{diag}(y_1, \dots, y_n) = \mathrm{diag}(x_1 E_{n_1}, \dots, x_s E_{n_s}),$$

and

$$\mathrm{diag}(y_{n+1}, \dots, y_{n+m}) = \mathrm{diag}(E_{m-l}, x_{s+1} E_{n_{s+1}}, \dots, x_{s+t} E_{n_{s+t}}).$$

We change the variables as follows:

$$x_{\alpha j} \longmapsto x_{\alpha j} y_j y_{n+\alpha}^{-1} \ (1 \le \alpha \le m,\ 1 \le j \le n), \quad x_j \longmapsto x_j \ (1 \le j \le s+t).$$

Then we have

$$\zeta(Z, w) = \sum_{r_1, \cdots, r_{s+t}=0}^{\infty} \prod_{i=1}^{s+t} (p^{-1} z_i)^{r_i} \int_{\boldsymbol{Z}_p^{*s+t} \times M_{mn}(\boldsymbol{Z}_p)} \prod |h_{ij}(p^{r_1}, \dots, p^{r_{s+t}}, X)|_p^{s_{ij}} dx dX$$

$$= (1 - p^{-1})^{s+t} \sum_{r_1, \cdots, r_{s+t}=0}^{\infty} \prod_{i=1}^{s+t} (p^{-1} z_i)^{r_i} \int_{M_{mn}(\boldsymbol{Z}_p)} \prod |h_{ij}(p^{r_1}, \dots, p^{r_{s+t}}, X)|_p^{s_{ij}} dX.$$

Now for non-negative integers $r_1, \dots, r_{s+t}, e_{ij} (1 \le i \le j \le n)$, put

$$v(r_1, \dots, r_{s+t}, (e_{ij})) = v(\{X \in M_{m,n}(\boldsymbol{Z}_p); h_{ij}(p^{r_1}, \dots, p^{r_{r+s}}, X) \in p^{e_{ij}} \boldsymbol{Z}_p^* \text{ for any } 1 \le i \le j \le n\}).$$

Then we have

$$\zeta(Z, w) = \sum_{r_1, \cdots, r_{s+t}, e_{ij}=0}^{\infty} v(r_1, \dots, r_{s+t}, (e_{ij})) \prod z_{ij}^{e_{ij}} \prod_{i=1}^{s+t} (p^{-1} z_i)^{r_i}.$$

We write $(r) = (r_1, \dots, r_{s+t})$ and $s((r)) = 4(n_1 r_1 + \cdots + n_{s+t} r_{s+t})$. Define

$$\tilde{\zeta}(Z, x_1, \dots, x_{s+t}) = \zeta(Z, p x_1 \prod (p^{-1} z_{ij})^{-4n_1}, \dots, p x_{s+t} \prod (p^{-1} z_{ij})^{-4n_{s+t}}).$$

Then we have

$$\tilde{\zeta}(Z, x_1, \ldots, x_{s+t}) = \sum_{r_1, \cdots, r_{s+t}=0}^{\infty} \sum_{(k_{ij})} v(r_1, \ldots, r_{s+t}, (s((r)) + k_{ij}))p^{\Sigma(s((r))+k_{ij})}$$

$$\times \prod (p^{-1}z_{ij})^{k_{ij}} \prod_{i=1}^{s+t} x_i^{r_i},$$

where (k_{ij}) runs through all elements of $\boldsymbol{Z}^{\langle n \rangle}$ such that $k_{ij} \geq -s((r))$ for any $1 \leq i \leq j \leq n$. For each (k_{ij}), define a formal power series $P((k_{ij}); x_1, \ldots, x_{s+t})$ by

$$P((k_{ij}); x_1, \ldots, x_{s+t}) = \sum_{r_1, \cdots, r_{s+t}} v(r_1, \ldots, r_{s+t}, (s((r)) + k_{ij}))p^{\Sigma(s((r))+k_{ij})} x_1^{r_1} \cdots x_{s+t}^{r_{s+t}},$$

where $r_1, \ldots, r_{s+t}$ run through all non-negative integers such that $r_1, \ldots, r_{s+t} \geq 0$, and $s((r)) \geq -k_{ij}$ for any $1 \leq i \leq j \leq n$. Since the right-hand side of $\tilde{\zeta}(Z, x_1, \ldots, x_{s+t})$ is absolutely convergent in the region $0 < |z_{ij}| < 1$, $0 < |px_i \prod (p^{-1}z_{ij})^{-4n_i}| < 1$, the formal power series $P((k_{ij}); x_1, \ldots, x_{s+t})$ is a convergent power series, and we have

$$\tilde{\zeta}(Z, x_1, \ldots, x_{s+t}) = (1-p^{-1})^{s+t} \sum_{(k_{ij})} P((k_{ij}); x_1, \ldots, x_{s+t}) \prod (p^{-1}z_{ij})^{k_{ij}}.$$

Further by Corollary to Proposition 2.3, for any $(k_{ij}) \in N(B)$ we have

$$P((k_{ij}); x_1, \ldots, x_{s+t}) = (1-p^{-1})^{\langle n \rangle} P(B, A, I; l; n_1, \ldots, n_{s+t}; x_1, \ldots, x_{s+t}).$$

Thus the assertion holds.

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