Tôhoku Math. J.
44 (1992), 425–431

ON NORMAL AND CONORMAL MAPS FOR AFFINE HYPERSURFACES

KATSUMI NOMIZU AND BARBARA OPOZDA*

(Received September 11, 1991, revised January 20, 1992)

Abstract. We prove two main results in affine differential geometry that characterize ellipsoids among the ovaloids. The first theorem states that an ovaloid in the 3-dimensional affine space is an ellipsoid if and only if the Laplacian of the normal map is proportional to the normal map. The second theorem says that a hyperovaloid in an affine space of any dimension is a hyperellipsoid if and only if the conormal image (or the normal image) is a hyperellipsoid with center at the origin.

Let $f: M^n \to R^{n+1}$ be a nondegenerate hypersurface with affine normal ξ in the affine space R^{n+1}. We then have the normal map $\phi: M^n \to R^{n+1}$ and the conormal immersion $v: M^n \to R_{n+1}$, where R_{n+1} is the coaffine space of R^{n+1} (for the terminology see [N-P]). Our main results are the following.

THEOREM 1. *An ovaloid $f: M^2 \to R^3$ is an ellipsoid if and only if the Laplacian of the normal map $\phi: M^2 \to R^3$ is proportional to ϕ.*

THEOREM 2. *For a hyperovaloid $f: M^n \to R^{n+1}$, $n \geq 2$, the following three conditions are equivalent*:
(1) *The conormal image $v(M^n)$ is a hyperellipsoid with center at the origin of R_{n+1}.*
(2) *The normal image $\phi(M^n)$ is a hyperellipsoid with center at the origin of R^{n+1}.*
(3) *$f(M^n)$ is a hyperellipsoid.*

In Section 1 we study the normal and conormal maps for nondegenerate hypersurfaces. By using the notion of conjugate connection we express the relationships between the three connections that arise when the normal map is an immersion. In Section 2 we compute the Laplacian of the normal map and prove Theorem 1. We may prove Theorem 2 in the case $n=2$ using the same method, but the general case of Theorem 2 requires a different approach and this is presented in Section 3.

We thank Professor U. Simon for calling our attention to the work of Shen [Sh] after the results in Sections 1 and 2 were presented in his seminar at Technische Universität Berlin.

1991 *Mathematics Subject Classification*. Primary 53A15; Secondary 53C45.

* The work of the first author is supported by an Alexander von Humboldt research award at Technische Universität Berlin and Max-Planck-Institut für Mathematik, Bonn. The work of the second author is supported by an Alexander von Humboldt research fellowship at Universität zu Köln and Max-Planck-Institut für Mathematik, Bonn.

1. Conormal and normal maps; conjugate connections. For a nondegenerate immersion $f: M^n \to R^{n+1}$ there is a transversal vector field ξ, unique up to sign and called the affine normal, so that we have the basic equations for any vector fields X, Y on M^n:

$$D_X(f_*(Y)) = f_*(\nabla_X Y) + h(X, Y)\xi, \tag{1}$$

$$D_X\xi = -f_*(SX), \tag{2}$$

where D is the canonical flat connection in R^{n+1}, ∇ the induced connection, h the affine metric and S the shape operator. The induced volume element θ given by

$$\theta(X_1, \ldots, X_n) = \det(f_*(X_1), \ldots, f_*(X_n), \xi)$$

satisfies $\nabla\theta = 0$ and $\theta = \omega_h$, where ω_h denotes the volume element for the metric h. The condition $\nabla\omega_h = 0$ is called apolarity. We have the fundamental equations of Gauss, Codazzi (for h and S), and Ricci, respectively, as follows:

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \tag{3}$$

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \tag{4}$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X), \tag{5}$$

$$h(SX, Y) = h(X, SY). \tag{6}$$

(For these equations, see, for example, [N-P].)

The conormal vector, say, v_x at $x \in M^n$ is a covector uniquely determined by the conditions $v_x(f_*(X)) = 0$ for all $X \in T_x(M^n)$ and $v_x(\xi_x) = 1$. The conormal map $v: M^n \to R_{n+1}$ is defined by $x \to v_x \in R_{n+1}$. By differentiating $v(f_*(Y)) = 0$ we obtain $v_*(X)(f_*(Y)) = -h(X, Y)$, which shows that v is an immersion. Regarding $v: M^n \to R_{n+1}$ as centro-affine hypersurface with transversal field $-v$, we get

$$D_X(v_*(Y)) = v_*(\bar{\nabla}_X Y) + \bar{h}(X, Y)(-v), \tag{7}$$

where $\bar{\nabla}$ is the induced connection and $\bar{h}$ is the fundamental form. It is known that the two connections ∇ and $\bar{\nabla}$ are related by the equation

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \bar{\nabla}_X Z), \tag{8}$$

where X, Y, Z are arbitrary vector fields on M^n. We say that ∇ and $\bar{\nabla}$ are conjugate to each other relative to the metric h. (See [N-P] as well as [D-N-V].)

We now define the normal map ϕ associated to $f: M^n \to R^{n+1}$. For each $x \in M^{n+1}$ let $\phi(x)$ be the end point of the vector ξ_x when it is displaced parallelly so as to have the starting point at origin, say, o of R^{n+1}. In this way, we get the map $\phi: M^n \to R^{n+1}$. Since $\phi_*(X) = D_X\phi = D_X\xi = -f_*(SX)$, if follows that ϕ is an immersion if and only if S is nonsingular. The following gives more specific information than that found in [Sch, pp. 142–3] about the normal maps in general.

PROPOSITION 1. *Assume that S is nonsingular. Then*

(i) *the normal map ϕ is an immersion;*

(ii) *as centro-affine hypersurface with $-\phi$ as transversal field we can write*

$$D_X(\phi_*(Y)) = \phi_*(\nabla'_X Y) + h'(X, Y)(-\phi), \tag{9}$$

where the induced connection ∇' is equal to

$$\nabla'_X Y = S^{-1}(\nabla_X S)Y + \nabla_X Y, \tag{10}$$

and the fundamental form h' is given by

$$h'(X, Y) = h(SX, Y); \tag{11}$$

(iii) *the connections ∇' and $\bar{\nabla}$ are conjugate relative to the metric $\bar{h} = h'$.*

PROOF. (ii) easily follows from $\phi_*(X) = -f_*(SX)$. The verification of (iii) is also straightforward computation. It can also be shown by a geometric observation that both $f: M^n \to R^{n+1}$ and $\phi: M^n \to R^{n+1}$ have the same conormal, because for each $x \in M^n$ the tangent hyperplane to $f(M^n)$ at $f(x)$ and the tangent hyperplane to $\phi(M^n)$ at $\phi(x)$ are parallel, and ξ at $f(x)$ and $-\phi(x)$ at $\phi(x)$ are also parallel to each other. It follows that the connection ∇' is the conjugate of the connection $\bar{\nabla}$ relative to the metric $\bar{h}$. (This geometric observation that f and ϕ are in Peterson correspondence is from [Sch, p. 142]).

REMARK. Proposition 1 is valid more generally for any nondegenerate immersion $M^n \to R^{n+1}$ with an equiaffine transversal vector field ξ (that is, $D_X\xi$ is tangential, without requiring that it be the affine normal).

2. Laplacian of the normal map. Now we compute the Laplacian of the normal map ϕ. In general, we recall that the Laplacian can be defined for any differentiable map $f: M \to \tilde{M}$, where M is a Riemannian or pseudo-Riemannian manifold with metric, say h, and $\tilde{M}$ is a manifold with a torsion-free connection D. First we define the Hessian Hess_f of f by setting $\mathrm{Hess}_f(X, Y) = D_X(f_*(Y)) - f_*(\hat{\nabla}_X Y)$, where X, Y are any vector fields on M and $\hat{\nabla}$ is the Levi-Civita connection for the metric h. Thus Hess_f is a bilinear symmetric mapping of $T_x(M) \times T_x(M)$ into $T_f(x)(\tilde{M})$. Then we take $\Delta f = \mathrm{trace}_h \mathrm{Hess}_f$ (that is, $\sum_{i,j=1}^n h^{ij}\mathrm{Hess}_f(e_i, e_j)$), where $\{e_1, \ldots, e_n\}$ is an arbitrary basis in $T_x(M)$. For the conormal immersion, we have $\Delta v = -(\mathrm{tr}\, S)v$. (See [N].) Thus Δv is always proportional to v, as is known in the Lelieuvre formula [B, p. 133] for the case of a nondegenerate surface.

We apply this definition to the normal map $\phi: M^n \to R^{n+1}$ (without assuming that it is an immersion). We get

$$\begin{aligned} D_X(\phi_*(Y)) &= D_X(f_*(-SY)) = f_*(-\nabla_X(SY)) + h(X, SY)(-\xi) \\ &= f_*(-(\nabla_X S)(Y) - S(\nabla_X Y)) - h(X, SY)\xi\,. \end{aligned}$$

Using the Levi-Civita connection $\hat{\nabla}$ for the affine metric h we have

$$\phi_*(\hat{\nabla}_X Y) = f_*(-S(\hat{\nabla}_X Y)) = f_*(-S(\nabla_X Y) + SK(X, Y)), \tag{13}$$

where K is the difference tensor: $K(X, Y) = \nabla_X Y - \hat{\nabla}_X Y$. Thus

$$\mathrm{Hess}_\phi(X, Y) = f_*(-(\nabla_X S)(Y) - SK(X, Y)) - h(X, SY)\xi .$$

Using apolarity in the form $\mathrm{tr}_h K = 0$ we get

$$\Delta\phi = -f_*(\mathrm{tr}_h(\nabla S)) - \mathrm{tr}(S)\phi . \tag{14}$$

We have thus

PROPOSITION 2. *For a nondegenerate hypersurface $f\colon M^n \to R^{n+1}$ with the affine normal, the Laplacian of the normal map is given by* (14). *Consequently,*

(1) *ϕ is harmonic if and only if* $\mathrm{tr}_h(\nabla S) = 0$ *and* $\mathrm{tr}\, S = 0$ (*the affine mean curvature H is* 0);

(2) *$\Delta\phi$ is proportional to ϕ if and only if* $\mathrm{tr}_h(\nabla S) = 0$. *In this case, the proportion factor is $-nH$.*

REMARK 1. We may establish the relationship

$$h(\mathrm{tr}_h(\nabla S), Y) = Y\,\mathrm{tr}\, S + 2\,\mathrm{tr}(K_Y S) \qquad \text{for every} \quad Y \in T_x(M^n),$$

where $K_Y Z = K(Y, Z)$.

REMARK 2. The terminology is somewhat different in [Sh], where the tangential component of our Laplacian is computed in the case where S is nonsingular.

We shall now prove Theorem 1. Suppose $\Delta\phi$ is proportional to ϕ. By Proposition 2, (2) we obtain $\mathrm{tr}_h(\nabla S) = 0$. Now around each point in M^2 we take an isothermal coordinate system (x, y) for the affine metric h and write $X = \partial/\partial x$, $Y = \partial/\partial y$ so that $h(X, X) = h(Y, Y) = E$, $h(X, Y) = 0$. Then we see that we have a globally defined quadratic form

$$\Psi = \{h(SX, X) - h(SY, Y) - 2ih(SX, Y)\}dz^2 , \qquad \text{where} \quad z = x + iy .$$

This form is holomorphic under the condition $\mathrm{tr}_h(\nabla S) = 0$, as we shall prove in the lemma below. Our surface being homeomorphic to S^2, Ψ must vanish everywhere. Thus $h(SX, X) = h(SY, Y)$ and $h(SX, Y) = 0$, which imply that $S = \lambda I$, that is, M^2 is an affine sphere. We note that S cannot be 0. By a classical theorem of Blaschke, it follows that $f(M^2)$ is an ellipsoid.

LEMMA. *The form Ψ defined above is holomorphic if and only if* $\mathrm{tr}_h(\nabla S)$ *is identically zero.*

PROOF. Let $u = h(SX, X) - h(SY, Y)$ and $v = -2h(SX, Y)$. We compute to show that the Cauchy-Riemann equations are valid. Since (x, y) is an isothermal coordinate system, we have $\hat{\nabla}_X X = -\hat{\nabla}_Y Y$. Using $K_X X + K_Y Y = 0$ (apolarity), we obtain

$$\nabla_X X = \hat{\nabla}_X X + K_X X = -\hat{\nabla}_Y Y - K_Y Y = -\nabla_Y Y. \tag{15}$$

Of course, we have also $\nabla_X Y = \nabla_Y X$. Now we have

$$\begin{aligned}\partial u/\partial x &= X(h(SX, X) - h(SY, Y)) \\ &= (\nabla_X h)(SX, X) - (\nabla_X h)(SY, Y) + h((\nabla_X S)X, X) + 2h(SX, \nabla_X X) \\ &\quad - h((\nabla_X S)Y, Y) - 2h(SY, \nabla_X Y).\end{aligned}$$

Similarly we get

$$\begin{aligned}-\partial v/\partial y &= Y(h(X, SY) + h(SX, Y)) \\ &= (\nabla_Y h)(X, SY) + (\nabla_Y h)(SX, Y) + h((\nabla_Y S)Y, X) + 2h(SX, \nabla_Y Y) \\ &\quad + h((\nabla_Y S)X, Y) + 2h(SY, \nabla_Y X).\end{aligned}$$

Using the Codazzi equations for S and h, the apolarity and (15), we get

$$\partial u/\partial x - \partial v/\partial y = h(X, (\nabla_X S)(X)) + h(X, (\nabla_Y S)(Y)) = Eh(\mathrm{tr}_h(\nabla S), X).$$

By similar computation we have

$$\partial u/\partial y + \partial v/\partial x = h(Y, (\nabla_X S)(X) + (\nabla_Y S)(Y)) = Eh(Y, \mathrm{tr}_h(\nabla S)).$$

From these equations it follows that the Cauchy-Riemann equations are satisfied (and the form Ψ is holomorphic) if and only if $\mathrm{tr}_h(\nabla S) = 0$, thus proving the lemma and completing the proof of Theorem 1.

REMARK. For a nondegenerate surface M^2 with affine normal ξ one can easily verify that $\mathrm{tr}_h(\nabla S) = 0$ holds if and only if $(\nabla_W R)(X, Y)Z$ is symmetric in Z and W. This fact can be used in order to prove a theorem in [O-V] that an ovaloid M^2 with $\nabla R = 0$ is an ellipsoid.

3. Proof of Theorem 2. We start with the following proposition which summarizes the relationships among the three connections induced by f, v, and ϕ.

PROPOSITION 3. *If $f: M^n \to R^{n+1}$ is a nondegenerate immersion with an equiaffine transversal field ξ and nonsingular S, then the following conditions are equivalent*:

(1a) $\overline{\nabla} h = 0$;

(1b) *$v(M^n)$ is an open part of a hyperquadric with its center at the origin of R_{n+1}.*

(2a) $\nabla' h' = 0$;

(2b) *$\phi(M^n)$ is an open part of a hyperquadric with its center at the origin of R_{n+1}.*

(3a) $\nabla' = \bar{\nabla}$;

(3b) $2K_X Y = -S^{-1}(\nabla_X S)(Y)$.

PROOF. The equivalence of (1a) and (1b) (as well as that of (2a) and (2b)) follows from a well-known theorem of Maschke-Pick-Berwald. The equivalence of (3a) and (3b) follows from (10) and $\nabla_X - \bar{\nabla}_X = 2K_X$. In order to prove the equivalence of (1a), (2a) and (3a), it is sufficient to note in general that if two connections ∇^1 and ∇^2 are conjugate relative to a nondegenerate metric g, then they coincide if and only if $\nabla^1 g=0$, or equivalently, if and only if $\nabla^2 g=0$. This completes the proof of Proposition 3.

The essential part of Theorem 2 in the case where $n=2$ and S is non-singular follows from this proposition. From (1) or (2) we get $2K_X Y=-S^{-1}(\nabla_X S)Y$, as in Proposition 3. By apolarity $\mathrm{tr}_h K=0$, then, we get $\mathrm{tr}_h(\nabla S)=0$. Now we can use the same argument as for Theorem 1 to conclude that $f(M^2)$ is an ellipsoid. We may prove Theorem 2 in the general case as follows. First, (3) obviously implies (1). If we assume (1), then we see that the immersion v is nondegenerate. This means that S is nonsingular. Thus ϕ is an immersion and, by the equivalence of (1a), (1b), (2a) and (2b) in Proposition 3 we see that (1) implies (2). It now remains to prove that (2) implies (3). For an arbitrary dimension n, we can do this as follows.

We start with the following observation.

PROPOSITION 4. *Let $f\colon M^n \to R^{n+1}$ be an isometric immersion of a pseudo-Riemannian manifold M^n with metric g into a pseudo-Euclidean space R^{n+1}. Let N be a field of (space-like or time-like) unit normal vectors. Suppose the Gauss-Kronecker curvature K_n (i.e. the determinant of the metric shape operator A) is nowhere 0. Then, f is nondegenerate and the affine normal has the same direction as N if and only if K_n is constant.*

PROOF. The metric second fundamental form h is equal to $g(AX, Y)$ and hence nondegenerate. The affine normal is obtained in the form $\lambda N+f_*(Z)$, where $\lambda=|K_n|^{1/(n+2)}$ and Z is a tangent vector field such that $h(X, Z)=-(d\lambda)(X)$ for all tangent vectors X. See [N]. (For the case $n=2$, cf. formula (186) in [B, p. 166].) Thus the affine normal has the same direction as N if and only if $Z=0$, that is, $|K_n|$ is constant.

Now assume that $\phi(M^n)$ is a hyperellipsoid with center at the origin. This means that there is a D-parallel, positive-definite scalar product g in R^{n+1} such that $\phi(M^n)$ is the unit sphere Σ relative to $g\colon g(\xi, \xi)=1$. Thus we get $g(D_X\xi, \xi)=0$. Since $D_X\xi=-f_*(SX)$, it follows that $g(f_*(SX), \xi)=0$.

Let U be an open subset of M^n on which $\det S$ is not 0. Then $f\colon U \to R^{n+1}$ is an immersion for which ξ is a unit vector field perpendicular to $f(U)$. By Proposition 4 we can conclude that the Gauss-Kronecker curvature, namely, $\det S$ in this case is constant on U.

Now go back to $\phi\colon M^n \to R^{n+1}$. Since $\phi(M^n)$ is a hyperellipsoid, we cannot have $\det S$ identically equal to 0 on M^n. Let $\det S=c\neq 0$ at some point and consider the set W of all points where $\det S=c$. Then W is a closed subset of M^n. On the other hand, if $x\in W$, then there exists an open neighborhood U of x on which $\det S$ is not zero and hence constant according to the assertion above. Thus W is an open subset. We conclude that $W = M^n$, that is, $\det S$ is constant on M^n. It is known ([Sü]; also [Si]) that for a hyperovaloid this condition implies that it is a hyperellipsoid. We have thus completed the proof of Theorem 2 in the general case.

REMARK 1. By using similar arguments we can prove, for a nondegenerate hypersurface $f: M^n \to R^{n+1}$ with affine normal, the equivalence of the statements (1b), (2b) given in Proposition 3 and the statement that $f(M^n) \to R^{n+1}$ is an open part of a pseudo-Riemannian hypersurface with constant Gauss-Kronecker curvature isometrically immersed in a pseudo-Euclidean space R^{n+1}.

REFERENCES

[B] W. BLASCHKE, Vorlesungen über Differentialgeometrie II, Affine Differentialgeometrie, Springer, Berlin, 1923.

[DNV] F. DILLEN, K. NOMIZU AND L. VRANCKEN, Conjugate connections and Radon's theorem in affine differential geometry, Monatsh. Math. 109 (1990), 221–235.

[N] K. NOMIZU, Introduction to Affine Differential Geometry, Part I, Lecture Notes, MPI preprint MPI 88-37, 1988; Revised: Department of Mathematics, Brown University, 1989.

[N-P] K. NOMIZU AND U. PINKALL, On the geometry of affine immersions, Math. Z. 195 (1987), 165–178.

[O-V] B. OPOZDA AND L. VERSTRAELEN, On a new curvature tensor in affine differential geometry, Geometry and Topology of Submanifolds II, Avignon, May 1988, World Scientific, 1990, pp. 271–293.

[Sch] P. A. SCHIROKOW AND A. P. SCHIROKOW, Affine Differentialgeometrie, Teubner, Leipzig, 1962.

[Sh] Y-B. SHEN, Harmonic Gauss maps and Codazzi tensors for affine hypersurfaces, Archiv der Math. 55 (1990), 298–305.

[Si] U. SIMON, Minkowskische Integralformeln und ihre Anwendungen in der Differentialgeometrie im Grossen, Math. Ann. 173 (1967), 307–321.

[Sü] W. SÜSS, Zur relativen Differentialgeometrie V: Über Eihyperflächen in R^{n+1}, Tôhoku Math. J. 37 (1929), 202–209.

DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RI 02912
U.S.A.

INSTYTUT MATEMATYKI UJ
UL. REYMONTA 4
30–059 KRAKÓW
POLAND