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ISOMETRIC DEFORMATIONS OF HYPERSURFACES IN A EUCLIDEAN SPACE PRESERVING MEAN CURVATURE

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Abstract. Under some conditions we classify hypersurfaces in a Euclidean space which admit isometric deformations preserving mean curvature.

1. Introduction. The isometric deformations of surfaces in a 3-dimensional space form preserving mean curvature (which are called H-deformations) have been studied by a number of mathematicians. Bonnet [2] showed over a century ago that all the surfaces with constant mean curvature in a Euclidean 3-space except planes or spheres are locally H-deformable. Referring to this problem in the case of surfaces with non-constant mean curvature, the work done by Cartan [3] is authoritative and Chern [4] gave an interesting characterization for their existence.

Recently, Colares and Kenmotsu [5] classified H-deformable surfaces with constant Gaussian curvature in a Euclidean 3-space. Umehara [9] proved that a compact surface in a 3-dimensional space form is locally H-deformable if and only if it has constant mean curvature.

In this paper, we study such a deformation of hypersurfaces in $\boldsymbol{R}^n$ (as a direct generalization of that of a surface in $\boldsymbol{R}^3$).

DEFINITION. Let $f: M^n \looparrowright \boldsymbol{R}^{n+1}$ be an isometric immersion as a hypersurface of an n-dimensional Riemannian manifold and H the mean curvature of f.

An *H-deformation* of the immersion f is a continuous mapping $F: (-\varepsilon, \varepsilon) \times M^n \to \boldsymbol{R}^{n+1}$ ($\varepsilon > 0$) such that

(1.1) $f_t := F(t, \cdot)$ for any fixed $t \in (-\varepsilon, \varepsilon)$ is an isometric immersion whose mean curvature is equal to H,

(1.2) $f_0 = f$.

An H-deformation is said to be *trivial* if for each $t \in (-\varepsilon, \varepsilon)$, there exists a motion T_t of $\boldsymbol{R}^{n+1}$ such that $f_t = T_t \circ f$. f is said to be *H-deformable* if there exists a non-trivial H-deformation. f is said to be *locally H-deformable* if for each point of M^n there exists a neighborhood U such that $f|_U$ is H-deformable.

First we mention the following well-known theorem:

THEOREM A (Beez [1], Killing [7]). *A hypersurface M is rigid in* $\boldsymbol{R}^{n+1}$ *if the type*

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number is greater than 2 *at every point* (*where the type number is the rank of the shape operator as a linear transformation of the tangent spaces of* M).

Consequently, such a hypersurface as that in Theorem A cannot admit a non-trivial H-deformation. From such a point of view, we deal only with hypersurfaces with type number ≤ 2.

Our main result is the following.

THEOREM. *Let* $f: M^n \hookrightarrow \boldsymbol{R}^{n+1}$ *be an isometric immersion with type number* 2 *which has distinct non-zero principal curvatures. If* f *is locally H-deformable, then* f *is one of the following*:

(a) *a minimal immersion*

(b) *an open piece of a cylinder* $M^2 \times \boldsymbol{R}^{n-2}$, *where* M^2 *is a locally H-deformable surface in* $\boldsymbol{R}^3$

(c) *an open piece of a cylinder* $CN \times \boldsymbol{R}^{n-3}$, *where* N *is a locally H-deformable surface in* $\boldsymbol{S}^3$ *and* CN *is a cone over* N *in* $\boldsymbol{R}^4$

(d) *a hypersurface mixed with that of* (a), (b) *and* (c).

Moreover, we assume the real analyticity of f *and* M^n. *Then* f *is H-deformable if and only if* f *is* (a) *or* (b) *or* (c).

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2. Preliminaries. We use the *moving frame method.*

Assume that M^n is immersed as a hypersurface in $\boldsymbol{R}^{n+1}$ with constant type number d $(0 \leq d \leq n)$ and that its non-zero principal curvatures are distinct. Let A denote the shape operator. From now on we shall use the following convention on the ranges of indices:

$$1 \leq A, B, C, \cdots \leq n$$

$$1 \leq i, j, k, \cdots \leq d$$

$$d+1 \leq p, q, r, \cdots \leq n .$$

Choose a local orthonormal frame field $\{e_1, \ldots, e_d, e_{d+1}, \ldots, e_n\}$ of M in such a way that $e_1, \ldots, e_n$ are principal vectors and their principal curvatures are $k_1, \ldots, k_d, 0, \ldots, 0$ respectively $(k_1 k_2 \cdots k_d \neq 0, k_i \neq k_j$ for $i \neq j)$. Let $\{\theta^1, \ldots, \theta^n\}$ be the dual frame field of $\{e_A\}$. The Levi-Civita connection of M is denoted by ∇ and the connection form of ∇ is denoted by

$$\omega = (\omega_B^A) \quad \text{with respect to} \quad \{e_A\} .$$

Then the shape operator A can be written as

$$A=\begin{pmatrix} k_1 & & & & & \\ & \ddots & & & 0 & \\ & & k_d & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix}. \tag{2.1}$$

Let $\Omega=(\Omega^A_B)$ be the curvature form of M. Then the Gauss equation implies

$$\Omega^i_j=k_ik_j\theta^i\wedge\theta^j\,,\quad \Omega^i_q=0\,,\quad \Omega^p_q=0 \tag{2.2}$$

and the Codazzi equation is given by

$$(\nabla_{e_A}A)e_B=(\nabla_{e_B}A)e_A\,. \tag{2.3}$$

LEMMA 2.1. *Under the above assumption, the Codazzi equation* (2.3) *is equivalent to the following equations.*

$$\begin{aligned} &e_ik_j=(k_i-k_j)\omega^j_i(e_j) \qquad \text{for}\quad i\neq j\\ &(k_j-k_l)\omega^l_j(e_i)=(k_i-k_l)\omega^l_i(e_j) \qquad \text{for}\quad l\neq i,j\\ &k_i\omega^p_i(e_j)=k_j\omega^p_j(e_i)\\ &e_pk_i=k_i\omega^p_i(e_i)\\ &k_j\omega^p_j(e_i)=(k_i-k_j)\omega^j_i(e_p) \qquad \text{for}\quad i\neq j\\ &\omega^q_i(e_p)=0\,. \end{aligned} \tag{2.4}$$

From the last three equations of (2.4), we have

$$\omega^i_p=-\frac{e_pk_i}{k_i}\,\theta^i+\sum_{j\neq i}\frac{k_i-k_j}{k_i}\,\omega^i_j(e_p)\theta^j\,. \tag{2.5}$$

Furthermore, it follows from (2.5) and the *structure equation* $d\theta^A=-\sum\omega^A_B\wedge\theta^B$ that

$$d\theta^i=-\sum_j\omega^i_j\wedge\theta^j+\sum_p\frac{e_pk_i}{k_i}\,\theta^i\wedge\theta^p-\sum_{p,j\neq i}\frac{k_i-k_j}{k_i}\,\omega^i_j(e_p)\theta^j\wedge\theta^p \tag{2.6}$$

$$d\theta^p=\sum_{i<j}\frac{(k_i-k_j)^2}{k_ik_j}\,\omega^i_j(e_p)\theta^i\wedge\theta^j-\sum_q\omega^p_q\wedge\theta^q\,. \tag{2.7}$$

Thus we have proved the following lemma.

LEMMA 2.2. (a) *The distribution* V^0 *defined by* $\theta^1=\cdots=\theta^d=0$ *is completely integrable.*

(b) *The distribution* V^1 *defined by* $\theta^{d+1}=\cdots=\theta^n=0$ *is completely integrable if and only if* $(k_i-k_j)\omega^i_j(e_p)=0$ *holds for all* i, j, p.

REMARK. It is easy to see that the integral manifolds of the distribution V^0 are totally geodesic submanifolds in $\boldsymbol{R}^{n+1}$. So we can choose $e_{d+1}, \dots, e_n$ in such a way that $\nabla_{e_p} e_q = 0$, i.e., $\omega_p^A(e_q) = 0$ holds. From now on we assume that e_p's are chosen in that way.

3. *H*-deformability of hypersurfaces of constant type number 2. We shall apply formulas obtained in the previous section to the case $d=2$.

Let $f: M^n \subsetneq \boldsymbol{R}^{n+1}$ be an immersed hypersurface with type number 2 and H the mean curvature. Assume that the two non-zero principal curvatures are distinct at every point.

Suppose that f is locally H-deformable. Then there exists a simply-connected neighborhood U for an arbitrary point x_0 in M^n and an isometric immersion $f': U \subsetneq \boldsymbol{R}^{n+1}$ with mean curvature H such that $f' \neq f$ on U. In other words, there exists a symmetric (1, 1)-tensor field A' on U satisfying the Gauss equation, the Codazzi equation and $\operatorname{tr} A' = nH$, because of the fundamental theorem for hypersurfaces. The type number of f' is also equal to 2. This follows from the fact that the null space of the shape operator does not depend on the immersion in this case, because of the following theorem.

THEOREM B ([8, Theorem 6.1]). *For an isometric immersion $M^n \subsetneq \boldsymbol{R}^{n+1}$, if the type number ≥ 2 at a point x, then* $\ker A_x = \{X \in T_xM\,; R(X, Y) = 0$ *for all* $Y \in T_xM\}$, *where R denotes the curvature tensor of M.*

Consider two orthonormal frame fields $\{e_A\}$ and $\{e'_A\}$ on U which consist of principal vectors of f and f', respectively, and let $k_1, k_2, 0, \dots, 0$ (resp. $k'_1, k'_2, 0, \dots, 0$) be principal curvatures with respect to f (resp. f'). From the Gauss equations and $\operatorname{tr} A = \operatorname{tr} A'$, we have $k_1 k_2 = k'_1 k'_2$, $k_1 + k_2 = k'_1 + k'_2$, so we may assume $k_1 = k'_1$, $k_2 = k'_2$. Then from the above consideration,

$$\ker A_x = \ker A'_x \qquad \text{at every} \quad x \in U\,,$$

i.e.,

$$\{e_1|_x, e_2|_x\} = \{e'_1|_x, e'_2|_x\} \qquad \text{at every} \quad x \in U\,,$$

$$\{e_3|_x, \dots, e_n|_x\} = \{e'_3|_x, \dots, e'_n|_x\} \qquad \text{at every} \quad x \in U\,,$$

where $\{\cdots\}$ denotes the subspace of T_xM spanned by $\cdots$. Thus the frame $\{e_A\}$ corresponds to $\{e'_A\}$ by an $SO(2) \times SO(n-2)$-valued function. Therefore A' can be written as

$$A'=\left(\begin{array}{c|c} P^{-1}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}P & \\ \hline & \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} \end{array}\right) \quad \text{with respect to } \{e_A\}$$

for some $SO(2)$-valued function P. Putting

$$P=\begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix}, \qquad \xi\in C^\infty(U),$$

we have

$$A'=\frac{1}{2}\left(\begin{array}{cc|c} k_1(1+\cos\tau)+k_2(1-\cos\tau) & (k_1-k_2)\sin\tau & \\ (k_1-k_2)\sin\tau & k_1(1-\cos\tau)+k_2(1+\cos\tau) & \\ \hline & & \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} \end{array}\right)$$

where $\tau=2\xi$.

LEMMA 3.1. *The Codazzi equation* $(\nabla_{e_A}A')e_B=(\nabla_{e_B}A')e_A$ *can be written as follows*:

(3.1) $(e_1(k_1+k_2))\sin\tau-(e_2(k_1+k_2))(1-\cos\tau)+(k_1-k_2)\{(e_1\tau)\cos\tau+(e_2\tau)\sin\tau\}=0$

(3.2) $(e_1(k_1+k_2))(1-\cos\tau)+(e_2(k_1+k_2))\sin\tau-(k_1-k_2)\{(e_1\tau)\sin\tau-(e_2\tau)\cos\tau\}=0$

(3.3) $$(1-\cos\tau)(k_1+k_2)(k_1-k_2)\omega_2^1(e_p)-\sin\tau\{(e_pk_2)k_1-(e_pk_1)k_2\}=0$$

(3.4) $$\frac{(1-\cos\tau)}{k_1}\{(e_pk_2)k_1-(e_pk_1)k_2\}+\frac{\sin\tau}{k_2}(k_1+k_2)(k_1-k_2)\omega_2^1(e_p)-(k_1-k_2)(e_p\tau)\sin\tau=0$$

(3.5) $$\frac{\sin\tau}{k_1}\{(e_pk_2)k_1-(e_pk_1)k_2\}-\frac{(1-\cos\tau)}{k_2}(k_1+k_2)(k_1-k_2)\omega_2^1(e_p)-(k_1-k_2)(e_p\tau)\cos\tau=0$$

(3.6) $$\frac{(1-\cos\tau)}{k_2}\{(e_pk_2)k_1-(e_pk_1)k_2\}+\frac{\sin\tau}{k_1}(k_1+k_2)(k_1-k_2)\omega_2^1(e_p)-(k_1-k_2)(e_p\tau)\sin\tau=0$$

(3.7) $$\frac{\sin\tau}{k_2}\{(e_pk_2)k_1-(e_pk_1)k_2\}-\frac{(1-\cos\tau)}{k_1}(k_1+k_2)(k_1-k_2)\omega_2^1(e_p)-(k_1-k_2)(e_p\tau)\cos\tau=0\,.$$

Furthermore, the equations (3.4)–(3.7) *are equivalent to*

$$(e_pk_2)k_1-(e_pk_1)k_2=0 \tag{3.8}$$

$$(k_1+k_2)\omega_2^1(e_p)=0 \tag{3.9}$$

$$e_p\tau=0\,. \tag{3.10}$$

PROOF. Apply the formulas (2.4), (2.5) to the case $d=2$. Then (3.1), (3.2), (3.3) are obtained from $(\nabla_{e_1}A')e_2=(\nabla_{e_2}A')e_1$ and (3.4), (3.5), (3.6), (3.7) from $(\nabla_{e_i}A')e_p=(\nabla_{e_p}A')e_i$ $(i=1, 2)$. $(\nabla_{e_p}A')e_q=(\nabla_{e_q}A')e_p$ holds automatically. The latter part is obvious. ∎

Let $(*)$ stand for the equations (3.1)–(3.3) and (3.8)–(3.10). Therefore we have:

PROPOSITION 3.2. *If $f|_U$ is H-deformable, then there exists $\tau\in C^\infty(U)$ satisfying the equations* $(*)$.

Conversely, if there exists a one-parameter family of functions τ_t $(\tau_0\equiv 0)$ satisfying $(*)$, *then $f|_U$ is H-deformable.*

REMARK. It is remarked that a necessary and sufficient condition for the H-deformability of a surface in a 3-dimensional space form is the existence of a one-parameter family of functions τ_t satisfying (3.1) and (3.2).

4. Proof of Theorem. We investigate the hypersurfaces satisfying $(*)$ in this section. From (3.9), we consider the following three cases:

Case 1. x_0 has a neighborhood such that $k_1+k_2\equiv 0$ holds, i.e., a piece of minimal hypersurface.

Case 2. x_0 has a neighborhood such that $\omega_2^1(e_p)\equiv 0$ holds.

Case 3. Neither $k_1+k_2\equiv 0$ nor $\omega_2^1(e_p)\equiv 0$ holds in any neighborhood of x_0, but $(k_1+k_2)\omega_2^1(e_p)\equiv 0$ holds in some neighborhood.

Case 1. Assume that $k_1+k_2=0$. Then (3.8) holds, and (3.1) and (3.2) hold if and only if $e_1\tau=e_2\tau\equiv 0$. Thus a necessary and sufficient condition for $(*)$ is $\tau=$constant. If we put $\tau_t=t$ for example, then τ_t defines an H-deformation. Therefore minimal hypersurfaces with type number 2 are locally H-deformable.

Case 2. It is remarked that the distribution V^1 generated by e_1 and e_2 is integrable in this case. It follows from (2.4) and (2.5) that ω_2^1 is generated by θ^1 and θ^2 and

$$\omega_p^i=-\frac{e_pk_i}{k_i}\,\theta^i \qquad (i=1, 2)\,. \tag{4.1}$$

On the other hand, from (3.8) we may put

$$\varphi_p:=\frac{e_pk_1}{k_1}=\frac{e_pk_2}{k_2}$$

and

$$\omega_p^i = -\varphi_p \theta^i . \tag{4.2}$$

LEMMA 4.1. *In Case 2, $f(U)$ is of the form $M^2 \times \boldsymbol{R}^{n-2}$ or $M^3 \times \boldsymbol{R}^{n-3}$, or it is mixed with $M^2 \times \boldsymbol{R}^{n-2}$ and $M^3 \times \boldsymbol{R}^{n-3}$, where M^2 and M^3 are immersed in $\boldsymbol{R}^3$ and $\boldsymbol{R}^4$, respectively.*

PROOF. Assume that $\varphi_p \equiv 0$ for all p. It is easy to see that $\nabla_{e_i} e_j \in V^1$ so the integral manifold of the distribution $\{e_1, e_2\}$ is totally geodesic in M^n. Let M^2 be an integral manifold of the distribution $\{e_1, e_2\}$. On the other hand, if the Levi-Civita connection of the ambient space $\boldsymbol{R}^{n+1}$ is denoted by D, then $D_{e_i} e_p \in V^0$ holds so that V^0 is parallel in $\boldsymbol{R}^{n+1}$. Therefore the integral manifolds of V^0 are prallel Euclidean subspaces, and

$$T_x M^2 = V_x^1 \perp V_x^0 \cong \boldsymbol{R}^{n-2}$$

at every point $x \in M^n$. So M^2 is contained in $\boldsymbol{R}^3$ and it is obvious that $f(U) = M^2 \times \boldsymbol{R}^{n-2}$.

Otherwise, x_0 is not an interior point of $B := \{x \in M^n \mid \varphi_p(x) = 0 \text{ for all } p\}$. That is,

$$x_0 \in B^c \quad \text{or} \quad x_0 \in \partial B$$

where B^c and ∂B denote the complementary set and the boundary set of B, respectively.

Whenever x_0 is a point of B^c, we can take a neighborhood where $\varphi_p \neq 0$ holds. We also write it as U. We can see that $\dim \Lambda = 1$ holds at each point of U if we set

$$\Lambda = \{\text{the } V^0\text{-component of } \nabla_X Y;\ X, Y \in V^1\} .$$

We choose a new frame field $\{e_3, \dots, e_n\}$ such that $e_3 \in \Lambda$. (The observations so far are applicable with respect to this frame.) Then $\varphi_3 \neq 0$, $\varphi_4 = \cdots = \varphi_n = 0$ holds. Putting $\varphi_3 =: \varphi$, we see that the connection form ω can be written as

$$\begin{pmatrix} 0 & \omega_2^1 & -\varphi\theta^1 & 0 & \cdots & 0 \\ -\omega_2^1 & 0 & -\varphi\theta^2 & 0 & \cdots & 0 \\ \varphi\theta^1 & \varphi\theta^2 & 0 & \omega_4^3 & \cdots & \omega_n^3 \\ 0 & 0 & -\omega_4^3 & & & \\ \vdots & \vdots & \vdots & & \omega_p^q & \\ 0 & 0 & -\omega_n^3 & & & \end{pmatrix} . \tag{4.3}$$

We compute $R(e_1, e_2)e_p$ for $p \geq 4$ using (4.3) directly:

$$R(e_1, e_2)e_p = \nabla_{e_1}\nabla_{e_2} e_p - \nabla_{e_2}\nabla_{e_1} e_p - \nabla_{[e_1, e_2]} e_p = -\varphi\omega_p^3(e_2)e_1 + \varphi\omega_p^3(e_1)e_2 + \cdots .$$

However, $R(e_1, e_2)e_p = 0$ holds by the Gauss equation. Thus $\omega_p^3 \equiv 0$ for $\omega_p^3(e_1) = \omega_p^3(e_2) = 0$.

Then it is easy to see that the distribution spanned by $\{e_1, e_2, e_3\}$ is integrable and $f(U)$ is of the form $M^3 \times \boldsymbol{R}^{n-3}$ by the argument analogous to that in the case $\varphi_p \equiv 0$.

Assume that x_0 is a boundary point of B. If x_0 has a neighborhood U such that $U \cap B$ is nowhere dense in U, then $\varphi \neq 0$ holds and $f(U) = M^3 \times \boldsymbol{R}^{n-3}$. Otherwise, x_0 is a boundary point of an open portion of $M^2 \times \boldsymbol{R}^{n-2}$ and an open portion of $M^3 \times \boldsymbol{R}^{n-3}$. ■

We examine the case $\varphi \neq 0$ in the above lemma. We only consider the case of 3-dimensional hypersurfaces M^3. We give necessary formulas. The structure equations are

$$\begin{aligned} d\theta^1 &= -\omega_2^1 \wedge \theta^2 + \varphi \theta^1 \wedge \theta^3 \\ d\theta^2 &= -\omega_1^2 \wedge \theta^1 + \varphi \theta^2 \wedge \theta^3 \\ d\theta^3 &= 0 . \end{aligned} \tag{4.4}$$

From the Gauss equations, we have

$$\begin{aligned} d\omega_2^1 &= (k_1 k_2 + (\varphi)^2)\theta^1 \wedge \theta^2 \\ 0 &= (-d\varphi + (\varphi)^2 \theta^3) \wedge \theta^1 \\ 0 &= (-d\varphi + (\varphi)^2 \theta^3) \wedge \theta^2 . \end{aligned} \tag{4.5}$$

It is immediately seen that

$$d\varphi = (\varphi)^2 \theta^3 , \tag{4.6}$$

i.e.,

$$e_1 \varphi = e_2 \varphi = 0 , \qquad e_3 \varphi = (\varphi)^2 .$$

Let l be the integral curve of e_3 through a point x and t the arc length parameter of l initiated at x, i.e., $e_3 = \partial/\partial t$, $l(0) = x$. The integral manifold of the distribution $\{e_1, e_2\}$ through $l(c)$ is denoted by M_c.

PROPOSITION 4.2.
(a) *φ is constant on each leaf M_c.*
(b) *M_c is contained in an embedded 3-sphere with curvature $|\varphi(c)|$ in $\boldsymbol{R}^4$.*
(c) *The spheres defined in (b) are concentric for all t.*
(d) *M^3 is a cone.*

PROOF. (a) is obvious from (4.6). Let y_c be the position vector of M_c in $\boldsymbol{R}^4$ and D the covariant derivative in $\boldsymbol{R}^4$. Then

$$D_{e_i}\left(y_c + \frac{1}{\varphi(c)} e_3\right) = e_i + \frac{1}{\varphi(c)} D_{e_i} e_3 = e_i + \frac{1}{\varphi(c)}(-\varphi(c) e_i) = 0 .$$

This means that the vector $y_c + \varphi(c)^{-1} e_3$ is parallel along M_c in $\boldsymbol{R}^4$. Let w be the vector in $\boldsymbol{R}^4$ which is obtained by parallel translation of $y_c + \varphi(c)^{-1} e_3$ to the origin in $\boldsymbol{R}^4$. Then

$$\langle y_c - w, y_c - w\rangle = \frac{1}{\varphi(c)^2}$$

holds. This proves (b). For (c) it is sufficient to show that $y_t + \varphi(t)^{-1}e_3$ is parallel along l in $\boldsymbol{R}^4$:

$$D_{e_3}\left(y_t + \frac{1}{\varphi(t)}e_3\right) = e_3 + e_3\left(\frac{1}{\varphi}\right)e_3 + \frac{1}{\varphi}D_{e_3}e_3 = e_3 - \frac{e_3\varphi}{\varphi^2}e_3 = 0\,.$$

If φ is restricted to l, then φ satisfies an ordinary differential equation $d\varphi/dt = \varphi^2$ from (4.6). The solution of this equation is

$$\varphi(t) = -\frac{1}{t - \varphi(0)^{-1}}\,.$$

Put $s = t - \varphi(0)^{-1}$. Then $\varphi(s) = -s^{-1}$. We may assume that the centers of the spheres in (c) are the origin in $\boldsymbol{R}^4$, that is, $y_s + \varphi(s)^{-1}e_3 = 0$. Thus $y_s = se_3$. On the other hand, it can be seen that e_3 is parallel along l. So $y_s = sy_1$. Therefore y_s's make up a cone over $y_1 \subsetneq \boldsymbol{S}^3(1)$ in $\boldsymbol{R}^4$. ■

Case 3. In this case, even if we take any neighborhood for the x_0, it contains an open portion where $k_1 + k_2 = 0$ holds and an open portion where $\omega^1_2(e_p) = 0$ holds. Thus the neighborhood of x_0 is of the form mixed with that of Case 1 and that of Case 2.

PROOF OF THEOREM. From the remaining condition of (∗), we can conclude that the surfaces appearing in Case 2 are H-deformable surfaces in $\boldsymbol{R}^3$ and $\boldsymbol{S}^3$, respectively. ■

REMARKS. 1. It can be seen that an H-deformation is a principal curvature preserving deformation.

2. Concerning minimal hypersurfaces with type number 2, Dajczer and Gromoll [6] proved that one can construct them by making use of a minimal surface S in $\boldsymbol{S}^n$ and an eigenfunction of the Laplace-Bertrami operator with eigenvalue 2 on S. In particular, if we take a fully immersed one as a minimal surface in $\boldsymbol{S}^n$ ($n \geq 4$), then a minimal hypersurface with type number 2 which is not of the form $M^2 \times \boldsymbol{R}^{n-2}$ or $CN \times \boldsymbol{R}^{n-3}$ is obtained.

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