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COHOMOLOGY OF INFINITESIMAL QUANTUM GROUPS, I

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Abstract. In this paper, we investigate the cohomology of infinitesimal quantum groups (and algebras) associated to classical quantum groups (and algebras) at a root of unity. A main result expresses the Ext-groups between irreducible modules in terms of those for the full quantum group. Under the assumption that certain module categories for the quantum group have a Kazhdan-Lusztig theory (in the sense of Cline, Parshall, and Scott), this permits explicit calculations of cohomology in terms of Kazhdan-Lusztig polynomials. This assumption in turn follows from recently announced results of Kazhdan and Lusztig.

Let $\mathscr{C}$ be a highest weight category, as introduced in [6]. In recent work [8], the first author, together with E. Cline and L. Scott, defined the concept of a *Kazhdan-Lusztig theory* for the category $\mathscr{C}$. The notion played a key rôle in various significant simplifications of Lusztig's famous conjecture [26] concerning the characters of simple modules for a semisimple algebraic group G over a field of positive characteristic. In addition, when this conjecture holds, one obtains a combinatorial calculation of the groups $\mathrm{Ext}^{\bullet}_{G}(L(\lambda), L(\nu))$, whenever λ, ν are regular dominant weights satisfying the Jantzen condition. In [9], this work was extended to include the representation theory of G_1T, the pull-back through the Frobenius morphism F of a maximal torus T of G.

Similar results apply to the category $\mathscr{C}_q$ of rational modules for quantum groups and quantum enveloping algebras at a root of unity. In particular, assuming the recently announced work of Kazhdan-Lusztig [24], it follows from [8] that the full subcategory of $\mathscr{C}_q$ whose objects have composition factors with l-regular highest weights has a Kazhdan-Lusztig theory. Thus, the groups $\mathrm{Ext}^{\bullet}_{\mathscr{C}_q}(L^q(\lambda), L^q(\nu))$ are completely determined in terms of Kazhdan-Lusztig polynomials when the highest weights λ, ν are l-regular.

As is well-known, the finite (or "infinitesimal") quantum groups (first introduced in the setting of quantum enveloping algebras by Lusztig [28]) play a rôle somewhat analogous to that of the restricted enveloping algebras of the Lie algebras of semisimple algebraic groups in positive characteristic (or, equivalently, that of the group scheme $G_1 \equiv \mathrm{Ker}\, F$). Also, the cohomology theory of these restricted enveloping algebras has interesting geometric interpretations. In fact, there are analogues of many results from the study of the spectrum of the cohomology ring of finite groups (although the proofs

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are often considerably different). See [13], [14], for example. This paper represents a first step in investigating the cohomology of the infinitesimal quantum groups with the eventual aim of obtaining a geometric interpretation of this cohomology. Using the characteristic p theory as a guide, we are able to give, often assuming [24], many calculations in the quantum case, although we do not consider here any precise geometric meaning. Also, we briefly take up the question of extending some of the results of [9] to the quantum analogue of G_1T (see Theorem 5.3).

This paper is organized as follows. In Section 1, we review some results from Kazhdan-Lusztig theory for highest weight categories. In §§2, 3, we describe the various module categories with which we work. We have tried to set things up so that they apply to the case both of quantum *groups* and quantum *enveloping algebras*. In the former, we rely heavily on results established in our previous paper [31]. In the latter, we make use of work of Andersen-Polo-Wen [2], [3]. In Section 4, we consider the cohomology of the "Frobenius kernel" $(G_q)_1$, while Section 5 treats that of the pull-back $(G_q)_1T$ of the maximal torus under the Frobenius morphism. The main computational device, which is very elementary, is given in Theorem 4.2. It expresses certain Ext^i groups for the Frobenius kernel in terms of the corresponding Ext^i groups for the full quantum group. Assuming that certain module categories for the quantum group have a Kazhdan-Lusztig thoery (a consequence of [24] and [8]), we readily obtain some explicit calculations. (For example, see Proposition 4.5, Corollary 4.6, and Remark 4.7e.) We obtain similar calculations for $(G_q)_1T$ in §5. Finally, since the cohomology of a quantum group is essentially the cohomology of comodules over a coalgebra, we have included an Appendix which collects together some elementary facts concerning comodule cohomology of a coalgebra.

In a sequel, we treat several questions left open by this paper. For example, although Theorem 5.3 proves (assuming the results of [24]) an "even-odd vanishing behavior" for the $\mathrm{Ext}^\bullet$ groups between simple $(G_1)_qT$-modules (having l-regular highest weights), we leave unanswered here the question as to whether the associated $(G_q)_1T$-module cagegory has a Kazhdan-Lusztig theory. Also, we will consider the geometric interpretation of the cohomology of infinitesimal quantum groups (i.e., the theory of support varieties) suggested by this paper.

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List of Notation.

- Φ Irreducible root system (finite, crystallographic) in a Euclidean space $\boldsymbol{E}$ having inner product (,). Usually, $\boldsymbol{E}$ is the span of Φ. However, if Φ has type A_{n-1}, the following exceptional case is also allowed: $\boldsymbol{E}$ has a basis $e_1, e_2, \ldots, e_n$ with $\Phi = \{e_i - e_j \mid i \neq j\}$.
- h The Coxeter number of Φ.
- $\Phi^\vee$ Root system dual to Φ.

Φ^+ (resp., Φ^-)	Fixed set of positive (resp., negative) roots.
Π	Set of simple roots with respect to the above choice of Φ^+.
W	Weyl group of Φ.
Q	The $\boldsymbol{Z}$-span of Φ.
X	Integral weight lattice of Φ (in the exceptional case above, X is the $\boldsymbol{Z}$-span of e_i's).
X_+	$\{\lambda \in X \mid (\lambda, \alpha^\vee) \geq 0, \forall \alpha \in \Pi\}$, the set of dominant integral weights.
ρ	$(1/2)\sum_{\alpha \in \Phi^+} \alpha$, the Weyl weight.
l	An odd integer >1 (in the case of quantum enveloping algebras, l is assumed to be a power of an odd prime p in [2], and l is assumed to equal p in [3]).
X_l	$\{\lambda \in X \mid 0 \leq (\lambda, \alpha^\vee) < l, \forall \alpha \in \Pi\}$, the set of l-restricted dominant weights.
C_l	$\{\lambda \in \boldsymbol{E} \mid 0 < (\lambda + \rho, \alpha^\vee) < l, \forall \alpha \in \Phi^+\}$, the bottom l-alcove.
$X^{l\text{-reg}} \equiv X^{\text{reg}}$	$\{\lambda \in X \mid (\lambda + \rho, \alpha^\vee) \not\equiv 0 \pmod{l}, \forall \alpha \in \Phi\}$, the set of l-regular integral weights. Thus, $X^{\text{reg}} \neq \varnothing \Leftrightarrow l \geq h$.
X_+^{reg}	$X_+ \cap X^{\text{reg}}$.
$\wp$	The Kostant partition function defined on X: for $\lambda \in X$, $\wp(\lambda)$ is the number of ways to write λ as a non-negative integral linear combination of positive roots.
$\wp_i$	The "i-part" partition function: for $\lambda \in X$, $\wp_i(\lambda)$ is the number of ways to write λ as a non-negative integral linear combination of positive roots with coefficient sum i.
$\mathscr{T}_l$ (resp., $\tilde{\mathscr{T}}_l$)	Group of translations $t_{l\lambda}\colon \boldsymbol{E} \to \boldsymbol{E}$, $x \mapsto x + l\lambda$ for $\lambda \in Q$ (resp., $\lambda \in X$).
W_l	$W \ltimes \mathscr{T}_l$, the affine Weyl group of Φ with parameter l.
$\tilde{W}_l$	$W \ltimes \tilde{\mathscr{T}}_l$, the extended affine Weyl group of Φ with parameter l. For $w \in \tilde{W}_l$ and $x \in \boldsymbol{E}$, write $w \cdot x = w(x + \rho) - \rho$.
$\uparrow$	Uparrow partial ordering on X induced by W_l, [19; §II.6].
$\leq$	Usual partial ordering on X: $\lambda \leq \mu \Leftrightarrow \wp(\mu - \lambda) > 0$.
ℓ	The standard length function on $\tilde{W}_l$, given by the formula $$\ell(wt_{l\lambda}) = \sum_{\alpha \in \Phi^+ \cap w^{-1}\Phi^+} \lvert (\lambda, \alpha^\vee) \rvert + \sum_{\alpha \in \Phi^+ \cap w^{-1}\Phi^-} \lvert 1 + (\lambda, \alpha^\vee) \rvert,$$ for $w \in W$ and $\lambda \in X$ (see [18; Prop. 1.23]). Also, we denote by ℓ the length function on X^{reg}: if $\lambda = wt_{l\mu} \cdot \tau$ for $\tau \in C_l$, $w \in W$ and $\mu \in X$, put $$\ell(\lambda) = -\ell(w) + \sum_{\alpha \in \Phi^+} (w\mu, \alpha^\vee).$$ If $\lambda \in X_+$, then $\ell(\lambda) = \ell(wt_{l\mu})$, see [9; 3.12.5].
$\chi(\lambda)$	$\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \lambda} / \sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot 0} \in \boldsymbol{Z}X$, the Weyl character.
$\cdot^{(l)}$	The R-linear endomorphism of the group algebra RX, for a commutative ring R, induced by the endomorphism $\lambda \mapsto l\lambda$ of X. (The notation also serves for "Frobenius twist", see §3.)

1. Kazhdan-Lusztig theories. Throughout this paper, let k be a fixed field. Let $\mathscr{C}$ be a highest weight category over k, having weight poset Λ (which is always assumed inteval-finite) and satisfying the following conditions (1.1)–(1.4):

(1.1) $\mathscr{C}$ is finite; i.e., objects in $\mathscr{C}$ have finite length. If $L \in \mathrm{Ob}(\mathscr{C})$ is simple, $\mathrm{End}_{\mathscr{C}}(L) \cong k$.

(1.2) The opposite category $\mathscr{C}^{\mathrm{op}}$ is a highest weight category with the same weight poset Λ. (For a weight $\lambda \in \Lambda$, let $A(\lambda)$, $V(\lambda)$, and $L(\lambda)$ denote, respectively, the "induced", the "Weyl" object, and the simple object corresponding to λ.)

(1.3) There is a fixed duality $D: \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ (in the sense of [7]). Thus, $DL(\lambda) \cong L(\lambda)$ and $DA(\lambda) \cong V(\lambda)$ for all $\lambda \in \Lambda$.

(1.4) There is given a fixed function $\ell: \Lambda \to \boldsymbol{Z}$, called the "length function" on the weight poset Λ which is *compatible* with Λ in the sense that $\lambda < \nu \Rightarrow \ell(\lambda) < \ell(\nu)$.

If $\mathscr{C}$ is a highest weight category as above and if Γ is an ideal in the weight poset Λ, let $\mathscr{C}[\Gamma]$ denote the full subcategory of $\mathscr{C}$ consisting of objects having composition factors $L(\gamma)$ for $\gamma \in \Gamma$. Then $\mathscr{C}[\Gamma]$ is a highest weight category with weight poset Γ. (Also, we will often use without further comment the fact that $D^b(\mathscr{C}[\Gamma])$ is a full subcategory of $D^b(\mathscr{C})$; see [6; Thm. 3.9] and [9; (1.2)].) For a general discussion of highest weight categories, see [6]–[9].

Consider the bounded derived category $D^b(\mathscr{C})$ of $\mathscr{C}$. Let $T: D^b(\mathscr{C}) \to D^b(\mathscr{C})$ be the translation operator, and write $X[n] = T^n X$ for $n \in \boldsymbol{Z}$ and $X \in \mathrm{Ob}(D^b(\mathscr{C}))$. For $X \in \mathrm{Ob}(D^b(\mathscr{C}))$ and $\nu \in \Lambda$, the *left Poincaré polynomial* $p_{\nu,X} \in \boldsymbol{Z}[t, t^{-1}]$ is defined by the expression:

$$p_{\nu,X} = \sum_n \dim \mathrm{Hom}^n_{D^b(\mathscr{C})}(X, A(\nu)) t^n \,, \tag{1.5}$$

where, for X, $Y \in D^b(\mathscr{C})$,

$$\mathrm{Hom}^n_{D^b(\mathscr{C})}(X, Y) \equiv \mathrm{Hom}_{D^b(\mathscr{C})}(X, Y[n]) \cong \mathrm{Hom}_{D^b(\mathscr{C})}(X[-n], Y) \,.$$

In particular, if X, $Y \in \mathscr{C}$, then

$$\mathrm{Hom}^n_{D^b(\mathscr{C})}(X, Y) \cong \mathrm{Ext}^n_{\mathscr{C}}(X, Y) \,. \tag{1.6}$$

The *right Poincaré polynomial* $p^R_{\nu,X}$ is defined by

$$p^R_{\nu,X} = p_{\nu,DX} \,, \tag{1.7}$$

using the fact that the duality D extends naturally to $D^b(\mathscr{C})$. Thus, if X is self-dual, we have $p^R_{\nu,X} = p_{\nu,X}$. If $X = L(\lambda)$, we write $p_{\nu,L(\lambda)} = p_{\nu,\lambda}$, so that $p^R_{\nu,L(\lambda)} = p_{\nu,\lambda}$ for all λ, $\nu \in \Lambda$. Thus, thanks to (1.6), we have

$$p^R_{\nu,\lambda} = p_{\nu,\lambda} = \sum_n \dim \mathrm{Ext}^n_{\mathscr{C}}(L(\lambda), A(\mu)) t^n \,. \tag{1.8}$$

Using the Poincaré polynomial $p_{\nu,\lambda}$, we define the corresponding *Kazhdan-Lusztig* *polynomial*

$$P_{\nu,\lambda} = t^{\ell(\lambda)-\ell(\nu)}\bar{p}_{\nu,\lambda}\,. \tag{1.9}$$

(Here, and in what follows, $\bar{f}$ for $f \in \mathbf{Z}[t, t^{-1}]$ denotes the image of f under the automorphism of $\mathbf{Z}[t, t^{-1}]$ defined by $t \mapsto t^{-1}$.) Thus, in case Λ is finite, we have

$$[L(\lambda)] = \sum_{\nu} (-1)^{\ell(\lambda)-\ell(\nu)} P_{\nu,\lambda}(-1)[A(\nu)] \tag{1.10}$$

in the Grothendieck group of $\mathscr{C}$ [8; Prop. 3.2]. (A similar, but more complicated formula holds when Λ is not finite [9; Prop. 3.11]. However, we will not use this here.) Observe that $P_{\nu,\lambda} = 0$ unless $\nu \le \lambda$.

Let $\mathscr{E}^L$ be the full subcategory of $D^b(\mathscr{C})$ with objects X such that, for any weight ν and integer n, if t^n has nonzero coefficient in $p_{\nu,X}$, then $n \equiv \ell(\nu) \pmod 2$. Put $\mathscr{E}^R = D\mathscr{E}^L$. Following [8; Defn. 2.1], $\mathscr{C}$ has a *Kazhdan-Lusztig theory* if and only if $L(\lambda)[-\ell(\lambda)] \in \mathscr{E}^L$ for all $\lambda \in \Lambda$. (When the weight poset Λ is finite, this condition is equivalent to the assertion that $L(\lambda)[-\ell(\lambda)] \in \mathrm{Ob}(D^b(\mathscr{C}))$ has a "filtration" (in a certain sense) by shifted Weyl modules $V(\nu)[k]$ with $k \equiv \ell(\nu) \pmod 2$. See [8; Thm. 2.4] and [10; §3].) The importance of these concepts is suggested by the fact, established in [8; §3], that if $X, Y \in \mathrm{Ob}(\mathscr{E}^L)$, then

$$\sum_n \dim \mathrm{Hom}^n_{D^b(\mathscr{C})}(X, DY)t^n = \sum_{\tau \in \Lambda} p_{\tau,X} p_{\tau,Y}\,. \tag{1.11}$$

Thus, if $\mathscr{C}$ has a Kazhdan-Lusztig theory, the groups $\mathrm{Ext}^\bullet_{\mathscr{C}}(L(\lambda), L(\nu))$ can be calculated in terms of the Poincaré polynomials. In turn, these polynomials are often recursively determined. (See Theorem (3.5) below, for example.)

2. Relations between representations of $GL_q(n)$ and $SL_q(n)$. The representations of quantum linear groups $\tilde{G}_q \equiv GL_q(n)$ and $G_q \equiv SL_q(n)$, q a nonzero element of k, are studied in detail in [31]. This section is devoted to making clear the connections between the representations of $\tilde{G}_q$ and those of G_q. Some results are quoted from [31], and some results are new (and complete those of [31]). (For further results on $GL_q(n)$, $SL_q(n)$ and their infinitesimal subgroups and the finite quantum enveloping algebras of [28], see [33].)

We adopt the "naïve" point of view of [31], identifying the category $\mathbf{QGr}_k$ of quantum groups over k with the dual of the category ***k*-Hopf** of k-Hopf algebras. The Hopf algebra corresponding to a quantum group G, denoted by $k[G]$, is usually called the *coördinate algebra* of G. From this point of view, a *rational G-module* is nothing but a $k[G]$-comodule. Recall the construction of the coordinate algebras $k[\tilde{G}_q]$ and $k[G_q]$ for a nonzero parameter $q \in k$. We begin with an associative algebra $k[M_q(n)]$ generated by n^2 elements X_{ij}, $i, j = 1, 2, \ldots, n$, with relations

$$
\begin{aligned}
&X_{ri}X_{rj}=q^{-1}X_{rj}X_{ri}\,, && \forall i<j\,;\\
(2.1)\qquad &X_{ri}X_{si}=q^{-1}X_{si}X_{ri}\,, && \forall r<s\,;\\
&X_{ri}X_{sj}=X_{sj}X_{ri}\,, && \text{if } r<s \text{ and } i>j\,;\\
&X_{ri}X_{sj}-X_{sj}X_{ri}=(q^{-1}-q)X_{si}X_{rj}\,, && \text{if } r<s \text{ and } i<j\,.
\end{aligned}
$$

It is known that $k[M_q(n)]$ is a noncommutative "polynomial" algebra in the abovementioned n^2 generators. It is also known that $k[M_q(n)]$ is a bialgebra whose comultiplication Δ and augmentation map ε are given by

$$\Delta(X_{ij})=\sum_k X_{ik}\otimes X_{kj}\,,\quad \varepsilon(X_{ij})=\delta_{ij}\,. \tag{2.2}$$

There is a central group-like element, called the *quantum determinant*, given by

$$D_q=\sum_{\sigma\in\mathfrak{S}_n}(-q)^{-\ell(\sigma)}X_{1\sigma(1)}X_{2\sigma(2)}\cdots X_{n\sigma(n)}\,, \tag{2.3}$$

where $\mathfrak{S}_n$ is the symmetric group in n letters, serving as the Weyl group. The coordinate algebra of $\tilde{G}_q$ (resp., G_q) is the localization of $k[M_q(n)]$ at D_q (resp., the quotient algebra of $k[M_q(n)]$ by the ideal generated by D_q-1). These algebras are Hopf algebras with the same comultiplication Δ and augmentation map ε as given in (2.2).

The closed subgroup $\tilde{T}_q$ (resp., T_q) of $\tilde{G}_q$ (resp., G_q) defined by the ideal generated by the X_{ij}, $i\neq j$, is an algebraic torus, in the usual sense, with character group

$$X(\tilde{T}_q)=\{X_{11}^{r_1}X_{22}^{r_2}\cdots X_{nn}^{r_n}\,|\,r_i\in\boldsymbol{Z}\}\quad(\text{resp., } X(T_q)=\{X_{11}^{r_1}X_{22}^{r_2}\cdots X_{n-1,n-1}^{r_{n-1}}\,|\,r_i\in\boldsymbol{Z}\})$$

(the operation in the group is the multiplication in the coordinate algebra). Let

$$\Phi=\{X_{ii}^{-1}X_{jj}\,|\,i\neq j\}\,,\quad \Phi^+=\{X_{ii}^{-1}X_{jj}\,|\,i>j\}$$

in both cases. Then Φ is a root system of type A_{n-1} with Φ^+ as its set of positive roots, and $X(\tilde{T}_q)$ (resp., $X(T_q)$) is identified with the weight lattice X as defined in the "List of Notation" (in the case of $\tilde{G}_q$, X must be the exceptional case as mentioned there). Recall that there is a surjective homomorphism $X(\tilde{T}_q)\to X(T_q)$, $\lambda\mapsto\bar{\lambda}$, which is the identity on X_{ii} for $i<n$ and sends X_{nn} to $X_{11}^{-1}X_{22}^{-1}\cdots X_{n-1,n-1}^{-1}$. Also, $\lambda\in X(\tilde{T}_q)_+$ if and only if $\bar{\lambda}\in X(T_q)_+$. As in the classical case, now we can define the weights and formal characters for rational $\tilde{G}_q$-modules or rational G_q-modules.

Now let q be an l-th primitive root of 1. (What we will say will be trivial if q is not a root of 1.) For $\lambda\in X(\tilde{T}_q)_+$ (resp., $X(T_q)_+$), we have the following rational modules for $\tilde{G}_q$ (resp., G_q):

$\tilde{L}^q(\lambda)$ (resp., $L^q(\lambda)$) — the irreducible $\tilde{G}_q$-module (resp., G_q-module) with highest weight λ.

$\tilde{V}^q(\lambda)$ (resp., $V^q(\lambda)$) — the Weyl module (i.e., the universal highest weight module) with highest weight λ for $\tilde{G}_q$ (resp., G_q).

$\tilde{A}^q(\lambda)$ (resp., $A^q(\lambda)$) — the "induced" module with highest weight λ for $\tilde{G}_q$ (resp., for G_q); the module can be described as the universal module with the property that it has an irreducible socle $L^q(\lambda)$ and all other composition factors have smaller (with respect to the ordering $\leq$) highest weights. Also, this module is denoted by $H^0(\lambda)$ in [31].

$\tilde{I}^q(\lambda)$ (resp., $I^q(\lambda)$) — the injective hull of $\tilde{L}^q(\lambda)$ (resp., $L^q(\lambda)$).

Since G_q is a closed subgroup of $\tilde{G}_q$ we have a restriction functor from the category of rational $\tilde{G}_q$-modules to the category of rational G_q-modules. Concerning this functor, we have the following theorem:

(2.4) THEOREM. *For $\lambda \in X(\tilde{T}_q)_+$, we have*
(1) $\tilde{L}^q(\lambda)|_{G_q} \cong L^q(\bar{\lambda})$;
(2) $\tilde{V}^q(\lambda)|_{G_q} \cong V^q(\bar{\lambda})$;
(3) $\tilde{A}^q(\lambda)|_{G_q} \cong A^q(\bar{\lambda})$;
(4) $\tilde{I}^q(\lambda)|_{G_q} \cong I^q(\bar{\lambda})$.

PROOF. This theorem, except for (4), is a special case of the connection between the representations of a parabolic subgroup of $\tilde{G}_q$ and its "semisimple part" developed in [31]. See [31; (8.4.8) and (8.4.6)] for (1) and (3); (2) then also follows, since the well-known duality between Weyl modules and "induced" modules. As for (4), [31; (8.4.4)] claims that $\tilde{I}^q(\lambda)|_{G_q}$ is injective. Then (4) follows from Lemma (2.5) below. □

The coordinate algebra $k[\tilde{G}_q]$ is $\boldsymbol{Z}$-graded in the usual manner, and all its homogeneous components are subcoalgebras. We call a rational $\tilde{G}_q$-module V with structure map τ *homogeneous of degree* r if $\tau(V) \subset V \otimes k[\tilde{G}_q]_r$. A standard argument shows that any $\tilde{G}_q$-module is a direct sum of its homogeneous components. In particular, an indecomposable $\tilde{G}_q$-module is homogeneous. This is the first conclusion of the following lemma. (One also obtains the conclusion from the linkage principle [31; (10.3.5)], noting that the affine Weyl group W_l is degree-preserving.)

(2.5) LEMMA. (1) *An indecomposable $\tilde{G}_q$-module is homogeneous and it remains indecomposable when restricted to G_q;*

(2) *For a homogeneous $\tilde{G}_q$-module V, a subspace is a $\tilde{G}_q$-submodule if and only if it is a G_q-submodule.*

PROOF. To prove (2), suppose our module V has degree r. Recall that we have a canonical epimorphism

$$\theta: G_q \times \boldsymbol{G}_m \to \tilde{G}_q$$

defined in [31; (6.2.2)], where $\boldsymbol{G}_m$, whose coordinate algebra is $k[\boldsymbol{G}_m] = k[t, t^{-1}]$, is viewed as the scalar subgroup of $\tilde{T}_q$. Thus, a subspace of V is a $\tilde{G}_q$-submodule if and only if it is a $G_q \times \boldsymbol{G}_m$-submodule. However, it is easy to see that the restriction of V to

$\boldsymbol{G}_m$ has the map $v \mapsto v \otimes t^r$, for all $v \in V$, as its structure map. Obviously, this implies (2) of the lemma. The second conclusion of (1) now also follows. □

Now we have the following result in the opposite direction.

(2.6) THEOREM. *Every indecomposable G_q-module lifts (not uniquely) to a homogeneous $\tilde{G}_q$-module.*

PROOF. Let V be a indecomposable G_q-module. Thanks to the linkage principle for G_q ([31; (10.3.5)]), the injective hull of V is of the form

$$I \cong \bigoplus_{\substack{w \in W_l \\ w \cdot \lambda \in X(T_q)_+}} I^q(w \cdot \lambda)^{\oplus r_w}, \qquad r_w \in \boldsymbol{Z}^+,$$

for a fixed $\lambda \in X(T_q)_+$. Let $\tilde{\lambda} \in X(\tilde{T}_q)_+$ be an inverse image of λ. Then

$$\tilde{I} \cong \bigoplus_{\substack{w \in W_l \\ w \cdot \lambda \in X(\tilde{T}_q)_+}} \tilde{I}^q(w \cdot \tilde{\lambda})^{\oplus r_w}$$

is a lifting of I, by Theorem (2.4). Since $\tilde{I}$ is homogeneous, Lemma (2.5) can be used to obtain a $\tilde{G}_q$-submodule of $\tilde{I}$ which is a lifting of V. □

It is known that the formal characters of Weyl modules (which are the same as the formal characters of "induced" modules, and are exactly the Weyl characters) form a $\boldsymbol{Z}$-basis for the subring of W-invariants in $\boldsymbol{Z}X(\tilde{T}_q)$ (resp., $\boldsymbol{Z}X(T_q)$). Also, any $\operatorname{ch} \tilde{L}^q(\lambda)$ or $\operatorname{ch} L^q(\lambda)$ is W-invariant. Thus, we can express $\operatorname{ch} \tilde{L}^q(\lambda)$ and $\operatorname{ch} L^q(\lambda)$ as $\boldsymbol{Z}$-linear combinations of the formal characters of certain Weyl modules. Suppose that $\lambda \in X(\tilde{T}_q)$ is a weight in the bottom l-alcove, and that $w \in W$ with $w \cdot \lambda \in X(\tilde{T}_q)_+$. Using the linkage principle again, we have

$$\operatorname{ch} \tilde{L}^q(w \cdot \lambda) = \sum_{\substack{y \in W_l \\ y \cdot \lambda \in X(\tilde{T}_q)_+}} \tilde{c}_{y,w} \operatorname{ch} \tilde{V}^q(y \cdot \lambda),$$

and

$$\operatorname{ch} L^q(w \cdot \bar{\lambda}) = \sum_{\substack{y \in W_l \\ y \cdot \bar{\lambda} \in X(T_q)_+}} c_{y,w} \operatorname{ch} V^q(y \cdot \bar{\lambda})$$

for $\tilde{c}_{y,w}, c_{y,w} \in \boldsymbol{Z}$.

From the above discussion, we have the following result:

(2.7) THEOREM. *With the above notation and assumption, $\tilde{c}_{y,w} = c_{y,w}$ for all y, w.*

As defined in [31;, Chapt. 7], there is a *Frobenius morphism* $F\colon \tilde{G}_q \to \tilde{G} \equiv GL(n)$, whose comorphism sends x_{ij}, the (i,j)-coordinate function of $\tilde{G}$, to X_{ij}^l. Let $(\tilde{G}_q)_1$, the *Frobenius kernel* of $\tilde{G}_q$, be the kernel of F in the categorical sense, and $(\tilde{G}_q)_1\tilde{T}$ be the pull-back under F of the diagonal torus $\tilde{T}$ of $\tilde{G}$. Moreover, F induces a morphism, also called the *Frobenius morphism*, $F: G_q \to G \equiv SL(n)$. We define the *Frobenius kernel* $(G_q)_1$ and the pull-back $(G_q)_1 T$ of the diagonal torus T of G similarly.

In [31; Chapt. 9], a representation theory for $(\tilde{G}_q)_1$ and $(\tilde{G}_q)_1 \tilde{T}$ is developed. It is also possible to develop a similar theory for $(G_q)_1$ and $(G_q)_1 T$. Also, an "infinitesimal version", i.e., a similar theory for $(\tilde{G}_q)_1 \tilde{T}$ and $(G_q)_1 T$, of the theory in this section can also be obtained. We will not go into details here. However, we would like to point out some key links to the theory. One of them is a "density theorem" for $(G_q)_1 T$. It may be obtained from the "density theorem" for $(\tilde{G}_q)_1 \tilde{T}$ ([31; (9.1.2)]) together with the canonical epimorphism $\theta_1: (G_q)_1 T \times \boldsymbol{G}_m \to (\tilde{G}_q)_1 \tilde{T}$, which is the restriction of the epimorphism θ used in the proof of Lemma (2.5). Also, in order that the infinitesimal "induced" module $\hat{A}_1^q(\lambda)$ (denoted by $\hat{Z}(\lambda)$ in [31]) for $(G_q)_1 T$ have "correct" dimension, we need a result similar to [31; (9.6.1)]. However, the proof for [31; (9.6.1)] works also for our case. Finally, for a result similar to Theorem (2.6) above, a linkage principle for $(G_q)_1 T$-modules is necessary. Here we give the statement and a proof of the strong version of the principle.

(2.8) THEOREM (infinitesimal strong linkage principle). *Let $\lambda, \mu \in X(\tilde{T}_q)$ (resp., $\lambda, \mu \in X(T_q)$) be such that the irreducible $(\tilde{G}_q)_1 \tilde{T}$-module (resp., $(G_q)_1 T$-module) with highest weight μ is a composition factor of the "induced" module with highest weight λ. Then $\mu \uparrow \lambda$.*

PROOF. Since we will have a result similar to Theorem (2.4) without using the linkage principle, we need only to consider the case of $(\tilde{G}_q)_1 \tilde{T}$-modules. Denote the irreducible and the "induced" $(\tilde{G}_q)_1 \tilde{T}$-modules with highest weight λ by $\hat{\tilde{L}}_1^q(\lambda)$ and $\hat{\tilde{A}}_1^q(\lambda)$, respectively. Note that in the "List of Noation" the uparrow partial ordering is defined as in [19], instead of that used in [31] (two orderings are the same in the dominant chamber, see [34]). This ordering is preserved by translations in $\tilde{W}_l$. Thus, we may assume that all composition factors $\hat{\tilde{A}}_1^q(\lambda)$ have dominant highest weights. Then by [31; (10.1.1)], together with the Kempf vanishing theorem for $\tilde{G}$, we see that if $\hat{\tilde{L}}_1^q(\mu)$ is a composition factor of $\hat{\tilde{A}}_1^q(\lambda)$, then $\tilde{L}^q(\mu)$ is a composition factor of $\tilde{A}^q(\lambda)$. Now the strong linkage principle for $\tilde{G}_q$ gives the needed result. □

The method used in the above proof is also used by [3; §2.9] in the circumstances of quantum enveloping algebras.

3. Quantum groups and quantum enveloping algebras. In the remainder of this paper, except for the Appendix, we assume that the field k has characteristics zero. Let $q \in k$ be a fixed primitive l-th root of 1. We are interested in the cohomology of quantum groups and quantum enveloping algebras with parameter q. In order to unify the treatment in both cases, we make the following general set-up.

We are working with four abelian categories: $\mathscr{C}$, $\mathscr{C}_q$, $\hat{\mathscr{C}}_q^1$ and $\mathscr{C}_q^1$. The main features of these categories are as follows:

(1) The category $\mathscr{C}$ is semisimple (i.e., any object in this category is a direct sum

of irreducibles), with X_+ as the indexing set of its irreducibles. Thus, $\mathscr{C}$ can be viewed as a highest weight category via a partial ordering on X_+, say "$\leq$". Denote by $L(\lambda)$ the irreducible object with highest weight $\lambda \in X_+$. Hence, $A(\lambda) = V(\lambda) = L(\lambda)$.

(2) The category $\mathscr{C}_q$ (resp., $\hat{\mathscr{C}}_q^1$) is a highest weight category with indexing poset $(X_+, \uparrow)$ (resp., $(X, \uparrow)$) and duality functor D (resp., $\hat{D}_1$). We will denote the irreducible, "induced" and "Weyl" objects in $\mathscr{C}_q$ with highest weight λ by $L^q(\lambda)$, $A^q(\lambda)$ and $V^q(\lambda)$, respectively, and denote those objects in $\hat{\mathscr{C}}_q^1$ by $\hat{L}_1^q(\lambda)$, $\hat{A}_1^q(\lambda)$ and $\hat{V}_1^q(\lambda)$, respectively.

(3) The full subcategory $\mathscr{C}_q^{\mathrm{reg}}$ (resp., $(\hat{\mathscr{C}}_q^1)^{\mathrm{reg}}$) of $\mathscr{C}_q$ (resp., $\hat{\mathscr{C}}_q^1$) consisting of all objects whose composition factors have regular highest weights satisfies the conditions (1.1)–(1.4) with length function ℓ as defined in the "List of Notation". (In order that these subcategories are nonempty, we must assume $l \geq h$.)

(4) The irreducible objects in $\mathscr{C}_q^1$ are indexed naturally by X/lX. For $\lambda \in X$, denote the irreducible object with "highest" weight λ (mod lX) by $L_1^q(\lambda)$.

(5) There is a "Frobenius twist" functor $F^*: \mathscr{C} \to \mathscr{C}_q$, $V \mapsto V^{(l)}$, such that $L(\lambda)^{(l)} = L^q(l\lambda)$ for all $\lambda \in X_+$.

(6) There are restriction functors $\mathscr{C}_q \to \hat{\mathscr{C}}_q^1$ and $\hat{\mathscr{C}}_q^1 \to \mathscr{C}_q^1$, both of them preserve the irreducibility of any irreducible object with l-restricted highest weight.

(7) There are additive functions (the formal character functions) ch from the categories $\mathscr{C}$, $\mathscr{C}_q$ and $\hat{\mathscr{C}}_q^1$ to the group ring $\boldsymbol{Z}X$ such that

$$\mathrm{ch}(V^{(l)}) = (\mathrm{ch}\, V)^{(l)}\,, \text{ for } V \in \mathrm{Ob}(\mathscr{C})\,; \quad \mathrm{ch}\, V = \mathrm{ch}(V|_{\hat{\mathscr{C}}_q^1})\,, \text{ for } V \in \mathrm{Ob}(\mathscr{C}_q)\,,$$

and $\mathrm{ch}\, L(\lambda) = \chi(\lambda)$ for $\lambda \in X_+$.

There are many other properties these categories and functors have. It is almost impossible to list all them. Instead, we will indicate what there categories and functors are in the case of quantum groups and in the case of quantum enveloping algebras, in order that we can freely use all known results related to our set-up.

A. *The case of quantum groups*: This case is fully discussed in §2. Let G_q be $GL_q(n)$ or $SL_q(n)$, G the corresponding reductive algebraic group over k, $(G_q)_1$ the Frobenius kernel, and $(G_q)_1 T$ is the pull-back of the diagonal subgroup T of G. Then we have the following four categories.

$\mathscr{C}$ — the category of finite dimensional, rational G-modules;
$\mathscr{C}_q$ — the category of finite dimensional, rational G_q-modules;
$\hat{\mathscr{C}}_q^1$ — the category of finite dimensional, rational $(G_q)_1 T$-modules;
$\mathscr{C}_q^1$ — the category of finite dimensional, rational $(G_q)_1$-modules.

Let the "Frobenius twist" functor F^* be the pull-back via the Frobenius morphism $F: G_q \to G$ as defined in §2, and the restriction functors $\mathscr{C}_q \to \hat{\mathscr{C}}_q^1$ and $\hat{\mathscr{C}}_q^1 \to \mathscr{C}_q^1$ be the ordinary restrictions. By the theory developed in [31] and §2, all conditions above are satisfied.

B. *The case of quantum enveloping algebras*: We adhere generally to the notation of [27], [2], [3]. We remind the reader that in using the results of [2] (resp., [3]) we are often required to assume that l is a power of an odd prime p (resp., equals an odd prime p). For the simple Lie algebra $\mathfrak{g}$ with root system Φ, denote by $U(\mathfrak{g})$ its universal enveloping algebra. We can consider the "arithmetic" quantum deformation $U_q(\mathfrak{g})$ as defined by Lusztig [27], see also [2], [3]. As an algebra, $U_q(\mathfrak{g})$ is generated by elements E_i, $E_i^{(l)}$, F_i, $F_i^{(l)}$ and $K_i^{\pm 1}$ ($i=1, 2, \ldots$, rank $\mathfrak{g}$) satisfying a set of now well-known relations. Also, $U_q(\mathfrak{g})$ decomposes as $U^+U^0U^-$, where U^+ (resp., U^-) is the subalgebra generated by the E_i, $E_i^{(l)}$ (resp., F_i, $F_i^{(l)}$), and U^0 is a subalgebra generated by the $K_i^{\pm 1}$ and certain other elements, denoted $\begin{bmatrix} K_i;c \\ t \end{bmatrix}$ for $c \in \boldsymbol{Z}$ and $t \in \boldsymbol{Z}^+$ in [27], [2], [3]. It is known that X can be identified with a subgroup of the group $\mathrm{Hom}_{\mathrm{alg}}(U^0, k)$.

As in [27; Prop. 7.5], there is a "Frobenius homomorphism" F: $U_q(\mathfrak{g}) \to U(\mathfrak{g})$, which sends E_i, F_i and $K_i^{\pm 1}-1$ to 0, and sends $E_i^{(l)}$ (resp., $F_i^{(l)}$) to the root vector of the i-th simple root (resp., negative simple root). The image of U^0 is $U(\mathfrak{h})$, the universal enveloping algebra of a Cartan subalgebra of $\mathfrak{g}$. On the other hand, the subalgebra $\boldsymbol{u}$ of $U_q(\mathfrak{g})$ generated by E_i, F_i and $K_i^{\pm 1}$, which is finite dimensional, serves as the "Frobenius kernel". We also consider the subalgebra of $\hat{\boldsymbol{u}}$ generated by $\boldsymbol{u}$ and U^0.

An $U_q(\mathfrak{g})$-module V is called integral if it is a direct sum of 1-dimensional U^0-submodules and if the E_i, F_i, $E_i^{(l)}$ and $F_i^{(l)}$ act locally finitely. If, in addition, the algebra homomorphisms $U^0 \to k$ (weights) determining the 1-dimensional U^0-submodules of V are all in X, V is said to have type 1. Similar definitions are applied to $\hat{\boldsymbol{u}}$-modules, as well as to $\boldsymbol{u}$-modules if we use $\boldsymbol{u}^0 = U^0 \cap \boldsymbol{u}$ instead of U^0. Then by weights of an integral $U_q(\mathfrak{g})$-module or a $\hat{\boldsymbol{u}}$-module V we mean the U^0-weights of V, and the formal character ch V is its U^0-formal character. The weights and formal character of a $U(\mathfrak{g})$-module are defined in the usual way.

As in the quantum group case, we have a list of categories:

$\mathscr{C}$ — the category of finite dimensional $U(\mathfrak{g})$-modules;
$\mathscr{C}_q$ — the category of finite dimensional, integral $U_q(\mathfrak{g})$-modules of type 1;
$\hat{\mathscr{C}}_q^1$ — the category of finite dimensional, integral $\hat{\boldsymbol{u}}$-modules of type 1;
$\mathscr{C}_q^1$ — the category of finite dimensional, integral $\boldsymbol{u}$-modules of type 1.

The "Frobenius twist" functor is the pull-back via F, and the restriction functors $\mathscr{C}_q \to \hat{\mathscr{C}}_q^1$ and $\hat{\mathscr{C}}_q^1 \to \mathscr{C}_q^1$ are the ordinary restrictions.

We now summarize some general facts concerning these various categories which will be used in this paper.

The following result for quantum linear groups is proved in [31; (7.4.1)] (together with the theory in §2). From the viewpoint of quantum enveloping algebras, see Andersen-Polo-Wen [3; Thm. 4.6].

(3.1) PROPOSITION. *The restriction to $\hat{\mathscr{C}}_q^1$ and $\mathscr{C}_q^1$ of an injective object in $\mathscr{C}_q$ remains injective.*

A proof of the following elementary result for quantum group $GL_q(n)$ is given by Chen [5]. It holds for $SL_q(n)$. For the representations of quantum enveloping algebras, the result is due to Andersen-Polo-Wen [3; Thm. 3.4]. (In fact, the proof given in [3] applies equally well in the case of quantum groups.)

(3.2) PROPOSITION. *For any object in $\mathscr{C}_q$, the $\mathscr{C}_q$-socle of V equals the $\hat{\mathscr{C}}_q^1$-socle and $\mathscr{C}_q^1$-socle of V.*

For $\lambda \in X$, we denote by $\mathscr{C}_q(\lambda)$ the full subcategory of $\mathscr{C}_q$ consisting of objects whose composition factors have highest weights in $W_l \cdot \lambda \cap X_+$, and denote by $\hat{\mathscr{C}}_q^1(\lambda)$ the full subcategory of $\hat{\mathscr{C}}_q^1$ consisting of objects whose composition factors have highest weights in $W_l \cdot \lambda$. Then $\mathscr{C}_q$ (resp., $\hat{\mathscr{C}}_q^1$) is the "direct sum" of $\mathscr{C}_q(\lambda)$'s (resp., $\hat{\mathscr{C}}_q^1(\lambda)$'s), λ running over $\bar{C}_l \cap X$. Moreover, we have the following result.

(3.3) PROPOSITION. *Assume that $l \geq h$, and fix $\lambda \in X_+ \cap C_l$. Then*

(1) *$\mathscr{C}_q(\lambda)$ is a highest weight category with weight poset $(W_l \cdot \lambda \cap X_+, \uparrow)$. It satisfies conditions* (1.1)–(1.4) *and has irreducible* (*resp., induced, Weyl*) *objects $L^q(\tau)$* (*resp., $A^q(\tau)$, $V^q(\tau)$*) *for $\tau \in W_l \cdot \lambda \cap X_+$.*

(2) *Similarly, $\hat{\mathscr{C}}_q^1(\lambda)$ is a highest weight category weight poset $(W_l \cdot \lambda, \uparrow)$. It also satisfies all the conditions* (1.1)–(1.4) *and has irreducible* (*resp., induced, Weyl*) *objects $L^q(\tau)$* (*resp., $A_1^q(\tau)$, $V_1^q(\tau)$*) *for $\tau \in W_l \cdot \lambda$.*

PROOF. For $GL_q(n)$, (1) (resp., (2)) follows easily from [31; (10.4.8)] (resp., [31; (9.8.3)]) together with the linkage principle for $\mathscr{C}_q$ (resp., $\hat{\mathscr{C}}_q^1$) proved in [31; (10.3.5)] (resp., (2.8)). For $SL_q(n)$, we can then use the discussion of §2. Note that the argument used in the proof of [31; (10.4.8)] works also for quantum enveloping algebras (see also [11; Thm. 3.2]). This, together with [2; Thm. 8.1], gives (1) for quantum enveloping algebras. Finally, by [3; Lemma 4.10 and Prop. 2.9], we also have (2) for quantum enveloping algebras. $\square$

Let $\mathscr{P}_{x,y}$ be the classical Kazhdan-Lusztig polynomial associated to $x, y \in W_l$ (as defined in [23]) or, more generally, associated to $x, y \in \tilde{W}_l$ (as defined in [21]). In this paper, it is convenient to regard the Kazhdan-Lusztig polynomials as polynomials in $t = q^{1/2}$, where q is the variable in [23]. (Thus, in (3.4) below, we evaluate our polynomials at $t = -1$.) Recall that Lusztig [27] has conjectured, in the context of quantum enveloping algebras, that for $\lambda \in \mathscr{C}_l \cap X_+$, we have, for any $w \in W_l$ with $w \cdot \lambda \in X_+$,

$$\operatorname{ch} L^q(w \cdot \lambda) = \sum_{y \in W_l} (-1)^{\ell(y) - \ell(w)} \mathscr{P}_{yw_0, ww_0}(-1) \operatorname{ch} V^q(y \cdot \lambda) . \tag{3.4}$$

In (3.4), the summation is over all $y \in W_l$ such that $yw_0 \leq ww_0$ and $y \cdot \lambda \in X_+$. (Here $w_0 \in W$ is the long word in the Weyl group W.) It follows easily from [20] (together with the theory of q-Schur algebras developed in [31; Chapt. 11]) that the validity of (3.4) in type A_{n-1} is equivalent to the analogous statement for the quantum group $SL_q(n)$. (See also results of Lin [25] for the relationship between the representation theory of the quantum enveloping algebra in type A_{n-1} and that of $SL_q(n)$.) Also, by Theorem (2.7), the character formula (3.4) is equivalent to the analogous statement for $GL_q(n)$.

The following result is due to Cline-Parshall-Scott, see [8; (5.8)], [10; §5]. (These results generally assume that the highest weight category has a finite weight poset. However, since X_+^{reg} is bounded below, the results clearly generalize to the present situation.)

(3.5) THEOREM. *Assume that $l \geq h$, and fix $\lambda \in \mathscr{C}_l \cap X_+$. Let Λ be an ideal in the poset $(W_l \cdot \lambda \cap X_+, \uparrow)$, and denote by $\mathscr{C}_q(\lambda)[\Lambda]$ the full subcategory of $\mathscr{C}_q(\lambda)$ consisting of objects whose composition factors have highest weights in Λ. Then the following statements are equivalent.*

(1) *The category $\mathscr{C}_q(\lambda)[\Lambda]$ has a Kazhdan-Lusztig theory* (*as defined in* §1).

(2) *For any $w \cdot \lambda \in \Lambda$, the character formula* (3.4) *is valid.*

Furthermore, when either of these conditions hold, the Kazhdan-Lusztig polynomials $P_{y\cdot\lambda, w\cdot\lambda}$ (*in the sense of* (1.9)) *are given in terms of the classical Kazhdan-Lusztig polynomials by the rule*

$$P_{y\cdot\lambda, w\cdot\lambda} = \mathscr{P}_{w_0 y, w_0 w}\,.$$

Also, for $y \cdot \lambda$, $w \cdot \lambda \in \Lambda$, we have (*by* (1.11))

$$\sum_n \dim \operatorname{Ext}^n_{\mathscr{C}_q}(L^q(y\cdot\lambda), L^q(w\cdot\lambda))t^n = \sum_{\substack{z\in W_l\\ z\cdot\lambda\in\Lambda}} t^{\ell(y)+\ell(w)-2\ell(z)} \overline{\mathscr{P}}_{w_0 z, w_0 y} \overline{\mathscr{P}}_{w_0 z, w_0 w}\,.$$

A straightforward argument, given in [10; Prop. 5.7], shows that in formula (3.4) one can replace the terms $\mathscr{P}_{yw_0, ww_0}(-1)$ by $\mathscr{P}_{w_0 y, w_0 w}(-1)$. Thus, the description of the $P_{y\cdot\lambda, w\cdot\lambda}$ given in (3.5) and the formula (1.9) are consistent with the character formula (3.4) above. Also, we adopt the convention that $\mathscr{P}_{y,w} = 0$ if $y \nleq w$.

We observe that in [24], Kazhdan-Lusztig announce a proof of the formula (3.4). Thus, for any $\lambda \in \mathscr{C}_l \cap X_+$, the category $\mathscr{C}_q(\lambda)$ has a Kazhdan-Lusztig theory by Theorem (3.5). Because the full details of the proof of [24] are not yet available, we often prefer in certain results in the next two sections to regard the condition that $\mathscr{C}_q(\lambda)$ has a Kazhdan-Lusztig theory as a *hypothesis*. Observe that under the hypothesis that (3.4) holds for all regular weights, Theorem (4.2), Corollary (4.3), and Theorem (5.2) below all give, in view of Theorem 3.5 above, explicit cohomology calculations in $\mathscr{C}_q^1$ and $\hat{\mathscr{C}}_q^1$.

(3.6) NOTATIONAL CONVENTION. Let $\mathscr{D}$ be one of the categories $\mathscr{C}$, $\mathscr{C}_q$, $\hat{\mathscr{C}}_q^1$ or $\mathscr{C}_q^1$. There is a trivial object $k \in \mathrm{Ob}(\mathscr{D})$. (That is, there is a 1-dimensional vector space on which a module structure over a Hopf algebra is given by the augmentation map, or a comodule structure over a Hopf algebra is given by the unit map. In our case this trivial object is exactly the irreducible object with 0 highest weight.) We will denote by $H^i(\mathscr{D}, V)$ the group $\operatorname{Ext}^i_{\mathscr{D}}(k, V)$ for any object V of $\mathscr{D}$.

4. Cohomology of $\mathscr{C}_q^1$. We begin with the following lemma which follows immediately from Propositions (3.1) and (3.2).

(4.1) LEMMA. *A minimal injective resolution for an object V in $\mathscr{C}_q$ remains a minimal injective resolution for V viewed as an object in $\hat{\mathscr{C}}_q^1$ or $\mathscr{C}_q^1$.*

Recall that for objects A and B in $\mathscr{C}_q^1$, $\mathrm{Ext}^i_{\mathscr{C}_q^1}(A, B) \cong H^i(\mathscr{C}_q^1, B \otimes A^*)$ is canonically an object in $\mathscr{C}$. (Clearly, we only need to consider the case $i=0$. For quantum linear groups, see [31; (2.8.2) and (2.11.1)]; for quantum enveloping algebras, this is almost trivial.) The following theorem gives the $\mathscr{C}$-object structure for certain Ext groups.

(4.2) THEOREM. *For any object V in $\mathscr{C}_q$ and any $\lambda \in X_l$, we have the following isomorphism in $\mathscr{C}$:*

$$\mathrm{Ext}^i_{\mathscr{C}_q^1}(L_1^q(\lambda), V) \cong \bigoplus_{\tau \in X_+} \mathrm{Ext}^i_{\mathscr{C}_q}(L^q(\lambda + l\tau), V) \otimes L(\tau),$$

where $\mathrm{Ext}^i_{\mathscr{C}_q}(L^q(\lambda + l\tau), V)$ is regarded as a trivial object in $\mathscr{C}$.

PROOF. Let $0 \to V \to I^\cdot$ be the minimal $\mathscr{C}_q$-injective resolution of V, which is also the minimal $\mathscr{C}_q^1$-injective resolution of V, by Lemma (4.1). Clearly, the minimality of the resolution and the irreducibility of $L^q(\lambda)$ and $L_1^q(\lambda)$ imply that both the complexes $0 \to \mathrm{Hom}_{\mathscr{C}_q}(L^q(\lambda), I^\cdot)$ and $0 \to \mathrm{Hom}_{\mathscr{C}_q^1}(L^q(\lambda), I^\cdot)$ have zero maps as their differentials. Thus,

$$\begin{aligned} \mathrm{Ext}^i_{\mathscr{C}_q^1}(L_1^q(\lambda), V) &\cong \mathrm{Hom}_{\mathscr{C}_q^1}(L_1^q(\lambda), I^i) \\ &\cong \bigoplus_\tau \mathrm{Hom}_{\mathscr{C}_q}(L^q(\lambda + l\tau), I^i) \otimes L(\tau) \\ &\cong \bigoplus_\tau \mathrm{Ext}^i_{\mathscr{C}_q}(L^q(\lambda + l\tau), V) \otimes L(\tau), \end{aligned}$$

as required. □

In many problems, we are mainly interested in the special case in which $V = L^q(\mu)$ for a dominant $\mu \in X_+$, and both λ, μ are in the $\tilde{W}_l$-orbit (under the dot action) of 0. In this situation, we can strengthen the above result to the following corollary.

(4.3) COROLLARY. *Suppose $l > h$. Let $w, w' \in W$ and $\mu \in X$ with $w' \cdot 0 + l\mu \in X_+$. Then we have the following $\mathscr{C}$-isomorphism*

$$\mathrm{Ext}^i_{\mathscr{C}_q^1}(L_1^q(w \cdot 0), L^q(w' \cdot 0 + l\mu)) \cong \bigoplus_{\substack{\tau \in X_+ \\ \lambda - \mu + \tau \in Q}} \mathrm{Ext}^i_{\mathscr{C}_q}(L^q(w \cdot 0 + l\lambda + l\tau), L^q(w' \cdot 0 + l\mu)) \otimes L(\tau),$$

where $\lambda \in X$ is uniquely determined by the condition that $w \cdot 0 + l\lambda \in X_l$. In particular,

$$H^i(\mathscr{C}_q^1, k) \cong \bigoplus_{\tau \in Q \cap X_+} \mathrm{Ext}^i_{\mathscr{C}_q}(L^q(l\tau), k) \otimes L(\tau).$$

PROOF. Since $L_1^q(w \cdot 0) \cong L^q(w \cdot 0 + l\lambda)$, we need only to show, by Theorem (4.2), that $\mathrm{Ext}^i_{\mathscr{C}_q}(L^q(w \cdot 0 + l\lambda + l\tau), L^q(w' \cdot 0 + l\mu)) \neq 0$ for some i implies $\lambda - \mu + \tau \in Q$. The linkage principle ([31; (10.3.5)] and [2; §8]) shows that, if $\mathrm{Ext}^i_{\mathscr{C}_q}(L^q(w\cdot 0+l\lambda+l\tau), L^q(w'\cdot 0+l\mu))\neq 0$, then $w\cdot 0+l\lambda+l\tau=z\cdot(w'\cdot 0+l\mu)+l\zeta$ for some $z\in W$ and $\zeta\in Q$. That is,

$$l(w^{-1}\lambda+w^{-1}\tau-w^{-1}z\mu-w^{-1}\zeta)=w^{-1}zw'\cdot 0\,.$$

This in turn implies that $w^{-1}zw'=1$ and $w^{-1}\lambda+w^{-1}\tau-w^{-1}z\mu-w^{-1}\zeta=0$, since from the condition $l>h$ we obtain that $|(y\cdot 0,\alpha^\vee)|<l$ for any $y\in W$ and $\alpha\in\Pi$. Thus,

$$\lambda-\mu+\tau=(\lambda-w^{-1}\lambda)-(\mu-w^{-1}z\mu)+(\tau-w^{-1}\tau)+w^{-1}\zeta\in Q\,,$$

as required. □

It is interesting that, assuming the character formula (3.4), well-known identities involving Kazhdan-Lusztig polynomials readily yield closed formulas for certain cohomology groups. We illustrate this with the following result, which uses the Kato-Lusztig interpretation of the q-analogue of Kostant's weight multiplicity formula.

(4.4) THEOREM. *Assume that $l>h$ and assume that the character formula* (3.4) *is valid for all regular dominant weights. Fix $z\in W$ and $\lambda\in X$ so that $z\cdot 0+l\lambda\in X_+$. Then for any integer i the $\mathscr{C}$-object $H^i(\mathscr{C}^1_q, A(z\cdot 0+l\lambda))$ has character given by*

$$\mathrm{ch}(H^i(\mathscr{C}^1_q, A^q(z\cdot 0+l\lambda)))=\sum_{\mu\in X_+}\sum_{w\in W}(-1)^{\ell(w)}\wp_{(i-\ell(z))/2}(w\cdot\mu-\lambda)\chi(\mu)\,.$$

PROOF. For a dominant weight $z\cdot 0+l\lambda$, we have

$$\mathrm{Ext}^i_{\mathscr{C}^1_q}(k, A^q(z\cdot 0+l\lambda))\cong\bigoplus_{\mu\in X_+}\mathrm{Ext}^i_{\mathscr{C}_q}(L^q(l\mu), A^q(z\cdot 0+l\lambda))\otimes L(\mu) \tag{4.4.1}$$

as an object in $\mathscr{C}$, by Theorem (4.2). Also, since $l>h$, $z\cdot 0+l\lambda\in X_+$ implies that $\lambda\in X_+$. Using the formula for the length function on X given in "List of Notation", together with (1.6) and (1.9), we conclude that

$$\sum_i\dim\mathrm{Ext}^i_{\mathscr{C}_q}(L^q(l\mu), A^q(z\cdot 0+l\lambda))t^i=p_{z\cdot 0+l\lambda,l\mu}=t^{2(\mu-\lambda,\rho^\vee)+\ell(z)}\bar{P}_{z\cdot 0+l\lambda,l\mu}\,. \tag{4.4.2}$$

Since $\ell(w_0t_{l\mu})=\ell(w_0)+\ell(t_{l\mu})$, the recursive relations [23; (2.3.g)] for Kazhdan-Lusztig polynomials, together with Theorem (3.5), give

$$P_{z\cdot 0+l\lambda,l\mu}=\mathscr{P}_{w_0zt_{lz^{-1}\lambda},w_0t_{l\mu}}=\mathscr{P}_{t_{lz^{-1}\lambda},w_0t_{l\mu}}=\mathscr{P}_{t_{l\lambda}z,w_0t_{l\mu}}\,. \tag{4.4.3}$$

If $\xi\mapsto\xi^*\equiv-w_0(\xi)$ is the opposition involution on X, $w_0t_{l\mu}=t_{-l\mu^*}w_0$. An easy application of the length formula in the "List of Notation" gives, since μ^* is dominant, that $\ell(t_{-l\mu^*})=\ell(t_{l\mu})$. Hence

$$\ell(t_{-l\mu^*}w_0)=\ell(t_{-l\mu^*})+\ell(w_0)\,,$$

so that

$$\ell(w_0t_{l\mu}s)<l(w_0t_{l\mu})$$

for all simple reflections $s \in W_l$. Therefore, by [23; (2.3.g)] again, we have that

$$(4.4.4) \qquad \mathscr{P}_{t_{l\lambda}z,\, w_0 t_{l\mu}} = \mathscr{P}_{t_{l\lambda},\, w_0 t_{l\mu}} = \mathscr{P}_{w_0 t_{l\lambda},\, w_0 t_{l\mu}}\,.$$

By [21; Thm. 1.8] (together with the length formula in the "List of Notation", which establishes that, for $\xi \in X_+$, the longest word w_ξ in $W t_{l\xi} W$ is $w_0 t_{l\xi}$),

$$(4.4.5) \qquad \bar{\mathscr{P}}_{w_0 t_{l\lambda},\, w_0 t_{l\mu}} = t^{2(\lambda-\mu,\rho^\vee)} \sum_i \sum_{w \in W} (-1)^{l(w)} \wp_i(w \cdot \mu - \lambda) t^{2i}$$

for $\lambda \le \mu$. Observe that if $\lambda \not\le \mu$, then $w_0 t_{l\lambda} \not\le w_0 t_{l\mu}$ (see [29; p. 210]), so, by our conventions, the left hand side of (4.4.5) vanishes. But the right hand side of (4.4.5) vanishes trivially when $\lambda \not\le \mu$, so it follows that (4.4.5) holds for all $\lambda, \mu \in X_+$. If M is a finite dimensional rational G-module, let $[M : L(\tau)]$ denote the multiplicity of $L(\tau)$ as a composition factor of M. By (4.4.1)–(4.4.5), we have

$$\sum_i [\mathrm{Ext}^i_{\mathscr{C}_q^1}(k, A^q(z \cdot 0 + l\lambda)) : L(\mu)] t^i = t^{2(\mu-\lambda,\rho^\vee)+\ell(z)} \bar{\mathscr{P}}_{w_0 t_{l\lambda},\, w_0 t_{l\mu}}$$

$$= \sum_i \sum_{w \in W} (-1)^{\ell(w)} \wp_{(i-\ell(z))/2}(w \cdot \mu - \lambda) t^i\,.$$

□

Let $\mathscr{D} = \mathscr{C}, \mathscr{C}_q$ or $\hat{\mathscr{C}}_q^1$, and denote by $\mathscr{D}^{\mathrm{grd}}$ the category in which objects are graded vector spaces $A = \bigoplus_{i \in \boldsymbol{Z}^+} A_i$ with $A_i \in \mathrm{Ob}(\mathscr{D})$ for all i and morphisms are homogeneous homomorphisms $\varphi = \sum_{i \in \boldsymbol{Z}^+} \varphi_i$ of degree 0 such that each φ_i is a morphisms in $\mathscr{D}$. Define the *graded formal character* of $A = \bigoplus_{i \in \boldsymbol{Z}^+} A_i \in \mathrm{Ob}(\mathscr{D}^{\mathrm{grd}})$ to be the element in the group algebra $\boldsymbol{Z}[[t]][X]$ of X over the ring of formal power series over $\boldsymbol{Z}$ in a variable t given by

$$\mathrm{CH}(A) = \sum_{i \in \boldsymbol{Z}^+} \mathrm{ch}(A_i) t^i\,.$$

Let $\mathfrak{g}$ be the semisimple (split) Lie algebra over k having root system Φ, and G the corresponding algebraic group. Let $\mathscr{N} \equiv \mathscr{N}(\mathfrak{g})$ be the G-variety of nilpotent elements in $\mathfrak{g}$. It is well-known that $k[\mathscr{N}]$, as a G-algebra, is isomorphic to $S^\bullet(\mathfrak{g}^*)/I$, where I is the ideal in the symmetric algebra $k[\mathfrak{g}] = S^\bullet(\mathfrak{g}^*)$ generated by the G-invariants $S^+(\mathfrak{g}^*)^G$ (i.e., G-invariants having 0 constant terms). Since the ideal I is homogeneous, $k[\mathscr{N}]$ is graded as a rational G-module. That is, it is an object in $\mathscr{C}^{\mathrm{grd}}$. It is also known that $\mathrm{CH}(k[\mathscr{N}])$ can be expressed in terms of the "i-part" partition function as follows ([16]):

$$(4.5) \qquad \mathrm{CH}(k[\mathscr{N}]) = \sum_{\substack{\mu \in X_+ \\ w \in W}} (-1)^{\ell(w)} \wp_i(w \cdot \mu) \chi(\mu) t^i\,.$$

As an immediate application of (4.4), (4.5) and (A.3.5) we make the following observation.

(4.6) Corollary. *The graded character of $H^\bullet(\mathscr{C}_q^1, k) \in \mathrm{Ob}(\mathscr{C}^{\mathrm{grd}})$ is given by*

$$\mathrm{CH}(H^{\bullet}(\mathscr{C}_q^1, k)) = \mathrm{CH}(k[\mathscr{N}](t^2)\,.$$

Thus, $H^{\bullet}(\mathscr{C}_q^1, k)$ *is a commutative, graded algebra concentrated in even degrees.*

(4.7) REMARKS. (a) In view of (4.4), it is natural to define the *cohomology variety* of $\mathscr{C}_q$ to be the scheme $\mathrm{Spec}(H^{\bullet}(\mathscr{C}_q^1, k))$. By (4.4), $H^2(\mathscr{C}_q^1, k) \cong \mathfrak{g} \cong \mathfrak{g}^*$ as an object in $\mathscr{C}$. Grade the symmetric algebra $S^{\bullet}(\mathfrak{g}^*)$ by giving $\mathfrak{g}^*$ homogeneous degree 2. Therefore, any $\mathscr{C}$-isomorphism $\Psi^2 : \mathfrak{g} \to H^2(\mathscr{C}_q^1, k)$ extends to a $\mathscr{C}^{\mathrm{grd}}$-morphism

$$\Psi^{\bullet} : S^{\bullet}(\mathfrak{g}^*) \to H^{\bullet}(\mathscr{C}_q^1, k)$$

of commutative $\mathscr{C}^{\mathrm{grd}}$-algebras. Since $\mathrm{Ext}^n_{\mathscr{C}_q}(k, k) = 0$ for $n > 0$ (see [31; (10.4.6)], where the argument works also for the case of quantum enveloping algebras; see also [2; (9.9)]), Corollary (4.3) implies that $H^n(\mathscr{C}_q^1, k)^G = 0$ for all such n. Thus, $\Psi^{\bullet}$ factors through the surjective map $S^{\bullet}(\mathfrak{g}) \to k[\mathscr{N}]$ to induce a $\mathscr{C}^{\mathrm{grd}}$-homomorphism

$$\bar{\Psi}^{\bullet} : k[\mathscr{N}] \to H^{\bullet}(\mathscr{C}_q^1, k)\,.$$

It seems likely that $\bar{\Psi}^{\bullet}$ is an isomorphism of $\mathscr{C}^{\mathrm{grd}}$-algebras.

(b) Let M be an object in $\mathscr{C}_q^1$. As discussed in §A.4, the natural algebra homomorphism $k \to \mathrm{End}(M) \cong M^* \otimes M$ defines an algebra homomorphism

$$\Psi_M : S^{\bullet}(\mathfrak{g}^*) \to H^{\bullet}(\mathscr{C}_q^1, \mathrm{End}(M))\,,$$

obtained by composing $\Psi^{\bullet}$ above with the natural algebra homomorphism $H^{\bullet}(\mathscr{C}_q^1, k) \to H^{\bullet}(\mathscr{C}_q^1, \mathrm{End}(M))$, see (A.4.2) in the Appendix. Then, mimicking [13], define, for an object M in $\mathscr{C}_q^1$ its *support variety* $|\mathfrak{g}|_M$ to be the algebraic subscheme of $\mathfrak{g} \equiv \mathrm{Spec}\, S^{\bullet}(\mathfrak{g}^*)$ defined by the ideal $\mathrm{Ker}(\Psi_M)$. (To keep in strict analogy with [13], one should replace $\mathrm{Ker}(\Psi_M)$ by its radical.)

(c) The conclusion of (4.4) is inspired by an analogous result for the cohomology of the Frobenius kernel of a reductive group in characteristic p proved in [1; 3.8] (under a sometimes restrictive condition on λ). Relative to (4.6), see [1] as well as in [12].

(d) Assuming that (3.4) holds, (4.2) yields explicit determinations of the $\mathrm{Ext}^{\bullet}$ groups between irreducible objects in $\mathscr{C}_q^1$, in view of (3.5). In particular, we have the following explicit description of Ext^1 groups for $\mathscr{C}_q^1$. Suppose $w, y \in W$ and $\xi, \eta \in X$ are such that $w \cdot 0 + l\xi \in X_l$ and $y \cdot 0 + l\eta \in X_l$. Then, as an object in $\mathscr{C}$, we have

$$\mathrm{Ext}^1_{\mathscr{C}_q^1}(L_1^q(w \cdot 0), L_1^q(y \cdot 0)) \cong \bigoplus_{\substack{\theta \\ \theta - \xi \in X_+ \\ \theta - \eta \in Q}} L(\theta - \xi)^{\oplus \boldsymbol{\mu}(t_{-l\theta^*} w_0 w,\, t_{-l\eta^*} w_0 y)}\,,$$

where $\boldsymbol{\mu} : \tilde{W}_l \times \tilde{W}_l \to \boldsymbol{Z}$ is (the natural generalization of) the "μ"-function of [23].

5. Cohomology of $\hat{\mathscr{C}}_q^1$. Let $\mu \in X_+$ and write $\mu = \mu_0 + l\mu_1$ with $\mu_0 \in X_l$. By the tensor product theorem for $\mathscr{C}_q$ ([31; (9.4.1)] and [27; Thm. 7.4]), $L^q(\mu) \cong L^q(\mu_0) \otimes L(\mu_1)^{(1)}$. The module $L(\mu_1)^{(1)}|_{\mathscr{C}_q^1}$ is a direct sum of 1-dimensional objects of the form

$\hat{L}_1^q(l\zeta)=l\zeta$ with $\zeta\in X$. The multiplicity of $l\zeta$ is the multiplicity of weight ζ in $L(\mu_1)$, which in turn equals

$$m_{\mu_1}(\zeta)=\sum_{w\in W}(-1)^{\ell(w)}\wp(w\cdot\mu_1-\zeta)\,.$$

We obtain the following result. (Recall that if $\lambda=\lambda_0+l\lambda_1$ with $\lambda_0\in X_l$, then $\hat{L}_1^q(\lambda)\cong L^q(\lambda_0)|_{\hat{\mathscr{C}}_q^1}\otimes l\lambda_1$.)

(5.1) LEMMA. *Let $\lambda=\lambda_0+l\lambda_1\in X$ and $\mu=\mu_0+l\mu_1\in X_+$ with $\lambda_0,\mu_0\in X_l$. Then*

$$\dim\operatorname{Hom}_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda),L^q(\mu))=\delta_{\lambda_0\mu_0}m_{\mu_1}(\lambda_1)\,.$$

Now we can prove the following theorem.

(5.2) THEOREM. *Let $\lambda=\lambda_0+l\lambda_1$, $\mu=\mu_0+l\mu_1\in X$ with $\lambda_0,\mu_0\in X_l$, and V be a $\mathscr{C}_q$-object. Then*

(1) $\dim\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda),V)=\sum_{\zeta\in X_+}m_\zeta(\lambda_1)\dim\operatorname{Ext}^i_{\mathscr{C}_q}(L^q(\lambda_0+l\zeta),V)\,.$

(2) $\dim\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda),\hat{L}_1^q(\mu))=\sum_{\zeta\in X_+}m_\zeta(\lambda_1-\mu_1)\dim\operatorname{Ext}^i_{\mathscr{C}_q}(L^q(\lambda_0+l\zeta),L^q(\mu_0)).$

PROOF. (1) As in the proof of (4.2), let $0\to V\to I^\bullet$ be the minimal $\mathscr{C}_q$-injective resolution of V, which is also the minimal $\hat{\mathscr{C}}_q^1$-injective resolution of V, by Lemma (4.1). Then

$$\begin{aligned}\dim\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda),V)&=\dim\operatorname{Hom}_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda),I^i)\\&=\sum_{\zeta\in X_+}\dim\operatorname{Hom}_{\mathscr{C}_q}(L^q(\lambda_0+l\zeta),I^i)\cdot\dim\operatorname{Hom}_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda),L^q(\lambda_0+l\zeta))\\&=\sum_{\zeta\in X_+}m_\zeta(\lambda_1)\dim\operatorname{Ext}^i_{\mathscr{C}_q}(L^q(\lambda_0+l\zeta),V)\,,\end{aligned}$$

as required. Note that in the last two steps Lemma (5.1) is used.

(2) Clearly, $\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda),\hat{L}_1^q(\mu))\cong\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda+l\tau),\hat{L}_1^q(\mu+l\tau))$ for any $\tau\in X$. In particular, we have

$$\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}((\hat{L}_1^q(\lambda),\hat{L}_1^q(\mu))\cong\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda-l\mu_1),L^q(\mu_0))\,.$$

Then we use (1) to deduce (2). □

Theorem (5.2) has some important consequences. One of them is the following "even-odd vanishing behavior" of the extension groups between irreducible $\hat{\mathscr{C}}_q^1$-objects.

(5.3) THEOREM. *Assume that $l\geq h$ and that the character formula* (3.4) *is valid for all regular dominant weights. Let $\lambda\in C_l\cap X_+$, and $w,w'\in W_l$. If $\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(w\cdot\lambda),\hat{L}_1^q(w'\cdot\lambda))\neq 0$, then $\ell(w)-\ell(w')\equiv i$ (mod 2).*

PROOF. By using a translation in $\mathscr{T}_l$, we may assume that $w' \cdot \lambda$ is dominant. In this case $\hat{L}_1^q(w' \cdot \lambda)$ is a $\hat{\mathscr{C}}_q^1$-direct summand of $L^q(w' \cdot \lambda)$. Therefore,

$$\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(w \cdot \lambda), \hat{L}_1^q(w' \cdot \lambda)) \neq 0 \Rightarrow \operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(w \cdot \lambda), L^q(w' \cdot \lambda)) \neq 0 .$$

By Theorem 5.2, this implies that $\operatorname{Ext}^i_{\mathscr{C}_q}(L^q(w \cdot \lambda + l\zeta), L^q(w' \cdot \lambda)) \neq 0$ and $m_{\zeta+\mu_1}(\mu_1) \neq 0$ for some ζ, where $w \cdot \lambda = \mu_0 + l\mu_1$ with $\mu_0 \in X_l$. This, in particular, implies that $\zeta \in Q$. Since, by our assumption and Theorem 3.5, $\mathscr{C}_q$ has a Kazhdan-Lusztig thoery, it follows that $\ell(t_{l\zeta} w) - \ell(w') \equiv i \pmod 2$. It is well-known that in W_l a translation has even length. Thus, $\ell(w) - \ell(w') \equiv i \pmod 2$. □

(5.4) REMARK. The same argument shows that, under the same hypothesis as in (5.3), for any $w, w' \in W_l$ with $w' \cdot \lambda \in X_+$, $\operatorname{Ext}^i_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(w \cdot \lambda), A^q(w' \cdot \lambda)) \neq 0$ implies that $\ell(w) - \ell(w') \equiv i \pmod 2$.

(5.5) THEOREM. *Assume that $l \geq h$ and that the character formula* (3.4) *is valid for all regular dominant weights. Let C and C' be a pair of adjacent l-alcoves, and let $\lambda \in C$, $\mu \in C'$ be W_l-conjugate. Then*

$$\operatorname{Ext}^1_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda), \hat{L}_1^q(\mu)) \neq 0 .$$

PROOF. As in the proof of (5.3), after applying a translation in $\tilde{\mathscr{T}}_l$, one of these alcoves, say C', can be assumed to be l-restricted, and the other one to be dominant. Thus, $\hat{L}_1^q(\mu) = L^q(\mu)|_{\hat{\mathscr{C}}_q^1}$, and $\lambda = \lambda_0 + l\lambda_1$ with $\lambda_0 \in X_l$ and $\lambda_1 \in X_+$. Then, by Theorem (5.3),

$$\dim \operatorname{Ext}^1_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda), \hat{L}_1^q(\mu)) = \sum_{\zeta \in X_+} m_\zeta(\lambda_1) \dim \operatorname{Ext}^1_{\mathscr{C}_q}(L^q(\lambda_0 + l\zeta), L^q(\mu)) .$$

Because of the assumption that $\mathscr{C}_q$ has a Kazhdan-Lusztig theory, the right hand side of the equality has at least a non-zero term—the term with $\zeta = \lambda_1$ by [8; (5.3) and (5.8)]. This proves the theorem. □

We conclude this section with the following easy result.

(5.6) PROPOSITION. *Suppose that $l \geq h$. For W_l-conjugate weights λ, μ in adjacent l-alcoves,*

$$\dim \operatorname{Ext}^1_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda), \hat{L}_1^q(\mu)) \leq 1 .$$

In particular, if the character formula (3.4) *is valid for all regular dominant weights, then*

$$\dim \operatorname{Ext}^1_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda), \hat{L}_1^q(\mu)) = 1 .$$

PROOF. We can assume that $\mu \uparrow \lambda$. Since

$$\operatorname{Hom}_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda), \hat{A}_1^q(\mu)/\hat{L}_1^q(\mu)) = 0 ,$$

the long exact sequence of cohomology yields an injection

$$\operatorname{Ext}^1_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda), \hat{L}_1^q(\mu)) \hookrightarrow \operatorname{Ext}^1_{\hat{\mathscr{C}}_q^1}(\hat{L}_1^q(\lambda), \hat{A}_1^q(\mu)) .$$

On the other hand, the standard arguments involving Jantzen translation operators (see [19; pp. 334–335]) apply to the category $\hat{\mathscr{C}}_q^1$, so that the right hand side of the above expression is 1-dimensional. This proves the first assertion of the proposition. The second assertion follows from Theorem (5.5). □

Appendix. Cohomology of comodules

In this appendix, A is a coalgebra over a field k with comultiplication $\Delta: A \to A \otimes A$ and counit $\varepsilon: A \to k$. It is also assumed that A is *unitary* in the sense that there is a coalgebra homomorphism $\kappa: k \to A$, viewing k as the trivial k-coalgebra. By abuse of notation, denote $\kappa(1)$ also by 1. We will give a brief discussion of the cohomology of A-comodules.

A.1. Cofixed points and cohomology. Since A is unitary, we can define the *cofixed point functor* $\mathscr{F}^A$, which is left exact, from the category of A-comodules to the category of k-vector spaces: For a (right) A-comodule V with structure map $\tau: V \to V \otimes A$, put

$$\mathscr{F}^A(V) = \{v \in V \mid \tau(v) = v \otimes 1\},$$

the *cofixed points* of V. Clearly, the restriction of a comodule homomorphism $V \to V'$ gives a k-linear map $\mathscr{F}^A(V) \to \mathscr{F}^A(V')$, which completes the definiton of the functor $\mathscr{F}^A$.

(A.1.1) LEMMA. (1) *The A-comodule A with comodule structure map Δ (and any direct summand of a direct sum of copies of A) is injective in the category of A-comodules*;

(2) *For an A-comodule V, its structure map $\tau: V \to V \otimes A$ is an injective A-comodule homomorphism, viewing $V \otimes A$ as an A-comodule via structure map $\mathrm{id}_V \otimes \Delta$.*

PROOF. It is easy to verify that the functor $V \mapsto V \otimes A$ from the category of k-vector spaces to the category of A-comodules (the comodule structure map of $V \otimes A$ is $\mathrm{id}_V \otimes \Delta$) is a right adjoint to the forgetful functor from the category of A-comodules to the category of k-vector spaces. Thus, a standard categorical argument proves (1). For (2), one of the axioms for defining A-comodules shows that τ is a comodule homomorphism, and the other axiom, $(\mathrm{id}_V \otimes \varepsilon) \circ \tau = \mathrm{id}_V$, ensures the injectivity of τ. □

Now the following corollary is immediate.

(A.1.2) COROLLARY. *The category of A-comodules has enough injectives.*

Because of Corollary (A.1.2), one can form the i-th right derived functor $H^i(?, A) \equiv R^i \mathscr{F}^A(?)$ for $i \geq 0$. For an A-comodule V, we call $H^i(V, A)$ the i-th *comodule cohomology* of A with coefficients in V.

A.2. Hochschild complex. It is desirable to construct a standard complex to compute $H^i(V, A)$ for an A-comodule V with structure map τ. For $n \geq 0$, let

$$I_V^n = V \otimes A_1^n \otimes A_2^n \otimes \cdots \otimes A_n^n \otimes A_{n+1}^n \quad \text{with} \quad A_i^n = A \quad \text{for } i = 1, 2, \ldots, n+1\,,$$

viewed as an A-comodule via the structure map

$$\tau_n = \mathrm{id}_V \otimes \mathrm{id}_{A_1^n} \otimes \cdots \otimes \mathrm{id}_{A_n^n} \otimes \Delta\,.$$

By (A.1.1), I_V^n is an injective A-comodule for $n \geq 0$. Define a map $d^n: I_V^n \to I_V^{n+1}$ by

$$d^n = \sum_{i=0}^{n+1} (-1)^i d_i^n\,,$$

where

$$d_i^n = \begin{cases} \tau \otimes \mathrm{id}_{A_1^n} \otimes \cdots \otimes \mathrm{id}_{A_{n+1}^n}\,, & i = 0\,; \\ \mathrm{id}_V \otimes \mathrm{id}_{A_1^n} \otimes \cdots \otimes \mathrm{id}_{A_{i-1}^n} \otimes \Delta \otimes \mathrm{id}_{A_{i+1}^n} \otimes \cdots \otimes \mathrm{id}_{A_{n+1}^n}\,, & i > 0\,. \end{cases}$$

(A.2.1) THEOREM. *The above-defined $(I_V^{\bullet}, d^{\bullet})$ with augmentation map $\tau: V \to I_V^0 = V \otimes A$ is an injective resolution for V as an A-comodules.*

PROOF. It is trivial that d_i^n for $i \leq n$ is an A-comodule homomorphism. Thanks to (A.1.1(2)), d_{n+1}^n is also an A-comodule homomorphism, and then so is d^n. Also, it is straightforward to check that $d^n \circ d^{n-1} = 0$ for $n > 0$. Thus, $(I_V^{\bullet}, d^{\bullet})$ is an A-comodule complex consisting of injective A-comodules.

To prove $(I_V^{\bullet}, d^{\bullet})$ is exact in positive degree, we define for $n > 0$ a linear map $\sigma^n: I_V^n \to I_V^{n-1}$ by

$$\sigma^n = (-1)^n \mathrm{id}_V \otimes \mathrm{id}_{A_1^n} \otimes \cdots \otimes \mathrm{id}_{A_n^n} \otimes \varepsilon\,.$$

Clearly, for $n > 0$ and $i \leq n$, $\sigma^{n+1} \circ d_i^n = -d_i^{n-1} \circ \sigma^n$. Also, $\sigma^{n+1} \circ d_{n+1}^n = \mathrm{id}_{I_V^n}$, by the axioms for coalgebras. Therefore,

$$\sigma^{n+1} \circ d^n + d^{n-1} \circ \sigma^n = \mathrm{id}_{I_V^n} \qquad \text{for} \quad n > 0\,,$$

which proves the required exactness. Finally, it is easy to verify the exactness of

$$0 \longrightarrow V \xrightarrow{\tau} I_V^0 \xrightarrow{d^0} I_V^1\,.$$

The theorem thus has been proved. □

Now we let $C^n(V, A) = \mathscr{F}^A(I_V^n)$. Obviously, $\mathscr{F}^A(A) = k$. Hence,

$$C^n(V, A) = V \otimes A_1^n \otimes A_2^n \otimes \cdots \otimes A_n^n \qquad \text{with} \quad A_i^n = A \quad \text{for } i = 1, 2, \ldots, n\,.$$

It is easy to see that the differential $\partial^n: C^n(V, A) \to C^{n+1}(V, A)$ is defined by

$$\partial^n = \sum_{i=0}^{n+1} (-1)^i \partial_i^n\,,$$

where

$$\partial_i^n = \begin{cases} \tau \otimes \mathrm{id}_{A_1^n} \otimes \cdots \otimes \mathrm{id}_{A_n^n}, & i=0; \\ \mathrm{id}_V \otimes \mathrm{id}_{A_1^n} \otimes \cdots \otimes \mathrm{id}_{A_{i-1}^n} \otimes \Delta \otimes \mathrm{id}_{A_{i+1}^n} \otimes \cdots \otimes \mathrm{id}_{A_n^n}, & 0<i\le n; \\ \mathrm{id}_V \otimes \mathrm{id}_{A_1^n} \otimes \cdots \otimes \mathrm{id}_{A_n^n} \otimes \kappa, & i=n+1. \end{cases}$$

The complex $(C^{\bullet}(V, A), \partial^{\bullet})$ is called the *Hochschild complex* of A with coefficients in V. By the construction, $H^n(V, A)$ is the n-th cohomology of the complex.

A.3. Cup product. The comodule cohomology of A with coefficients in the trivial A-comodule k plays an important rôle in the cohomology theory of A. One reason for this is that we can define a product on the vector space $\mathscr{H}(A) \equiv \bigoplus_n H^n(k, A)$, making it into a graded k-algebra. To do this, we form the tensor product of the Hochschild complex with itself. By definition, the tensor product of a complex $(X^{\bullet}, \delta^{\bullet})$ with itself it the complex $(\tilde{X}^{\bullet}, \tilde{\delta}^{\bullet})$ with

$$\tilde{X}^n = \bigoplus_{s+t=n} X^s \otimes X^t$$

and

$$\tilde{\delta}^{s+t}|_{X^s \otimes X^t} = \delta^s \otimes \mathrm{id}_{X^t} + (-1)^s \mathrm{id}_{X^s} \otimes \delta^t .$$

Note that one can identify $C^s \otimes C^t$ with C^{s+t} canonically (where, and in the sequel, we denote $C^n(k, A)$ simply by C^n). Thus, we obtain a linear map $\theta^n : \tilde{C}^n \to C^n$.

(A.3.1) LEMMA. *The above θ^n's give a cochain map $\theta^{\bullet} : (\tilde{C}^{\bullet}, \tilde{\partial}^{\bullet}) \to (C^{\bullet}, \partial^{\bullet})$.*

Thanks to the Künneth theorem, there is a canonical isomorphism

$$H^n(\tilde{C}^{\bullet}, \tilde{\partial}^{\bullet}) \cong \bigoplus_{s+t=n} H^s(k, A) \otimes H^t(k, A) .$$

Therefore, the cochain map $\theta^{\bullet}$ induces a product $\bar{\theta}$: $H^s(k, A) \otimes H^t(k, A) \to H^{s+t}(k, A)$, which is clearly associative. This product is called the *cup product* of $\mathscr{H}(A)$, and the cup product of $x \in H^s(k, A)$ and $y \in H^t(k, A)$ is usually denoted by $x \cup y$.

Next, we will give a second interpretation of the cup product.

Consider two unitary coalgebras A and B and a unit-preserving coalgebra homomorphism $\varphi : A \to B$. Then an A-comodule V can be given a B-comodule structure via φ, and $\mathscr{F}^A(V) \subset F^B(V)$. Moreover, a standard homological algebra argument shows that there is a natural homomorphism $\bar{\varphi}$: $H^n(V, A) \to H^n(V, B)$ induced by φ. Now suppose that we have a unit-preserving coalgebra homomorphism $\varphi : A \otimes A \to A$. Then we obtain a linear map $H^n(k, A \otimes A) \to H^n(k, A)$. On the other hand, a standard homological argument using Künneth theorem shows that

$$H^n(k, A \otimes A) \cong \bigoplus_{s+t=n} H^s(k, A) \otimes H^t(k, A) .$$

Therefore, φ induces a "product" $\bar{\varphi} : H^s(k, A) \otimes H^t(k, A) \to H^{s+t}(k, A)$. Generally speaking, $\bar{\varphi}$ will vary when φ varies. However, if we strengthen the condition "φ preserves the unit" to the condition

$$\varphi(1\otimes y)=y=\varphi(y\otimes 1) \qquad \text{for all} \quad y\in A\,, \tag{A.3.2}$$

then we have the following surprising result.

(A.3.3) THEOREM. *Suppose that* $\varphi: A\otimes A\to A$ *is a coalgebra homomorphism satisfying* (A.3.2). *Then the map* $\bar{\varphi}: H^s(k, A)\otimes H^t(k, A)\to H^{s+t}(k, A)$ *induced by* φ *is the cup product.*

PROOF. We use the standard resolution $I^\bullet\equiv I_k^\bullet$ given in Theorem (A.2.1). Define a linear map $\psi^{s,t}: I^s\otimes I^t\to I^{s+t}$ by sending $(x_1\otimes\cdots\otimes x_s\otimes x)\otimes(y_1\otimes\cdots\otimes y_t\otimes y)(x, x_i, y, y_j\in A)$ to

$$x_1\otimes\cdots\otimes x_s\otimes\sum_x x_{(1)}y_1\otimes\cdots\otimes x_{(t)}y_t\otimes x_{(t+1)}y\,.$$

Here we use Sweedler's notation [32], i.e. let

$$(\Delta\otimes\mathrm{id}_{A^{\otimes(r-1)}})\circ\cdots\circ(\Delta\otimes\mathrm{id}_{A\otimes A})\circ(\Delta\otimes\mathrm{id}_A)\circ\Delta(x)=\sum_x x_{(1)}\otimes x_{(2)}\otimes\cdots\otimes x_{(r+1)}$$

for $x\in A$ and $r>1$. Also, we denote $\varphi(x\otimes y)$ by xy for simplicity.

Clearly, in order to verify that $\psi^{s,t}$ is an A-comodule homomorphism, it is enough to do this for the case $s=0$. Let τ be the A-comodule structure map on $I^0\otimes I^t$ induced by φ, and σ the A-comodule structure map on I^t. Then

$$\begin{aligned}
&((\psi^{0,t}\otimes\mathrm{id}_A)\circ\tau)(x\otimes y_1\otimes\cdots\otimes y_t\otimes y)\\
&=(\psi^{0,t}\otimes\mathrm{id}_A)\Big(\sum_{x,y} x_{(1)}\otimes y_1\otimes\cdots\otimes y_t\otimes y_{(1)}\otimes x_{(2)}y_{(2)}\Big)\\
&=\sum_{x,y} x_{(1)}y_1\otimes\cdots\otimes x_{(t)}y_t\otimes x_{(t+1)}y_{(1)}\otimes x_{(t+2)}y_{(2)}\\
&=\sigma\Big(\sum_x x_{(1)}y_1\otimes\cdots\otimes x_{(t)}y_t\otimes x_{(t+1)}y\Big)\\
&=(\sigma\circ\psi^{0,t})(x\otimes y_1\otimes\cdots\otimes y_t\otimes y)\,.
\end{aligned}$$

Thus, $\psi^{s,t}$ is an A-comodule homomorphism.

Define a map $\psi^n: \tilde{I}^n\to I^n$ by $\psi^n|_{I^s\otimes I^t}=\psi^{s,t}$. We claim that $\psi^\bullet$ is a cochain map from $(\tilde{I}^\bullet, \tilde{d}^\bullet)$ to $(I^\bullet, d^\bullet)$. As above, we need only to consider elements in $I^0\otimes I^t$. We have

$$(d^0\otimes\mathrm{id}_{I^t}+\mathrm{id}_{I^0}\otimes d^t)(x\otimes y_1\otimes\cdots\otimes y_t\otimes y_{t+1})=X_1-X_2+Y_1+Y_2$$

with

$$X_1=1\otimes x\otimes y_1\otimes\cdots\otimes y_{t+1}\,,$$

$$X_2=\sum_x x_{(1)}\otimes x_{(2)}\otimes y_1\otimes\cdots\otimes y_{t+1}\,,$$

which are elements in $I^1 \otimes I^t$, and

$$Y_1 = x \otimes 1 \otimes y_1 \otimes \cdots \otimes y_{t+1},$$

$$Y_2 = \sum_{i=1}^{t+1} \sum_{y_i} (-1)^i x \otimes y_1 \otimes \cdots \otimes y_{i-1} \otimes y_{i,(1)} \otimes y_{i,(2)} \otimes y_{i+1} \otimes \cdots \otimes y_{t+1},$$

which are elements in $I^0 \otimes I^{t+1}$. Note that

$$\psi^{1,t}(X_2) = \psi^{0,t+1}(Y_1) = \sum_x x_{(1)} \otimes x_{(2)} y_1 \otimes \cdots \otimes x_{(t+2)} y_{t+1}.$$

Thus we obtain that

$$\begin{aligned}
&(\psi^{t+1} \circ (d^0 \otimes \mathrm{id}_{I^t} + \mathrm{id}_{I^0} \otimes d^t))(x \otimes y_1 \otimes \cdots \otimes y_t \otimes y_{t+1}) \\
&= \sum_x 1 \otimes x_{(1)} y_1 \otimes \cdots \otimes x_{(t+1)} y_{t+1} + \sum_{i=1}^{t+1} \sum_{x, y_i} (-1)^i x_{(1)} y_1 \otimes \cdots \\
&\quad \otimes x_{(i-1)} y_{i-1} \otimes x_{(i)} y_{i,(1)} \otimes x_{(i+1)} y_{i,(2)} \otimes x_{(i+2)} y_{i+1} \otimes \cdots \otimes x_{(t+2)} y_{t+1} \\
&= d^t\left(\sum_x x_{(1)} y_1 \otimes \cdots \otimes x_{(t+1)} y_{t+1}\right) \\
&= (d^t \circ \psi^t)(x \otimes y_1 \otimes \cdots \otimes y_t \otimes y_{t+1}).
\end{aligned}$$

Therefore, $\psi^\bullet : \tilde{I}^\bullet \to I^\bullet$ is a cochain map. Clearly, $\psi^\bullet$ is an extension of the identity map on the augmentation term k. Hence $\bar{\varphi}$ is induced by $\psi^\bullet$.

Because of the condition (A.3.2), the restriction of $\psi^{s,t}$ on $C^s \otimes C^t$, which is the subspace of $I^s \otimes I^t$ spanned by vectors of the form $x_1 \otimes \cdots \otimes x_s \otimes 1 \otimes y_1 \otimes \cdots \otimes y_t \otimes 1$, is exactly the canonical identification of $C^s \otimes C^t$ with C^{s+t}. Thus, the map $\bar{\varphi}$ gives the cup product on $\mathscr{H}(A)$. □

(A.3.4) Remark. If A is a bialgebra, the multiplication of A satisfies condition (A.3.2). Thus, Theorem (A.3.3) in particular implies that if there are two different bialgebra structures on a coalgebra A, then the algebra structures on $\mathscr{H}(A)$ induced by two different multiplications are the same.

Now we can prove the following result.

(A.3.5) Theorem. *Let A be a coalgebra with a coalgebra homomorphism μ: $A \otimes A \to A$ satisfying condition* (A.3.2) (*e.g., A is a bialgebra*). *Then the cohomology algebra* ($\mathscr{H}(A)$, $\cup$) *is skew-commutative in the sense that*

$$x \cup y = (-1)^{st} y \cup x, \qquad \textit{for} \quad x \in H^s(k, A), \quad y \in H^t(k, A).$$

Proof. Clearly, μ^{op}: $A \otimes A \to A$, sending $x \otimes y$ to $\mu(y \otimes x)$, is also a coalgebra homomorphism satisfying (A.3.2). For an injective resolution $0 \to k \to J^\bullet$ of k as the trivial A-comodule, denote by $\tilde{J}^\bullet_\mu$ (resp., $\tilde{J}^\bullet_{\mu^{\mathrm{op}}}$) the complex $\tilde{J}^\bullet$ viewed as an A-comodule complex via μ (resp., μ^{op}). Let $\theta^\bullet_\mu : \tilde{J}^\bullet_\mu \to J^\bullet$ (resp., $\theta_{\mu^{\mathrm{op}}} : \tilde{J}^\bullet_{\mu^{\mathrm{op}}} \to J^\bullet$) be a cochain map extending the identity map on k. It is well-known that there is an automorphism $\xi^\cdot$: $\tilde{J}^\cdot \to \tilde{J}^\cdot$ given by

$$\xi^{s+t}(x \otimes y) = (-1)^{st}(y \otimes x), \qquad \text{for} \quad x \in J^s, \quad y \in J^t,$$

if we regard $\tilde{J}^\cdot$ only as a complex of linear spaces. It is clear that $\xi^\cdot$ is an A-comodule isomorphism from $\tilde{J}^\cdot_\mu$ to $\tilde{J}^\cdot_{\mu^{\mathrm{op}}}$. Thus, we have two chain maps

$$\theta^\cdot_\mu,\ \theta^\cdot_{\mu^{\mathrm{op}}} \circ \xi^\cdot : \tilde{J}^\cdot_\mu \to J^\cdot$$

extending the identity map on k. Since $J^\cdot$ is an injective resolution of k and $0 \to k \to \tilde{J}^\cdot_\mu$ is exact, we see that $\theta^\cdot_\mu$ and $\theta^\cdot_{\mu^{\mathrm{op}}} \circ \xi^\cdot$ are homotopic. Now it is clear that the restrictions $\theta^\cdot_\mu$ and $\theta_{\mu^{\mathrm{op}}} \circ \xi^\cdot$ are homotopic cochain maps from $\widetilde{\mathscr{F}^A(J^\cdot)}$ to $\mathscr{F}^A(J^\cdot)$. Hence the induced homomorphisms $\bar{\mu}$, $\overline{\mu^{\mathrm{op}}} \circ \bar{\xi}$: $\mathscr{H}(A) \otimes \mathscr{H}(A) \to \mathscr{H}(A)$ are identical, where

$$\bar{\xi}(x \otimes y) = (-1)^{st} y \otimes x, \qquad \text{for} \quad x \in H^s(k, A), \quad y \in H^t(k, A).$$

Thanks to Theorem (A.3.3), both $\bar{\mu}$ and $\overline{\mu^{\mathrm{op}}}$ are the cup product. The theorem is therefore proved. □

(A.3.6) Remark. When A is a commutative bialgebra, the above theorem is well-known. In particular, the result is well-known in the context of rational modules for affine group schemes. For a discussion of the cup product in the dual situation of the cohomology of an *algebra* A, see [30; Chap. VIII, §9]. It essentially follows from the discussion given there that the cup product is induced by the comultiplication Δ of any bialgebra structure on A. When Δ is *cocommutative*, a dual version of (A.3.5) is proved in [4; Cor. 3.2.2].

A.4. Generalized cup product. In a homology theory, cup product is usually defined in a more general context. This section is devoted to a discussion in this direction.

Suppose A is a bialgebra with product μ. Then the tensor product $V \otimes W$ of two A-comodules V and W, which is an $A \otimes A$-comodule, is given an A-comodule structure via μ. Thus, as in §A.3, there is a natural homomorphism

$$\bar{\mu}: H^n(V \otimes W, A \otimes A) \to H^n(V \otimes W, A).$$

On the other hand, we have, by Künneth theorem, $H^n(V \otimes W, A \otimes A) \cong \bigoplus_{s+t=n} H^s(V, A) \otimes H^t(W, A)$. Thus, we obtain a natural homomorphism $\bar{\mu}_{V,W}$: $H^s(V, A) \otimes H^t(W, A) \to H^{s+t}(V \otimes W, A)$, which is associative in the sense that $\bar{\mu}_{U \otimes V, W} \circ (\bar{\mu}_{U,V} \otimes \mathrm{id}_W) = \bar{\mu}_{U, V \otimes W} \circ (\mathrm{id}_U \otimes \bar{\mu}_{V,W})$. Clearly, the cup product $\bar{\mu}$ defined in §A.3 is exactly $\bar{\mu}_{k,k}$. Thus, $\bar{\mu}_{V,W}$ is usually also called the *cup product*.

Let S be a subgroup of the multiplicative group of group-like elements in A, and denote the 1-dimensional A-comodule corresponding to an element $\lambda \in S$ still by λ, and denote its basis element by 1_λ. Let

$$\mathscr{H}(S, A) = \bigoplus_{\lambda \in S} H^\cdot(\lambda, A).$$

Then the cup product $\bar{\mu}_{\lambda,\zeta}\colon H^s(\lambda, A)\otimes H^t(\zeta, A)\to H^{s+t}(\lambda\zeta, A)$ for $\lambda, \zeta\in S$ defines a graded associative algebra structure on $\mathscr{H}(S, A)$.

As in the case of the algebra $\mathscr{H}(A)$, we can also give a description of the cup product on $\mathscr{H}(S, A)$ in terms of Hochschild complexes. Recall from §A.2 that the cohomology group $H^n(\lambda, A)$ can be calculated by using the Hochschild complex $(C^\cdot(\lambda, A), \partial^\cdot)$, where $C^n(\lambda, A)=\lambda\otimes A^{\otimes n}$, which will be identified with $A^{\otimes n}$, and

$$\partial^n(x_1\otimes\cdots\otimes x_n)=\lambda\otimes x_1\otimes\cdots\otimes x_n$$
$$+\sum_{i=1}^{n}(-1)^i x_1\otimes x_{i-1}\otimes\Delta(x_i)\otimes x_{i+1}\otimes\cdots\otimes x_n+(-1)^{n+1}x_1\otimes\cdots\otimes x_n\otimes 1\,.$$

(A.4.1) PROPOSITION. *Using the above notation and assumption, the cup product $\bar{\mu}_{\lambda,\zeta}$ for group-like elements $\lambda, \zeta\in A$ is induced by the map $\bar{\psi}^{s,t}=\bar{\psi}^{s,t}_{\lambda,\zeta}\colon C^s(\lambda, A)\otimes C^t(\zeta, A)\to C^{s+t}(\lambda\zeta, A)$ given by*

$$\bar{\psi}^{s,t}(x_1\otimes\cdots\otimes x_s\otimes y_1\otimes\cdots\otimes y_t)=x_1\zeta\otimes\cdots\otimes x_s\zeta\otimes y_1\otimes\cdots\otimes y_t\,.$$

PROOF. We use the injective resolutions given in (A.2.1) and define $\psi^{s,t}\colon I^s_\lambda\otimes I^t_\zeta\to I^{s+t}_{\lambda\zeta}$ by sending $1_\lambda\otimes x_1\otimes\cdots\otimes x_s\otimes x\otimes 1_\zeta\otimes y_1\otimes\cdots\otimes y_{t+1}$ to

$$1_{\lambda\zeta}\otimes x_1\zeta\otimes\cdots\otimes x_s\zeta\otimes\sum_x x_{(1)}y_1\otimes\cdots\otimes x_{(t+1)}y_{t+1}\,.$$

A calculation similar to that we carried out in the proof of Theorem (A.3.3) shows that $\psi^{s,t}$ is an A-comodule homomorphism, and we obtain a cochain map by putting all these homomorphisms together. The details are left to the interested reader.

It is easy to see that if we identify $\lambda\otimes A$, $\zeta\otimes A$ and $\lambda\zeta\otimes A$ with A canonically, the restriction of $\psi^{s,t}$ is exactly $\bar{\psi}^{s,t}$. □

Let S be as before, and let $\mathfrak{S}$ be the subspace of A spanned by S. Then $\mathfrak{S}$ is a subbialgebra (thus, a subcomodule) of A. Clearly, $\mathscr{H}(S, A)=H^\cdot(\mathfrak{S}, A)$. The algebra $\mathfrak{S}$ is an example of A-comodule algebras. In general, an *A-comodule algebra* is an algebra which is also an A-comodule, and these two structures are compatible in the sense that the multiplication and the unit of the algebra are A-comodule homomorphisms.

It $\mathfrak{B}$ is an A-comodule algebra, the cup product $\bar{\mu}_{\mathfrak{B},\mathfrak{B}}\colon H^\cdot(\mathfrak{B}, A)\otimes H^\cdot(\mathfrak{B}, A)\to H^\cdot(\mathfrak{B}\otimes\mathfrak{B}, A)$ can be combined with the homomorphism $H^\cdot(\mathfrak{B}\otimes\mathfrak{B}, A)\to H^\cdot(\mathfrak{B}, A)$ induced by the multiplication of $\mathfrak{B}$ to obtain a multiplication on $\mathscr{H}(\mathfrak{B}, A)\equiv H^\cdot(\mathfrak{B}, A)$, making it into a graded algebra. (Thus, $\mathscr{H}(S, A)$ is a special case of these.) The product on $\mathscr{H}(\mathfrak{B}, A)$ can also be called the *cup product*.

If $\mathfrak{B}$ and $\mathfrak{C}$ are A-comodule algebras, and $\theta\colon\mathfrak{B}\to\mathfrak{C}$ is an A-comodule algebra homomorphism (i.e., it is an A-comodule homomorphism, and also an algebra homomorphism), then clearly we have a natural algebra homomorphism $\bar{\theta}\colon\mathscr{H}(\mathfrak{B}, A)\to\mathscr{H}(\mathfrak{C}, A)$.

(A.4.2) EXAMPLE. In this example we use some results stated in [31; §§2.3–2.4]. Let A be a Hopf algebra, and V a finite dimensional A-comodule with structure map τ. Then $\mathrm{End}(V)\cong V\otimes V^*$ is an A-comodule algebra. To see this, let $\{v_i\}$ and $\{f_i\}$ be a pair of dual bases for V and V^*. Write $X_{ij}=v_i\otimes f_j$, viewed as a left operator on V. Let $\tau(v_j)=\sum_i v_i\otimes a_{ij}$. Then the structure map τ^* on V^* is given by $\tau^*(f_i)=\sum_j f_j\otimes\gamma(a_{ij})$. Thus, the structure map σ of $V\otimes V^*$ satisfies

$$\sigma(X_{si})=\sum_{r,j} X_{rj}\otimes a_{rs}\gamma(a_{ij})\,.$$

We have $X_{si}X_{s'i'}=\delta_{is'}X_{si'}$. Thus,

$$\begin{aligned}\sigma(X_{si})\sigma(X_{s'i'})&=\sum_{r,j'} X_{rj'}\otimes a_{rs}\left(\sum_i \gamma(a_{ij})a_{js'}\right)\gamma(a_{i'j'})\\&=\delta_{is'}\sum_{r,j'} X_{rj'}\otimes a_{rs}\gamma(a_{i'j'})=\sigma(\delta_{is'}X_{si'})\\&=\sigma(X_{si}X_{s'i'})\,.\end{aligned}$$

The unit of the algebra $V\otimes V^*$ is given by $1_A\mapsto\sum_j X_{jj}$. We have

$$\sigma\left(\sum_j X_{jj}\right)=\sum_{i,s} X_{is}\otimes\left(\sum_j a_{ij}\gamma(a_{js})\right)=\sum_{i,s} X_{is}\otimes\delta_{is}=\left(\sum_j X_{jj}\right)\otimes 1_A\,.$$

(A.4.4) NOTATIONAL CONVENTION. A bialgebra A can be viewed as the coordinate algebra $k[G]$ of a quantum semigroup G, and an A-comodule V is regarded as a rational G-module. In this context, the cohomology $H^\bullet(V, A)$ is usually denoted by $H^\bullet(G, V)$, and is called the *rational cohomology* of G with coefficients in V.

(A.4.5) EXAMPLE. Let $G_q=GL_q(n)$ or $SL_q(n)$ as defined in §2. Let B_q be the Borel subgroup of G_q defined by the ideal in $k[G_q]$ generated by all X_{ij} with $i<j$. Let $X=X(B_q)\cong X(T_q)$. Then we can form the cup product algebra $H^\bullet(B_q, X)$. On the other hand, consider the *right* ideal I of $k[B_q]$ generated by all $X_{ii}-1$. It is easy to verify that I is a coideal. Thus, we obtain a quotient coalgebra $k[U_q]\equiv k[B_q]/I$. (The usage of the notation $k[U_q]$ is to show some analogy with the corresponding situation of algebraic groups: In the case $q=1$, U_q is a closed subgroup—the unipotent radical of B_q. However, if $q\neq 1$, $k[U_q]$ is no longer an algebra. So, in fact, U_q does not, in general, represent an object in $\mathbf{QGr}_k$.) One can show, using the description of injective B_q-modules given in [31; §8.9], that the restriction of any indecomposable injective B_q-module is the unique indecomposable injective $k[U_q]$-comodule—the "regular" comodule $k[U_q]$. Moreover, it then can be proved that the natural homomorphism

$$H^\bullet(B_q, X)\to H^\bullet(k, k[U_q])$$

is an isomorphism of k-algebras. If $H^\bullet(B_q, \lambda)$ is given a T_q-module structure via weight λ, then $H^\bullet(B_q, X)$, and thus $H^\bullet(k, k[U_q])$, is a T_q-algebra. If q is not a root of 1, the above observation, together with the Borel-Weil-Bott Theorem for G_q ([2] and [25]), will give a T_q-module isomorphism

$$H^i(k, k[U_q]) \cong \bigoplus_{\substack{w \in W \\ \ell(w)=i}} w \cdot 0 .$$

If q is an l-th primitive root of 1 with l odd, the conclusion is completely different. For example, one can check easily that $-l\alpha$ for any simple root α is a weight of $H^1(k, k[U_q])$.

Furthermore, in the case in which q is a primitive l-th root of 1 with l odd, we may consider $(B_q)_1 T$ and the quotient coalgebra $k[(U_q)_1]$. In this case we have a natural isomorphism of k-algebras:

$$H^\bullet((B_q)_1 T, X) \to H^\bullet(k, k[(U_q)_1]) .$$

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