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UNIRATIONALITY OF CERTAIN COMPLETE INTERSECTIONS IN POSITIVE CHARACTERISTICS

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Abstract. We prove, under a certain condition on the dimension, the unirationality of general complete intersections of hypersurfaces which are defined over an algebraically closed field of characteristic $p>0$ and projectively isomorphic to the Fermat hypersurface of degree $q+1$ where q is a power of p.

Introduction. The Fermat variety

$$X_0^{q+1}+X_1^{q+1}+\cdots+X_n^{q+1}=0$$

of degree $q+1$ ($q=p^\nu$) defined over a field of characteristic $p>0$ has a lot of interesting peculiarities of positive characteristic, such as supersingularity (Tate [T], Shioda [Sh], Shioda-Katsura [S-K]), unirationality (Shioda [Sh], Shioda-Katsura [S-K], Schoen [Sch]), and constancy of moduli of hyperplane sections (Beauville [B]). On the other hand, in characteristics $p>0$, hypersurfaces which are projectively isomorphic to the Fermat variety of degree $q+1$ constitute an open dense subset of a *linear* system $\mathscr{F}$. (See Beauville [B] and below.) Then it is very likely that the complete intersections defined by linear subsystems of $\mathscr{F}$ also possess those interesting peculiarities. In this paper, we shall study the unirationality of such complete intersections.

Let k be a field of characteristic $p>0$, $\bar{k}$ its algebraic closure, and q a power of p.

First we state our results over $\bar{k}$. Let $\mathscr{F}$ denote the linear subsystem of $|\mathcal{O}_{\boldsymbol{P}^n_{\bar{k}}}(q+1)|$ which consists of hypersurfaces whose defining equations are of the form

$$(0.1)\qquad \sum_{\mu,\nu=0}^{n} a_{\mu\nu}X_\mu X_\nu^q=0\,.$$

As is shown in Beauville [B], a hypersurface of degree $q+1$ in $\boldsymbol{P}^n_{\bar{k}}$ is projectively isomorphic to the Fermat variety if and only if it is a nonsingular member of $\mathscr{F}$.

THEOREM 1. *Suppose* $n\geq r^2+2r$. *Let* $V_1,\dots,V_r$ *be members of* $\mathscr{F}$. *We put* $W=V_1\cap\cdots\cap V_r$. *If* $V_1,\dots,V_r$ *are chosen generally, then there is a purely inseparable dominant rational map* $\boldsymbol{P}^{n-r}_{\bar{k}}\cdots\to W$ *of degree* $q^{r(r+1)/2}$. *In particular,* W *is unirational.*

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Because there is a surjective morphism from the Fermat variety of degree $q+1$ to the Fermat variety of degree m if $m|(q+1)$, our result implies the following:

COROLLARY. *Suppose $n \geq 3$. Then the Fermat variety*

$$X_0^m + X_1^m + \cdots + X_n^m = 0$$

of degree m defined over an algebraically closed field of characteristic $p>0$ is unirational, provided that $p^\nu \equiv -1 \pmod{m}$ for some integer ν.

Schoen [Sch] has also proved this Corollary. In case n is odd, this result had already been shown in Shioda [Sh] and Shioda-Katsura [S-K], by means of the inductive structure of Fermat varieties.

The same argument can be applied to the complete intersection of hypersurfaces of *diagonal type*. We shall prove the following:

THEOREM 2. *Suppose $n \geq r^2+3r$. Suppose also that $p^\nu \equiv -1 \pmod{m}$ for some integer ν. Let V_i $(i=1, \dots, r)$ be hypersurfaces of diagonal type*

$$b_{i0}X_0^m + \cdots + b_{in}X_n^m = 0$$

defined over $\bar{k}$. If the coefficients $b_{i\nu}$ are general enough, then the complete intersection $W = V_1 \cap \cdots \cap V_r$ is unirational.

Note that, since Theorem 1 states the unirationality only for general $V_1, \dots, V_r$, Theorem 2 does not follow directly from Theorem 1 if $r \geq 2$. We have to strengthen the condition on n from $\geq r^2+2r$ to $\geq r^2+3r$, as far as we adopt the method of the proof in this article.

In fact, we shall prove a stronger result. From now on, we work over k, which is not necessarily algebraically closed. We fix an r-dimensional linear subspace $L \subset \boldsymbol{P}_k^n$ defined over k. We denote by $\mathscr{F}_L$ the variety of all hypersurfaces which are defined by equations of the form (0.1) and contain L. Then $\mathscr{F}_L$ is defined over k and isomorphic to the projective space of dimension $(n+1)^2-(r+1)^2-1$.

THEOREM 3. *Suppose $n \geq r^2+r+1$. Then there is an open dense subvariety U of $\mathscr{F}_L \times \cdots \times \mathscr{F}_L$ (r-times) which has the following property. Let K/k be an arbitrary field extension and let $U(K)$ denote the set of K-valued points of U. Then, for every $(V_1, \dots, V_r) \in U(K)$, there is a purely inseparable dominant rational map of degree $q^{r(r+1)/2}$ defined over K^{1/q^r} from the $(n-r)$-dimensional projective space to $W = V_1 \cap \cdots \cap V_r$. In particular, W is K^{1/q^r}-unirational.*

The idea of the proof of Theorem 3 is as follows. We proceed by induction on r. Suppose that $V_1, \dots, V_r \in \mathscr{F}_L(K)$ are "general", by which we mean that they satisfy certain open conditions. Let $T_{\eta(L), W} \subset \boldsymbol{P}_{K(L)}^n$ be the tangent space to W at the generic point of L. Then there is a purely inseparable dominant rational map $T_{\eta(L), W} \cap (W \times_K K(L)) \cdots \to W$ defined over K. We shall show that $T_{\eta(L), W} \cap (W \times_K K(L))$ is birational over $K(L)^{1/q}$ to a complete intersection of $r-1$ hypersurfaces $V_1^{(1)}, \ldots, V_{r-1}^{(1)} \subset \boldsymbol{P}_{K^{(1)}}^{n-2r}$ defined over a field $K^{(1)}$ which is a purely transcendental extension of dimension $r-1$ over $K(L)^{1/q}$, and each $V_i^{(1)}$ is projectively isomorphic over $\overline{K^{(1)}}$ to the Fermat variety of degree $q+1$, and contains a $K^{(1)}$-rational $(r-1)$-dimensional linear subspace $L^{(1)} \subset \boldsymbol{P}_{K^{(1)}}^{n-2r}$. Moreover, if $V_1, \ldots, V_r$ are "general", then $V_1^{(1)}, \ldots, V_{r-1}^{(1)}$ are also "general". Since $(K^{(1)})^{1/q^{r-1}}$ is a purely transcendental extension of dimension $2r-1$ over K^{1/q^r}, the $(K^{(1)})^{1/q^{r-1}}$-unirationality of $V_1^{(1)} \cap \cdots \cap V_{r-1}^{(1)}$ implies the K^{1/q^r}-unirationality of $W = V_1 \cap \cdots \cap V_r$.

This paper is organized as follows. In §1, we give a finite set of open conditions on $V_1, \ldots, V_r \in \mathscr{F}_L(K)$ which is sufficient for the K^{1/q^r}-unirationality of $W = V_1 \cap \cdots \cap V_r$. In §2, we show the existence of an example of $V_1, \ldots, V_r \in \mathscr{F}_L(\bar{k})$ which satisfies those conditions and thus complete the proof of Theorem 3. In §3, we prove lemmas about linear subspaces contained in W and derive Theorem 1 from Theorem 3. In §4, we shall prove Theorem 2 by showing that there is such an element $(V_1, \ldots, V_r)$ in $U(\bar{k})$ that each V_i is a hypersurface of diagonal type.

CONVENTIONS AND NOTATION. Let V be a variety over a field E and let F/E be a field extension. Then $V(F)$ denotes the set of F-valued points of V, V_F denotes the fiber product $V \times_{\operatorname{Spec} E} \operatorname{Spec} F$, and $F(V)$ denotes the function field of V_F. Let $\bar{E}$ be the algebraic closure of E. Then $E^{1/q}$ is the field $\{x \in \bar{E} \mid x^q \in E\}$, and E^q is the field $\{x^q \mid x \in E\}$. The binary relation $\simeq$ means that varieties are birational, while $\cong$ means that they are isomorphic.

1. Open conditions sufficient for the unirationality.

We start to prove Theorem 3. Let $V_1, \ldots, V_r$ be members of $\mathscr{F}_L(K)$. Suppose that

(C1) $W := V_1 \cap \cdots \cap V_r$ is a complete intersection of dimension $n-r$ which is geometrically reduced irreducible and nonsingular along L.

Let $(X_0, \ldots, X_n)$ be homogeneous coordinates of $\boldsymbol{P}_K^n$ such that $L_K = \{X_{r+1} = \cdots = X_n = 0\}$, and let

$$\sum_{\mu, \nu=0}^{n} a_{i\mu\nu} X_\mu X_\nu^q = 0 \qquad \text{where} \quad a_{i\mu\nu} = 0 \quad \text{if} \quad 0 \le \mu, \nu \le r$$

be the defining equation of V_i. The tangent space to V_i at $(Y_0, \ldots, Y_n) \in V_i$ is given by

$$(1.1)_i \qquad \sum_{\mu=0}^{n} \left(\sum_{\nu=0}^{n} a_{i\mu\nu} Y_\nu^q \right) X_\mu = 0 .$$

Let $T_{L,W}$ be the variety $\{(Q, R) \in L_K \times \boldsymbol{P}_K^n \mid T_{Q,W} \ni R$, where $T_{Q,W} \subset \boldsymbol{P}_K^n$ is the tangent space to W at $Q\}$, which is defined by $(1.1)_1$–$(1.1)_r$ with $Y_{r+1} = \cdots = Y_n = 0$. Let

$$\begin{array}{ccc} T_{L,W} & \xrightarrow{\phi} & \boldsymbol{P}_K^n \\ \downarrow & & \\ L_K & & \end{array}$$

be the natural projections. The second projection ϕ is a surjection which is generically finite and purely inseparable of degree q^r. Indeed, as is seen from $(1.1)_i$, the polar divisor $\{Q\in V_i | T_{Q,V_i}\ni R\}$ of V_i with respect to $R\in \boldsymbol{P}_K^n$ is a q-th multiple of a hyperplane section $P_{R,V_i}\cap V_i$, where P_{R,V_i} is the hyperplane. Then $\phi^{-1}(R)$ is *set-theoretically* equal to $\{(Q,R) | Q\in P_{R,V_1}\cap\cdots\cap P_{R,V_r}\cap L\}$, which is always nonempty. Hence ϕ is surjective, and comparing the dimensions of $T_{L,W}$ and $\boldsymbol{P}_K^n$, we see that ϕ is generically finite; that is, the intersection $P_{R,V_1}\cap\cdots\cap P_{R,V_r}\cap L$ consists of one point for a general point $R\in\boldsymbol{P}_K^n$. Since each of r polar divisors has multiplicity q, the degree of ϕ is q^r. Let Γ_W denote the closed subset of $\boldsymbol{P}_K^n$ such that $\boldsymbol{P}_K^n\setminus\Gamma_W$ is the maximal open subset over which ϕ is finite. We suppose that

(C2) W is not contained in Γ_W, and the closure of $\phi^{-1}(W\setminus\Gamma_W)$ is mapped surjectively onto L_K by the first projection.

We denote by $\tilde{Z}$ the inverse image $\phi^{-1}(W)$. Then (C2) implies that

(1.2) a geometrically irreducible component of the generic fiber of $\tilde{Z}\to L_K$ (that is, the component which is obtained as the closure of $\phi^{-1}(W\setminus\Gamma_W)$) is mapped dominantly onto W by a purely inseparable rational map of degree q^r.

(It will turn out that, for general $V_1,\ldots,V_r$, the generic fiber of $\tilde{Z}\to L_K$ is again geometrically reduced irreducible unless $n=3$ and $r=1$. If $n=3$ and $r=1$, the generic fiber is a union of a line $\phi^{-1}(L_K)\times_{L_K}K(L)$ and a geometrically reduced irreducible curve in $\boldsymbol{P}^2_{K(L)}$.)

Let E be an arbitrary extension field of K, and let ρ: Spec $E\to L_K$ be an E-valued point of L_K. (Later on in this section, ρ will be the q-th root of the generic point Spec $K(L)^{1/q}\to L_K$, and in the next section, ρ will be a geometric point with respect to $\bar{K}$.) Then there are homogeneous coordinates, which we shall denote by $(X_0,\ldots,X_n)$ again, of $\boldsymbol{P}_E^n$ such that

$$\rho=(1,0,\ldots,0)\,,$$
$$L_E=\{X_{r+1}=\cdots=X_n=0\}\quad\text{and}$$
$$T_{\rho,W}=\{X_{n-r+1}=\cdots=X_n=0\}\,,$$

where $T_{\rho,W}$ is the tangent space to W at ρ. Suppose that E contains $K^{1/q}$ and ρ factors as Spec $E\to$ Spec $E^q\to L_K$, which is satisfied in the two cases mentioned in the parenthesis above. Then the defining equation of $(V_i)_E$ can be put into a form

$$\sum_{\nu=0}^{n} X_\nu l_{i\nu}(X_0, \ldots, X_n)^q = 0$$

where $l_{i\nu}$ are linear forms over E. We put $x_i = X_i/X_0$ $(i=1, \ldots, n-r)$ and consider $(x_1, \ldots, x_{n-r})$ as affine coordinates of $T_{\rho,W}$ with the origin ρ. Because $T_{\rho,W} \cap (V_i)_E$ is singular at ρ, its defining equation in $T_{\rho,W}$ is of a form

$$f_i^q + \sum_{\nu=1}^{n-r} x_\nu g_{i\nu}^q = 0 ,$$

where f_i and $g_{i\nu}$ are linear forms in $(x_1, \ldots, x_{n-r})$ over E. For simplicity, we put

$$h_i := \sum_{\nu=1}^{n-r} x_\nu g_{i\nu}^q . \tag{1.3}$$

We also put

$$Z_\rho := \tilde{Z} \times_{L_K} \operatorname{Spec} E = T_{\rho,W} \cap W_E = \bigcap_{i=1}^{r} T_{\rho,W} \cap (V_i)_E ,$$

$$L_\rho := \phi^{-1}(L_K) \times_{L_K} \operatorname{Spec} E \subset T_{\rho,W} .$$

Then L_ρ is contained in $T_{\rho,W} \cap (V_i)_E$ and hence in Z_ρ. We assume that the following condition is satisfied:

(C3. ρ) Z_ρ is a complete intersection of codimension r in $T_{\rho,W}$. Moreover, unless $n=3$ and $r=1$, Z_ρ is geometrically reduced irreducible. If $n=3$ and $r=1$, Z_ρ is a union of the line L_ρ and a geometrically reduced irreducible curve.

Let $D_\rho \cong \boldsymbol{P}_E^{n-r-1}$ be the variety of all lines on $T_{\rho,W}$ which pass through ρ, and let $\pi: T_{\rho,W} \cdots \rightarrow D_\rho$ be the natural projection. We may regard $(x_1, \ldots, x_{n-r})$ as homogeneous coordinates on D_ρ, and f_i and h_i as defining equations of hyperplanes and hypersurfaces in D_ρ. Then $L_K \subset V_i$ implies

$$\pi(L_\rho) \subset \{f_i = 0\} , \qquad \pi(L_\rho) \subset \{h_i = 0\} , \tag{1.4}$$

where $\pi(L_\rho) \cong \boldsymbol{P}_E^{r-1}$ is the linear subspace of D_ρ defined by $\{x_{r+1} = \cdots = x_{n-r} = 0\}$. Here again we assume that the following are satisfied:

(C4. ρ) $f_1, \ldots, f_r$ are linearly independent, and

(C5. ρ) unless $n=3$ and $r=1$, at least one of h_i's is not constantly zero on $\{f_1 = \cdots = f_r = 0\} \subset D_\rho$; if $n=3$ and $r=1$, then $f_1^2 \nmid h_1$. (Note that if $n=3$ and $r=1$, then (1.4) implies $f_1 | h_1$.)

Note that, unless $f_i(a_1, \ldots, a_{n-r}) = h_i(a_1, \ldots, a_{n-r}) = 0$, a line

$$\{((x_1, \ldots, x_{n-r}) = (\lambda a_1, \ldots, \lambda a_{n-r}) \,|\, \lambda \text{ is an affine parameter}\} \subset T_{\rho,W}$$

intersects $T_{\rho,W}\cap(V_i)_E$ at $\lambda=0$ with multiplicity q and at

$$\lambda=-\frac{f_i(a_1,\dots,a_{n-r})^q}{h_i(a_1,\dots,a_{n-r})}$$

with multiplicity 1. Thus, if f_i does not divide h_i, π gives a birational map over E between $T_{\rho,W}\cap(V_i)_E$ and D_ρ. In particular, if $r=1$ and $n>3$, then Z_ρ is birational to $D_\rho\cong \boldsymbol{P}_E^{n-2}$ over E. When $r=1$ and $n=3$, then $Z_\rho\setminus L_\rho$ is birational to $D_\rho\cong\boldsymbol{P}_E^1$. Hence in case $r=1$, (C4.ρ) and (C5.ρ) imply (C3.ρ) automatically. Now suppose $r\geq 2$. Let $Y_\rho\subset D_\rho$ be the variety defined by

$$\frac{f_1^q}{h_1}=\cdots=\frac{f_r^q}{h_r}.$$

Then we see that

(1.5) Z_ρ is mapped birationally by π to Y_ρ.

Indeed, the lines contained in Z_ρ and passing through ρ are parametrized by $\{f_1=\cdots=f_r=h_1=\cdots=h_r=0\}\subset D_\rho$. By dimension counting, (C4.ρ) and (C5.ρ) imply that Z_ρ is not a cone with the vertex ρ. Hence (1.5) holds. We denote by $U_{i,\rho}$ ($i=1,\dots,r-1$) the hypersurface defined in D_ρ by

$$f_i^q h_r - f_r^q h_i = 0\,.$$

Then $Y_\rho=\bigcap_{i=1}^{r-1}U_{i,\rho}$ is a geometrically reduced irreducible complete intersection of codimension $r-1$ by (C3.ρ) and (1.5).

By (C4.ρ), $\{f_1=\cdots=f_r=0\}$ defines an $(n-2r-1)$-dimensional linear subspace $M_\rho\subset D_\rho$, which contains $\pi(L_\rho)$ by (1.4). Let $G_\rho\cong\boldsymbol{P}_E^{r-1}$ be the variety of all $(n-2r)$-dimensional linear subspaces containing M_ρ, and let

$$\begin{array}{ccc} H_\rho & \xrightarrow{\ b\ } & D_\rho \\ \downarrow & & \\ G_\rho & & \end{array}$$

be the universal family. The morphism b is a blow-up along the center M_ρ. From the defining equation of $U_{i,\rho}$, we see that the total transform $b^{-1}(U_{i,\rho})$ contains the exceptional divisor $b^{-1}(M_\rho)$ with multiplicity at least q. We denote by $\tilde{V}_{i,\rho}^{(1)}$ the effective divisor $b^{-1}(U_{i,\rho})-q\cdot b^{-1}(M_\rho)$. The last condition we assume is

(C6.ρ) $\tilde{V}_{i,\rho}^{(1)}$ does not contain the exceptional divisor $b^{-1}(M_\rho)$ any more, and the projection $\tilde{V}_{i,\rho}^{(1)}\to G_\rho$ is surjective.

Then $\tilde{V}_{i,\rho}^{(1)}$ coincides with the strict transform of $U_{i,\rho}$, and the intersection $\tilde{W}_\rho^{(1)}:=\tilde{V}_{1,\rho}^{(1)}\cap\cdots\cap\tilde{V}_{r-1,\rho}^{(1)}$ is the strict transform of Y_ρ; hence $\tilde{W}_\rho^{(1)}\to Y_\rho$ is birational.

Moreover, the projection $\tilde{W}_\rho^{(1)} \to G_\rho$ is surjective. This implies that

(1.6) the generic fiber of $\tilde{W}_\rho^{(1)} \to G_\rho$ is mapped birationally onto Y_ρ.

Let F/E be an arbitrary field extension and let σ: $\operatorname{Spec} F \to G_\rho$ be an F-valued point, (which will be the generic point later in this section, and a geometric point with respect to $\bar{K}$ in the next section). Noting that the restriction of an equation of the form (0.1) to a linear subspace still has the form (0.1), we see from (1.3) that the defining equation of $V_{i,\rho,\sigma}^{(1)} := \tilde{V}_{i,\rho}^{(1)} \times_{G_\rho} \operatorname{Spec} F$ in $H_{\rho,\sigma} := H_\rho \times_{G_\rho} \operatorname{Spec} F \cong \boldsymbol{P}_F^{n-2r}$ is of the form (0.1). Moreover we see from (1.4) that $V_{i,\rho,\sigma}^{(1)}$ contains an $(r-1)$-dimensional linear subspace $L_{\rho,\sigma}^{(1)} := b^{-1}(\pi(L_\rho)) \times_{G_\rho} \operatorname{Spec} F$.

Now we take ρ to be the q-th root of the generic point η: $\operatorname{Spec} K(L)^{1/q} \to L_K$, and σ the generic point η': $\operatorname{Spec} K(L)^{1/q}(G_\eta) \to G_\eta$. In this case, we omit the η in the conditions and simply write (C3), etc. instead of (C3.η), etc. We also write $V_i^{(1)}$ and $L^{(1)}$ instead of $V_{i,\eta,\eta'}^{(1)}$ and $L_{\eta,\eta'}^{(1)}$. The field $F = K(L_\eta)^{1/q}(G_\eta)$ is a purely transcendental extension of dimension $2r-1$ over the constant field $K^{1/q}$, which we shall denote by $K^{(1)}$.

We summarize the construction above:

When $r=1$ and $n>3$ (resp. $n=3$), we get a dominant rational map $D_\eta \simeq Z_\eta$ (resp. $Z_\eta \setminus L_\eta) \cdots \to W$ defined over $K^{1/q}$ and purely inseparable of degree q, assuming (C1), (C2) and (C3). (Note that when $n=3$, $Z_\eta \setminus L_\eta \cdots \to W$ is still dominant.) Since $K(L)^{1/q}$ is a purely transcendental extension of dimension 1 over $K^{1/q}$, $D_\eta \cong \boldsymbol{P}_{K(L)^{1/q}}^{n-2}$ is birational to $\boldsymbol{P}_{K^{1/q}}^{n-1}$. Hence W is $K^{1/q}$-unirational.

When $r \geq 2$, starting from hypersurfaces $V_1, \ldots, V_r \in \mathscr{F}_L(K)$ in $\boldsymbol{P}_K^n$ and assuming (C1)–(C6), we get $V_1^{(1)}, \ldots, V_{r-1}^{(1)} \in \mathscr{F}_{L^{(1)}}(K^{(1)})$ in $\boldsymbol{P}_{K^{(1)}}^{n-2r}$, where $\mathscr{F}_{L^{(1)}}$ is the variety defined in the same way as $\mathscr{F}_L$ with k replaced by $K^{(1)}$, L replaced by $L^{(1)}$, and n replaced by $n-2r$. Moreover, putting $W^{(1)} := V_1^{(1)} \cap \cdots \cap V_{r-1}^{(1)}$, we get a dominant rational map $W^{(1)} \cdots \to W$ defined over $K^{1/q}$ and purely inseparable of degree q^r by composing

$$W^{(1)} \underset{(1.6)}{\simeq} Y_\eta \underset{(1.5)}{\simeq} Z_\eta \underset{(1.2)}{\cdots \to} W.$$

Let $(\mathrm{C1})^{(1)}, \ldots, (\mathrm{C6})^{(1)}$ be the conditions obtained from (C1), ..., (C6) by replacing K by $K^{(1)}$, n by $n^{(1)} := n-2r$, r by $r^{(1)} := r-1$, L by $L^{(1)}$ and V_i $(i=1, \ldots, r)$ by $V_i^{(1)}$ $(i=1, \ldots, r^{(1)})$. Inductively, assuming $(\mathrm{C1})^{(\nu-1)}$–$(\mathrm{C6})^{(\nu-1)}$ ($(\mathrm{C1})^{(\nu-1)}$–$(\mathrm{C3})^{(\nu-1)}$ when $\nu=r$), we get $r^{(\nu)} = r^{(\nu-1)}-1$ hypersurfaces $V_i^{(\nu)}$ $(i=1, \ldots, r^{(\nu)})$ in a projective space of dimension $n^{(\nu)} = n^{(\nu-1)} - 2r^{(\nu-1)}$ such that each $V_i^{(\nu)}$ is

(i) defined over the field $K^{(\nu)}$, which is a purely transcendental extension of dimension $2r^{(\nu-1)}-1$ over the constant field $(K^{(\nu-1)})^{1/q}$,

(ii) defined by an equation of the form (0.1), and

(iii) containing an $r^{(\nu)}$-dimensional linear subspace $L^{(\nu)}$ defined over $K^{(\nu)}$;

and moreover

(iv) there is a dominant rational map $W^{(\nu)} := V_1^{(\nu)} \cap \cdots \cap V_{r^{(\nu)}}^{(\nu)} \cdots \to W^{(\nu-1)}$ defined over $(K^{(\nu-1)})^{1/q}$ and purely inseparable of degree $q^{r+1-\nu}$.

Then we define the conditions $(C1)^{(\nu)}$–$(C6)^{(\nu)}$ in the obvious way. Note that if $n \geq r^2+r+1$, then $n^{(\nu)} \geq r^{(\nu)2}+r^{(\nu)}+1$ for $\nu=1,\ldots,r$. Thus if (C1)–(C6), $(C1)^{(1)}$–$(C6)^{(1)}, \ldots, (C1)^{(r-2)}$–$(C6)^{(r-2)}$ and $(C1)^{(r-1)}$–$(C3)^{(r-1)}$ are satisfied, we get a dominant rational map

$$\boldsymbol{P}^{n^{(r)}}_{K^{(r)}} \cdots \to W$$

defined over K^{1/q^r} and purely inseparable of degree $q^{r(r+1)/2}$. Noting that $K^{(r)}$ is a purely transcendental extension over K^{1/q^r} of dimension r^2 and that $n^{(r)}=n-r^2-r$, we see that $\boldsymbol{P}^{n^{(r)}}_{K^{(r)}}$ is birational to $\boldsymbol{P}^{n-r}_{K^{1/q^r}}$ over K^{1/q^r}. Hence W is K^{1/q^r}-unirational.

It is obvious that (C1)–(C6), $(C1)^{(1)}$–$(C6)^{(1)}, \ldots, (C1)^{(r-2)}$–$(C6)^{(r-2)}$ and $(C1)^{(r-1)}$–$(C3)^{(r-1)}$ impose open conditions on the initial choice of $V_1, \ldots, V_r \in \mathscr{F}_L(K)$. Moreover, these conditions are independent of the field K. Thus there is such an open subvariety $U \subset \mathscr{F}_L \times \cdots \times \mathscr{F}_L$ that for arbitrary K/k and $(V_1, \ldots, V_r) \in U(K)$, $W = V_1 \cap \cdots \cap V_r$ is K^{1/q^r}-unirational. Our next task is to show that U is dense, or equivalently, $U(\bar{k})$ is nonempty.

2. Non-emptiness of $U(\bar{k})$. In showing $U(\bar{k}) \neq \varnothing$, we may assume that k itself is algebraically closed. Therefore we will asume $k=\bar{k}=K$ in this section.

Let $\bar{\rho}$: Spec $k \to L$ be a closed point of L. It is easy to see from the openness of the conditions that

(2.1) if $(C3.\bar{\rho})$–$(C6.\bar{\rho})$ hold, then (C3)–(C6) also hold.

Moreover, let $\bar{\sigma}$: Spec $k \to G_{\bar{\rho}}$ be a closed point of $G_{\bar{\rho}}$, and let $\overline{(C1)}^{(1)}$–$\overline{(C6)}^{(1)}$ be the conditions obtained from $(C1)^{(1)}$–$(C6)^{(1)}$ replacing $K^{(1)}$ by k, $L^{(1)}$ by $L^{(1)}_{\bar{\rho},\bar{\sigma}}$, and $V_1^{(1)}, \ldots, V_{r-1}^{(1)}$ by $V^{(1)}_{1,\bar{\rho},\bar{\sigma}}, \ldots, V^{(1)}_{r-1,\bar{\rho},\bar{\sigma}}$. It is also easy to see that

(2.2) if $\overline{(C1)}^{(1)}$–$\overline{(C6)}^{(1)}$ hold, then $(C1)^{(1)}$–$(C6)^{(1)}$ also hold.

Now we replace $(C1)^{(1)}$–$(C6)^{(1)}$ by $\overline{(C1)}^{(1)}$–$\overline{(C6)}^{(1)}$, fix closed points of $L^{(1)}_{\bar{\rho},\bar{\sigma}}$ and $G^{(1)}_{\bar{\rho},\bar{\sigma}}$, and repeat the whole process above again to check $(C1)^{(2)}$–$(C6)^{(2)}$.

Thus, making repeated use of the stability (2.1) and (2.2) of the conditions under generizations, we can prove the non-emptiness of $U(k)$ by induction on r, provided that we prove the following two statements:

(a) For general $V_1, \ldots, V_r \in \mathscr{F}_L(k)$, (C1), (C2) and $(C3, \bar{\rho})$–$(C6, \bar{\rho})$ hold.

(b) If $V_1, \ldots, V_r \in \mathscr{F}_L(k)$ are general, then $V^{(1)}_{1,\bar{\rho},\bar{\sigma}}, \ldots, V^{(1)}_{r-1,\bar{\rho},\bar{\sigma}}$ are also general.

Let us state (b) more precisely. We fix the following data:

(i) homogeneous coordinates $(X_0, \ldots, X_n)$ of $\boldsymbol{P}^n_k$ such that $\bar{\rho}=(1,0,\ldots,0)$ and

$L=\{X_{r+1}=\cdots=X_n=0\}$,

(ii) an $(n-r)$-dimensional linear subspace $T=\{X_{n-r+1}=\cdots=X_n=0\}$ of $\boldsymbol{P}_k^n$, which contains L,

(iii) the variety $D\cong\boldsymbol{P}_k^{n-r-1}$ of lines on T passing through $\bar{\rho}$, equipped with homogeneous coordinates $x_\nu=X_\nu/X_0$ $(\nu=1,\ldots,n-r)$,

(iv) an $(n-2r-1)$-dimensional linear subspace $M=\{x_{n-2r+1}=\cdots=x_{n-r}=0\}$ of D containing $\pi(L)=\{x_{r+1}=\cdots=x_{n-r}=0\}$, where $\pi: T\cdots\to D$ is the natural projection,

(v) the variety $G\cong\boldsymbol{P}_k^{r-1}$ of all $(n-2r)$-dimensional linear subspaces of D containing M,

and

(vi) the closed point $\bar{\sigma}$ of G corresponding to $H=\{x_{n-2r+2}=\cdots=x_{n-r}=0\}$.

Let $\mathscr{G}\subset\mathscr{F}_L\times\cdots\times\mathscr{F}_L$ (r-times) be the subvariety consisting of all $(V_1,\ldots,V_r)$ such that

(α) $W=V_1\cap\cdots\cap V_r$ is nonsingular at $\bar{\rho}$ and $T_{\bar{\rho},W}$ coincides with T, and

(β) $(V_1,\ldots,V_r)$ satisfies (C4.$\bar{\rho}$) and $M_{\bar{\rho}}$ coincides with M.

Let $\mathscr{F}_L^{(1)}$ be the variety of hypersurfaces in $H\cong\boldsymbol{P}_k^{n-2r}$ containing $\pi(L)\cong\boldsymbol{P}_k^{r-1}$ and defined by the equations of the form (0.1). We have a rational map

$$\Psi:\mathscr{G}\to\overbrace{\mathscr{F}_L^{(1)}\times\cdots\times\mathscr{F}_L^{(1)}}^{(r-1)\text{-times}}$$
$$(V_1,\ldots,V_r)\mapsto(V_{1,\bar{\rho},\bar{\sigma}}^{(1)},\ldots,V_{r-1,\bar{\rho},\bar{\sigma}}^{(1)})\,.$$

The precise meaning of (b) is that

(b′) Ψ is dominant.

Now we start to prove (a). Invoking the openness of the conditions again, it is enough to show that

(a′) for each of the conditions (C1), (C2), (C3.$\bar{\rho}$)–(C6.$\bar{\rho}$), there exists $(V_1,\ldots,V_r)$ which satisfies it.

(C1): Note that we have $n\geq 3r$. Consider the complete intersection of hypersurfaces $V_i\in\mathscr{F}_L(k)$ $(i=1,\ldots,r)$ given by

$$X_{r+i}\cdot X_0^q+X_{r+i+1}\cdot X_1^q+\cdots+X_{2r+i}\cdot X_r^q=0\,,$$

which contain $L=\{X_{r+1}=\cdots=X_n=0\}$. The singular locus of $W=V_1\cap\cdots\cap V_r$ is $\operatorname{Sing}W=\{X_0=\cdots=X_r=0\}$, hence W is nonsingular along L. Moreover W is a reduced complete intersection of codimension r at least locally around L. Let $\tilde{W}$ be the strict transform of W by the blow-up of $\boldsymbol{P}_k^n$ along $\operatorname{Sing}W$. Then $\tilde{W}$ has the structure of a smooth fiber space over the variety of all $(n-r)$-dimensional linear subspaces containing Sing W with every fiber isomorphic to an $(n-2r)$-dimensional linear space. Hence W is irreducible.

(C2): Recall that for a closed point $R \in \boldsymbol{P}_k^n$ and a member V of $\mathscr{F}$, the reduced part of the polar divisor of V with respect to R is a hyperplane section $P_{R,V} \cap V$. Hence we see that

$$\Gamma_W = \{R \in \boldsymbol{P}_k^n \mid \dim(P_{R,V_1} \cap \cdots \cap P_{R,V_r} \cap L) \geq 1\} .$$

Suppose that $R \in W \setminus \Gamma_W$, and let Q be the intersection point $P_{R,V_1} \cap \cdots \cap P_{R,V_r} \cap L$. If $W \cap T_{Q,W}$ is a complete intersection of codimension r in $T_{Q,W}$ locally at R, then, by dimension counting, we can conclude that the closure of $\phi^{-1}(W \setminus \Gamma_W)$ is mapped surjectively onto L. It is not difficult to construct such an example of R and W.

(C3. $\bar{\rho}$)–(C6. $\bar{\rho}$): We use the data (i)–(vi) above. Suppose that $V_i \in \mathscr{F}_L(k)$ is defined by

$$\sum_{\mu,\nu=0}^{n} a_{i\mu\nu} X_\mu X_\nu^q = 0 \qquad \text{with} \quad (*) \begin{cases} a_{i\mu\nu} = 0 & \text{if} \quad 0 \leq \mu,\ \nu \leq r \\ a_{i\mu 0} = \delta_{i,\mu-n+r} \end{cases} .$$

Then $T_{\bar{\rho},W}$ coincides with T, and f_i, h_i are given by

$$f_i = \sum_{\nu=1}^{n-r} a_{i0\nu}^{1/q} x_\nu , \quad h_i = \sum_{\mu,\nu=1}^{n-r} a_{i\mu\nu} x_\mu x_\nu^q .$$

We can choose the coefficients $(a_{i\mu\nu})$ arbitrarily except for the condition $(*)$ above. Hence (C4. $\bar{\rho}$) and (C5. $\bar{\rho}$) hold obviously. Thus (C3. $\bar{\rho}$) also holds when $r=1$. Suppose $r \geq 2$. To construct an example for which (C3. $\bar{\rho}$) holds, we choose the coefficients such that

$$f_i = 0 \qquad \text{for} \quad i = 1, \ldots, r \text{ and}$$

$$h_i = x_i \cdot x_{r+1}^q + x_{i+1} \cdot x_{r+2}^q + \cdots + x_{i+n-2r-1} \cdot x_{n-r}^q .$$

Then $Z_{\bar{\rho}}$ is a cone with the vertex $\bar{\rho}$ over the variety $\{h_1 = \cdots = h_r = 0\} \subset D_{\bar{\rho}}$, which can be seen to be a reduced irreducible complete intersection of codimension r by blowing it up along $\{x_{r+1} = \cdots = x_{n-r} = 0\}$. Hence (C3. $\bar{\rho}$) holds. Now we check (C6. $\bar{\rho}$). Again by the openness of the condition, if $V_{i,\bar{\rho},\bar{\sigma}}^{(1)}$ is a hypersurface in H and does not contain the hyperplane $M \subset H$, then (C6. $\bar{\rho}$) holds. We choose $(a_{i\mu\nu})$ so that $f_i = x_{n-r+1-i}$ for $i = 1, \ldots, r$. Then M_ρ coincides with M. We consider $(x_1, \ldots, x_{n-2r+1})$ as homogeneous coordinates of H. Then the defining equations of M and $V_{i,\bar{\rho},\bar{\sigma}}^{(1)}$ in H is given by

$$M = \{x_{n-2r+1} = 0\} ,$$

$$V_{i,\bar{\rho},\bar{\sigma}}^{(1)} = \left\{ h_i(x_1, \ldots, x_{n-2r+1}, 0, \ldots, 0) = \sum_{\mu,\nu=1}^{n-2r+1} a_{i\mu\nu} x_\mu x_\nu^q = 0 \right\} ,$$

because $f_i = 0$ on H except for $i = r$, and $f_r = 0$ is the equation of M.

We can choose the coefficients $(a_{i\mu\nu})_{1 \leq \mu,\nu \leq n-2r+1}$ of the equation of $V_{i,\bar{\rho},\bar{\sigma}}^{(1)} \subset H$ still arbitrarily except for the condition $a_{i\mu\nu} = 0$ for $1 \leq \mu,\ \nu \leq r$, which is equivalent to $\pi(L) = L_{\bar{\rho},\bar{\sigma}}^{(1)} \subset V_{i,\bar{\rho},\bar{\sigma}}^{(1)}$. Hence (C6. $\bar{\rho}$) holds.

The paragraph just above says nothing but the surjectivity of Ψ. Hence (b′) is true, and the proof of Theorem 3 is completed.

3. Linear subspaces in the complete intersections. In this section we prove the following two lemmas. We still assume $k=\bar{k}$.

LEMMA 1. *Suppose that $n \geq sr+s+r$. Then, for every $V_1, \dots, V_r \in \mathscr{F}$, the intersection $V_1 \cap \cdots \cap V_r$ contains an s-dimensional linear subspace.*

LEMMA 2. *Suppose that $n \geq sr+s+2r$. Then, for every $V_1, \dots, V_r \in \mathscr{F}$ and every closed point $Q \in V_1 \cap \cdots \cap V_r$, there is an s-dimensional linear subspace contained in $V_1 \cap \cdots \cap V_r$ and passing through Q.*

Theorem 1 follows immediately from Lemma 1 and Theorem 3. Lemma 2 will be used in the next section.

PROOF OF LEMMA 1. Let I be the incidence correspondence

$$\left\{(N, \Lambda) \in \mathrm{Grass}\,(\boldsymbol{P}^s, \boldsymbol{P}^n) \times \mathrm{Grass}\,(\boldsymbol{P}^{r-1}, \mathscr{F}) \,\middle|\, \begin{array}{l}\text{the base locus } W_\Lambda \text{ of a linear} \\ \text{system } \Lambda \text{ contains } N\end{array}\right\} \tag{3.1}$$

with the natural projections

$$\begin{array}{ccc} I & \xrightarrow{\beta} & \mathrm{Grass}\,(\boldsymbol{P}^{r-1}, \mathscr{F}) \\ {\scriptstyle\alpha}\downarrow & & \\ \mathrm{Grass}\,(\boldsymbol{P}^s, \boldsymbol{P}^n)\,. & & \end{array}$$

Since $\dim \alpha^{-1}(N) = \dim \mathrm{Grass}(\boldsymbol{P}^{r-1}, \mathscr{F}) - r(s+1)^2$ for $N \in \mathrm{Grass}\,(\boldsymbol{P}^s, \boldsymbol{P}^n)$, we have

$$\dim I - \dim \mathrm{Grass}\,(\boldsymbol{P}^{r-1}, \mathscr{F}) = (s+1)(n-sr-s-r)\,.$$

Hence it is enough to show that when $n=sr+s+r$, the second projection β is generically finite. Let (N, Λ) be a general closed point of I. Let $k[\varepsilon]$ be the ring of dual numbers with $\varepsilon^2=0$. In order to show that β is generically finite, it is enough to show that any deformation of the first order $N_\varepsilon \to \mathrm{Spec}\, k[\varepsilon]$ of N which keeps N being contained in W_Λ is trivial. We fix homogeneous coordinates $(X_0, \dots, X_n)$ of $\boldsymbol{P}_k^n$ such that $N=\{X_{s+1}=\cdots=X_n=0\}$. Let $V_1, \dots, V_r \in \Lambda$ be hypersurfaces which span Λ and let

$$\sum_{\mu,\nu=0}^{n} a_{i\mu\nu} X_\mu X_\nu^q = 0 \qquad \text{where} \quad a_{i\mu\nu}=0 \quad \text{if} \quad 0 \leq \mu, \nu \leq s$$

be the equation of V_i. A deformation of N given by

$$N_\varepsilon = \left\{X_{s+1} = \left(\sum_{\lambda=0}^{s} X_\lambda c_{\lambda s+1}\right)\varepsilon, \dots, X_n = \left(\sum_{\lambda=0}^{s} X_\lambda c_{\lambda n}\right)\varepsilon\right\}$$

keeps N being contained in W_A if and only if

$$C \cdot A_i = 0 ,$$

where C denotes the $(s+1)\times(n-s)$ matrix $(c_{\lambda\mu})_{0\le\lambda\le s, s+1\le\mu\le n}$ and A_i denotes the $(n-s)\times(s+1)$ matrix $(a_{i\mu\nu})_{s+1\le\mu\le n, 0\le\nu\le s}$. When $n=sr+s+r$, the matrix $A := (A_1, \dots, A_r)$ is a square matrix of size $n-s$, and by the generality of the point $(N, A)\in I$, we can choose coefficients $(a_{i\mu\nu})_{s+1\le\mu\le n, 0\le\nu\le s}$ so that $\det A\neq 0$. Hence if N_ε is contained in W_A, we get $C=0$. □

PROOF OF LEMMA 2. Let $(X_0, \dots, X_n)$ be homogeneous coordinates of $\boldsymbol{P}_k^n$ such that $Q=(1, 0, \dots, 0)$, and let $x_i = X_i/X_0$ $(i=1, \dots, n)$ be affine coordinates of $\boldsymbol{P}_k^n$ with the origin Q. Then the equation of V_i is of the form

$$l_i + \tilde{f}_i^q + \sum_{\nu=1}^{n} x_\nu \tilde{g}_{i\nu}^q = 0 ,$$

where l_i, $\tilde{f}_i$ and $\tilde{g}_{i\nu}$ are linear forms in $(x_1, \dots, x_n)$. Regarding $(x_1, \dots, x_n)$ as homogeneous coordinates of the variety E_Q of lines in $\boldsymbol{P}_k^n$ passing through Q, we see that the reduced part of the variety $W'_Q\subset E_Q$ of lines in W passing through Q is given by

$$l_1 = \cdots = l_r = \tilde{f}_1 = \cdots = \tilde{f}_r = \sum_{\nu=1}^{n} x_\nu \tilde{g}_{1\nu}^q = \cdots = \sum_{\nu=1}^{n} x_\nu \tilde{g}_{r\nu}^q = 0 ,$$

which is an intersection of r hypersurfaces of the form (0.1) in $\boldsymbol{P}_k^m = \{l_1 = \cdots = l_r = \tilde{f}_1 = \cdots = \tilde{f}_r = 0\}\subset E_Q \cong \boldsymbol{P}_k^{n-1}$, where $m\ge n-2r-1$. By Lemma 1, W'_Q contains an $(s-1)$-dimensional linear subspace. Hence W contains an s-dimensional linear subspace passing through Q. □

4. Complete intersections of diagonal type. In this section, we shall prove Theorem 2. It is enough to show it when $m=q+1$. We still work over $k=\bar{k}$.

We fix homogeneous coordinates $(X_0, \dots, X_n)$ of $\boldsymbol{P}_k^n$ once for all and denote by $\mathscr{D}$ the linear system of hypersurfaces of diagonal type

$$b_0 X_0^{q+1} + \cdots + b_n X_n^{q+1} = 0 . \tag{4.1}$$

Let $I_{\mathscr{D}}\subset \mathrm{Grass}(\boldsymbol{P}^r, \boldsymbol{P}^n)\times \mathrm{Grass}(\boldsymbol{P}^{r-1}, \mathscr{D})$ be the incidence correspondence defined in the same way as in (3.1). We shall prove the following five statements:

(1) For general $V_1, \dots, V_r\in\mathscr{D}$, (C1) holds. Moreover, there is an r-dimensional linear subspace L contained in $W=V_1\cap\cdots\cap V_r$ such that (C2) holds with respect to L.

(2) For general $V_1, \dots, V_r\in\mathscr{D}$, there is a closed point $Q\in W=V_1\cap\cdots\cap V_r$ such that (C3.Q) holds.

(3) We fix a closed point

$$R = (\underbrace{1, \dots, 1}_{(2r+1)\text{-times}}, 0, \dots, 0) .$$

Let $\mathscr{D}_R \subset \mathscr{D}$ be the linear subsystem of $\mathscr{D}$ consisting of hypersurfaces passing through R. Then there are members $V_1, \ldots, V_r \in \mathscr{D}_R$ which satisfy (C4. R)–(C6. R).

Note that by Lemma 2 and the assumption $n \geq r^2 + 3r$, for any closed point $Q \in W$ of an intersection of any members $V_1, \ldots, V_r \in \mathscr{D}$, there is always an r-dimensional linear subspace contained in W and passing through Q. Note also that $I_{\mathscr{D}}$ is irreducible, and that the conditions (C1)–(C6) are open not only on $(V_1, \ldots, V_r)$ but also on L. Then, combining (1), (2) and (3), and invoking the openness of the conditions, we see that if $(L, \Lambda) \in I_{\mathscr{D}}$ is general, W_Λ satisfies (C1)–(C6) with respect to L. Now the following two statements allow us to show by induction on r that if $(L, \Lambda) \in I_{\mathscr{D}}$ is general, then W_Λ is a member of $U(k)$ with respect to L. Hence Theorem 2 will be proved.

(4) Let R be as in (3) and let $V_1, \ldots, V_r \in \mathscr{D}_R$ be general members. By (3), we can construct the variety D_R and G_R as in Section 1 taking ρ to be R. Let $S \in G_R$ be the closed point corresponding to the $(n-2r)$-dimensional linear subspace $H_{R,S} \subset D_R$ defined by $f_1 = \cdots = f_{r-1} = 0$. We shall show that there is a canonical identification between $H_{R,S}$ and an $(n-2r)$-dimensional projective space $\boldsymbol{P}_k^{n-2r}$, equipped with canonical homogeneous coordinates $(x_{2r}, \ldots, x_n)$ which are independent of $V_1, \ldots, V_r$, such that the equations of $V_{1,R,S}^{(1)}, \ldots, V_{r-1,R,S}^{(1)} \subset H_{R,S}$ with respect to these coordinates are of diagonal type (4.1).

(5) Let $\mathscr{D}^{(1)}$ be the variety of hypersurfaces in $\boldsymbol{P}_k^{n-2r}$ of diagonal type with respect to the homogeneous coordinates in (4). We get a rational map

$$\overbrace{\mathscr{D}_R \times \cdots \times \mathscr{D}_R}^{r\text{-times}} \to \overbrace{\mathscr{D}^{(1)} \times \cdots \times \mathscr{D}^{(1)}}^{(r-1)\text{-times}},$$
$$(V_1, \ldots, V_r) \mapsto (V_{1,R,S}^{(1)}, \ldots, V_{r-1,R,S}^{(1)}).$$

This map is dominant.

Proof of (1) and (2). It is easy to see that if $V_1, \ldots, V_r \in \mathscr{D}$ are general members, then $W = V_1 \cap \cdots \cap V_r$ is nonsingular of codimension r, hence (C1) holds. Let Q_j $(j = 0, \ldots, r)$ be a point of the intersection of W and the r-dimensional linear subspace defined by

$$X_\nu = 0 \qquad \text{unless} \quad j(r+1) \leq \nu \leq j(r+1) + r\,.$$

Since each V_i is diagonal, W contains the r-dimensional linear subspace L spanned by $Q_0, \ldots, Q_r$. Before showing that a general $(V_1, \ldots, V_r)$ satisfies (C2) with respect to this L, we make an observation about certain special points on W. We take a point on W such that $n-2r$ of its homogeneous coordinates are zero; for example $Q_0 = (\xi_0, \xi_1, \ldots, \xi_r, 0, \ldots, 0)$. Then it is easy to see that $T_{Q_0, W}$ and the intersection $P_{Q_0, W} := P_{Q_0, V_1} \cap \cdots \cap P_{Q_0, V_r}$ of polar hyperplanes coincide and they are both given by

$$X_0 : X_1 : \cdots : X_r = \xi_0 : \xi_1 : \cdots : \xi_r\,.$$

Let Q' be a point on W with coordinates

$$(\zeta_0, \underbrace{0, \dots, 0}_{r\text{-times}}, \zeta_1, \underbrace{0, \dots, 0}_{r\text{-times}}, \zeta_2, 0, \dots\dots, 0, \zeta_r, 0, \dots\dots, 0).$$

Then it can be easily checked that $P_{Q',W} \cap L$ consists of one point Q'', hence $W \not\subseteq \Gamma_W$. By the generality of $(V_1, \dots, V_r)$, we see that $\dim(T_{Q',W} \cap T_{Q'',W}) = n-2r$, which shows that $T_{Q'',W} \cap W$ is codimension r in $T_{Q'',W}$ at Q'. Hence, by dimension counting, (C2) holds. (By coordinate change of the type $X_i \mapsto c_i X_i$ $(c_i \neq 0)$, which preserves $\mathscr{D}$, we may assume that nonzero coefficients of Q_i and Q' are all 1. Then the point $P_{Q',W} \cap L$ is

$$Q'' = (\underbrace{1, \dots\dots, 1}_{(r+1)^2\text{-times}}, 0, \dots\dots, 0).$$

This will make the checking considerably less cumbersome.) Now we shall show that (C3. Q_0) holds. Let V_i be defined by $\sum_{\nu=0}^{n} b_{i\nu} X_\nu^{q+1} = 0$, and let $(Y_r, \dots, Y_n)$ be the homogeneous coordinates of $T_{Q_0,W}$ such that $T_{Q_0,W} \subsetneq \boldsymbol{P}_k^n$ be given by $(Y_r, \dots, Y_n) \mapsto (\xi_0 Y_r, \dots, \xi_r Y_r, Y_{r+1}, \dots, Y_n)$. Then $T_{Q_0,W} \cap W$ is given by $\sum_{\nu=r+1}^{n} b_{i\nu} Y_\nu^{q+1} = 0$ $(i = 1, \dots, r)$. (Note that $f_1, \dots, f_r$ are constantly zero at $\rho = Q_0$, and hence $T_{Q_0,W} \cap W$ is a cone with vertex $Q_0 = (1, 0, \dots, 0) \in T_{Q_0,W}$.) Since we can choose $(b_{i\nu})$ arbitrarily, $T_{Q_0,W} \cap W$ is a reduced irreducible complete intersection of codimension r in $T_{Q_0,W}$. Hence (C3. Q_0) holds.

PROOF OF (3), (4) AND (5). We consider $V_i \in \mathscr{D}_R$ $(i = 1, \dots, r)$ which are defined by

$$-(\alpha_i + \beta_i + \gamma_i) X_0^{q+1} + \alpha_i X_i^{q+1} + \beta_i X_{r+i}^{q+1} + \gamma_i X_{2r}^{q+1} + \sum_{\nu=2r+1}^{n} b_{i\nu} X_\nu^{q+1} = 0,$$

where the coefficients α_i, β_i $(i = 1, \dots, r)$, γ_i $(i = 1, \dots, r-1)$ and $b_{i\nu}$ are general enough. (We put $\gamma_r = 0$). We put

$$x_i = \begin{cases} X_i/X_0 - 1 & (i = 1, \dots, 2r), \\ X_i/X_0 & (i > 2r), \end{cases}$$

and, as before, regard $(x_1, \dots, x_n)$ as affine coordinates of $\boldsymbol{P}_k^n$ with the origin R or homogeneous coordinates of the variety E_R of lines in $\boldsymbol{P}_k^n$ passing through R. Then $T_{R,W}$ is given by

$$l_1 = \cdots = l_r = 0 \qquad \text{where} \quad l_i = \alpha_i x_i + \beta_i x_{r+i} + \gamma_i x_{2r}$$

and f_i and h_i are the restrictions to $T_{R,W}$ of

$$\tilde{f}_i = \alpha_i^{1/q} x_i + \beta_i^{1/q} x_{r+i} + \gamma_i^{1/q} x_{2r},$$

$$\tilde{h}_i = \alpha_i x_i^{q+1} + \beta_i x_{r+i}^{q+1} + \gamma_i x_{2r}^{q+1} + \sum_{\nu=2r+1}^{n} b_{i\nu} x_\nu^{q+1},$$

respectively. Since α_i, β_i, γ_i and $b_{i\nu}$ are general, it is easy to check (C4. R) and (C5. R). The $(n-2r)$-dimensional linear space $H_{R,S} (\subset D_R \subset E_R)$ is given by

$$l_1 = \cdots = l_r = \tilde{f}_1 = \cdots = \tilde{f}_{r-1} = 0\,,$$

which is equivalent to

$$x_j = \lambda_j x_{2r} \qquad \text{for} \quad j = 1, \dots, 2r-1\,,$$

where

$$\begin{pmatrix} \lambda_i \\ \lambda_{r+i} \end{pmatrix} = -\begin{pmatrix} \alpha_i & \beta_i \\ \alpha_i^{1/q} & \beta_i^{1/q} \end{pmatrix}^{-1} \begin{pmatrix} \gamma_i \\ \gamma_i^{1/q} \end{pmatrix} \qquad \text{for} \quad i = 1, \dots, r-1 \text{ and } \lambda_r = -\beta_r/\alpha_r\,.$$

Hence we can regard $(x_{2r}, \dots, x_n)$ as homogeneous coordinates of $H_{R,S}$. These are the canonical coordinates mentioned in (4). The hypersurface $V_{i,R,S}^{(1)} \subset H_{R,S}$ is given by

$$(\alpha_i \lambda_i^{q+1} + \beta_i \lambda_{r+i}^{q+1} + \gamma_i) x_{2r}^{q+1} + \sum_{\nu=2r+1}^{n} b_{i\nu} x_\nu^{q+1} = 0\,. \tag{4.2}$$

Thus (C6. R) holds and hence the proof of (3) is completed. The statements (4) and (5) are obvious by (4.2). q.e.d.

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