Tôhoku Math. J.
44 (1992), 365–378

INTEGRO-DIFFERENTIAL EQUATIONS AND DELAY INTEGRAL INEQUALITIES

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(Received July 29, 1991, revised February 18, 1992)

Abstract. In this paper sufficient conditions for the boundedness, asymptotic properties and exponential decay are first obtained for solutions of linear systems of integral inequalities with infinite delay. Then nonlinear integro-differential equations are reduced to delay integral inequalities by the variation of parameter formula, and some criteria are given for asymptotic stability, uniformly asymptotic stability and exponential asymptotic stability. The results obtained here are illustrated by examples which have been particularly difficult to treat by means of the standard Lyapunov theory.

1. Introduction. This paper is concerned with asymptotic behavior and stability of solutions of the integro-differential equation

$$\dot{x}(t)=A(t)x(t)+f[t, x(r_1(t))]+\int_\alpha^t G[t, s, x(r_2(s))]ds, \tag{1}$$

where $A(t)$ is a continuous $n\times n$ matrix on $[0, \infty)$, $r(t)\le r_1(t)$, $r_2(t)\le t$ and $r(t)\to\infty$ as $t\to\infty$.

In this discussion, R^n denotes the n-dimensional Euclidean space, $R^+=[0, \infty)$ and $C[X, Y]$ the class of continuous mappings from the topological space X to the topological space Y. $C=C([\alpha, 0], R^n)$, in which $\alpha\le t$ could be $-\infty$. For $\phi\in C$ we define $\|\phi\|_\alpha=\sup_{\alpha\le u\le t}|\phi(u)|$, where $|\cdot|$ is a norm in R^n.

It is assumed that $f\in C[R^+\times C, R^n]$ and $G\in C[R^+\times R\times C, R^n]$. For any $t_0\ge 0$ and any $\phi\in C$, a solution of (1) is a function $x: R\to R^n$ satisfying (1) for $t\ge t_0$ and that $x(t)=\phi(t)$ for $-\infty<t\le t_0$. Throughout this paper we always assume that (1) has a continuous solution denoted by $x(t, t_0, \phi)$ or simply $x(t)$ if no confusion should arise.

We refer the reader to [1] or [6] for the definitions of the terms we use on stability. We always assume that $f(t, 0)\equiv G(t, s, 0)\equiv 0$ in our discussion of stability.

If $r(t)\equiv t$ in (1), then (1) becomes a familiar integro-differential equation investigated extensively by a number of authors (see Burton [1], Hara, Yoneyama and Itoh [6], Kato [8] and Murakami [10] and their bibliographies). To avoid difficulty in constructing the Lyapunov functional, Gopalsamy [5] dealt with the systems of the type (1) with $r_1(t)=t-r$ (r is a constant) and $r_2(t)=t$ using the inequality technique, while Hara, Yoneyama and Itoh [6] dealt with the case with $r_2(t)=t$ and $f\equiv 0$ using the "variation of parameters" formula. Some "easily verifiable" sufficient conditions

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1991 *Mathematics Subject Classification*. Primary 34K20; Secondary 26D10, 45J05.

[1] Part of this work was supported by the National Natural Science Foundation of China.

were given by [5], [6] for the asymptotic stability (AS), uniform asymptotic stability (USA) and exponential asymptotic stability (EAS) of the zero solution of (1). In this paper the asymptotic properties of solutions of delay integral inequalities are first discussed. Then, we obtain some generalizations of the results in [5], [6] based on the inequalities. The criteria obtained are effective for the UAS and EAS of linear equations with both unbounded coefficients and infinite delay. The results are illustrated by examples which have been particularly difficult to treat by means of the standard Lyapunov theory.

ACKNOWLEDGEMENT. The author would like to thank the referee for his careful reading of this paper and making a number of helpful suggestions.

2. Delay integral inequalities.

DEFINITION 1. $f(t, s)\in UC_t$ means that $f\in C[R^+\times R, R^+]$ and that for any given α and any $\varepsilon>0$ there exist positive numbers B, T and A satisfying

$$\int_\alpha^t f(t,s)ds\le B\,,\qquad \int_\alpha^{t-T} f(t,s)ds<\varepsilon\,,\qquad \forall t\ge A\,. \tag{2}$$

Especially, $f\in UC_t$ if $f(t,s)=f(t-s)$ and $\int_0^\infty f(u)du<\infty$.

THEOREM 1. *Let $y_i\in C[R, R^+]$ be a solution of the delay integral inequality*

$$y_i(t)\le h_i(t)+\sum_{j=1}^m\left[a_{ij}(t)\|y_{jt}\|_s+\int_{\alpha_1}^t b_{ij}(t,u)\|y_{ju}\|_s du\right.$$
$$\left.+\int_{\alpha_2}^t \psi_{ij}(t,u)\int_{\alpha_3}^u c_{ij}(u,v)\|y_{jv}\|_s dvdu\right]+g_i(t,t_0)\phi_i(t_0)\,, \tag{3}$$

$$y_i(t)\le\phi_i(t)\,,\qquad \forall t\in(-\infty,t_0]\,,\quad i=1,\dots,m\,, \tag{4}$$

where $a_{ij}(t)$, $h_i(t)\in C(R^+, R^+)$, $g_i(t,t_0)\in C[R^+\times R^+, R^+]$, $\phi_i(t)\in C[(-\infty,t_0], R^+]$, $t_0\ge\tau\in R^+$, $-\infty\le\alpha\le\alpha_i\le 0$ $(i=1,2,3)$, $\|y_{jt}\|_s=\sup_{s\le u\le t}|y_j(u)|$, $s=r(t)\le t$ and $r(t)\to\infty$ as $t\to\infty$. Assume that the following conditions are satisfied:

(H1) $g_i(t,t_0)\le b=b(t_0)$ $(\forall t\ge t_0)$, $h_i(t)+g_i(t,t_0)\phi_i(t_0)\to 0$ *as* $t\to\infty$, $b_{ij}(t,u)$, $c_{ij}(t,u)$, $\psi_{ij}(t,u)\in UC_t$, *and there are constants $\pi_{ij}\ge 0$ such that*

$$a_{ij}(t)+\int_{\alpha_1}^t b_{ij}(t,u)du+\int_{\alpha_2}^t\psi_{ij}(t,u)\int_{\alpha_3}^u c_{ij}(u,v)dvdu\le\pi_{ij}\qquad\forall t\ge\tau\,. \tag{5}$$

(H2) *The spectral radius $\rho(\Pi)$ of the matrix $\Pi=(\pi_{ij})$ is less than one. Then*

(i) *$y_i(t)\to 0$ as $t\to\infty$.*

(ii) *When $h_i(t)\equiv 0$, for any $N>0$ there are constants $\delta=\delta(t_0)>0$ and $w_i>0$ such that the solution $y_i(t)$ of* (3) *with the initial condition* (4) *satisfies*

$$y_i(t)<w_i^{-1}N\,,\qquad\forall t\ge t_0\,,\quad\forall\phi\in C\,,\quad\|\phi\|_\alpha<\delta\,. \tag{6}$$

Furthermore, if b is independent of t_0, so is δ in (6); and if $t-r(t)\leq r$ ($r>0$ constant) and $g_i(t+t_0, t_0)\to 0$ as $t\to\infty$ uniformly with respect to t_0, then $\|\phi\|_\alpha<\delta$ yields that $y_i(t)\to 0$ as $t\to\infty$ uniformly, that is, for any $\varepsilon>0$, there is a positive number T, independent of t_0, such that

$$y_i(t)\leq\varepsilon\,,\qquad \forall t\geq t_0+T\,,\quad \|\phi\|_\alpha\leq\delta\,,\quad t_0\geq\tau\,. \tag{7}$$

PROOF. (i) Applying [9, Theorem 9.16] and using $\rho(\Pi)<1$, we can find positive numbers w_i such that

$$\Delta_i\equiv\sum_{j=1}^{m} w_i w_j^{-1}\pi_{ij}<1\,. \tag{8}$$

We first show that $y_i(t)$ is bounded. From $h_i(t)+g_i(t, t_0)\phi_i(t_0)\to 0$, there is a $T_1>0$ such that $w_ih_i(t)+w_ig_i(t, t_0)\phi_i(t_0)<(1-\Delta_i)/2$ for $t\geq t_0+T_1$. By the continuity of $y_i(t)$, there exists $N\geq\max\{1, w_i\|\phi\|_\alpha\}$ such that $y_i(t)<w_i^{-1}N$ for all $t\in[\alpha, t_0+T_1]$. We now prove that

$$y_i(t)<w_i^{-1}N\leq\max_i\{w_i^{-1}N\}=\Omega\,,\qquad \forall t\in[\alpha, \infty)\,. \tag{9}$$

Assume, on the contrary, there are $c>t_0+T_1$ and some i such that

$$w_iy_i(c)=N\,,\quad w_jy_j(t)\leq N\,,\qquad \forall t\in(-\infty, c]\,,\quad j=1,\dots,m\,. \tag{10}$$

By (3) and (10) we get

$$\begin{aligned} w_iy_i(c)&\leq w_ih_i(c)+w_i\sum_{j=1}^{m}\left\{a_{ij}(c)w_j^{-1}N+\int_{\alpha_1}^{c}b_{ij}(c, u)w_j^{-1}Ndu\right.\\ &\quad\left.+\int_{\alpha_2}^{c}\psi_{ij}(c, u)\int_{\alpha_3}^{u}c_{ij}(u, v)w_j^{-1}Ndvdu\right\}+w_ig_i(c)\phi_i(t_0)\\ &\leq N\Delta_i+(1-\Delta_i)/2\leq N\Delta_i+N(1-\Delta_i)/2<N\,, \end{aligned} \tag{11}$$

which contradicts the equality in (10). So (9) is true.

Next we will show that for any $\phi\in C$, $\lim_{t\to\infty}y_i(t, t_0, \phi)=0$. For any $\varepsilon>0$, let $\beta=\varepsilon\min_i\{w_i\}(1-\Delta)/2\max_i\{w_i\}$, $\Delta=\max_i\{\Delta_i\}$. From $h_i(t)+g_i(t, t_0)\phi_i(t_0)\to 0$ and b_{ij}, c_{ij}, $\psi_{ij}\in UC_t$, there exist positive numbers R and $A=A(t_0)\geq T_1$ such that for all $t\geq t_0+A$

$$\begin{gathered} h_i(t)+g_i(t, t_0)\phi_i(t_0)<\frac{\beta}{3}\,,\quad \int_{\alpha_2}^{t}\psi_{ij}(t, u)du\leq R\,,\quad \Omega\int_{\alpha_3}^{u}c_{ij}(u, v)dv\leq R\,,\\ \Omega\int_{\alpha_1}^{t-A}b_{ij}(t, u)du\leq\frac{\beta}{3m}\,,\quad \int_{\alpha_2}^{t-A}\psi_{ij}(t, u)du\leq\frac{\beta}{6Rm}\,,\quad \Omega\int_{\alpha_3}^{t-A}c_{ij}(t, v)dv\leq\frac{\beta}{6Rm}\,. \end{gathered} \tag{12}$$

Then from (9) and (12), we have

$$\int_{\alpha_1}^{t} b_{ij}(t,u)\|y_{ju}\|_s du = \left(\int_{\alpha_1}^{t-A} + \int_{t-A}^{t}\right) b_{ij}(t,u)\|y_{ju}\|_s du \tag{13}$$

$$\leq \Omega \int_{\alpha_1}^{t-A} b_{ij}(t,u)du + \int_{t-A}^{t} b_{ij}(t,u)\|y_{ju}\|_s du \leq \int_{t-A}^{t} b_{ij}(t,u)\|y_{ju}\|_s du + \frac{\beta}{3m},$$

$$\int_{\alpha_2}^{t} \psi_{ij}(t,u)\int_{\alpha_3}^{u} c_{ij}(u,v)\|y_{jv}\|_s dv du = \int_{\alpha_2}^{t-A} \psi_{ij}(t,u)\int_{\alpha_3}^{u} c_{ij}(u,v)\|y_{jv}\|_s dv du \tag{14}$$

$$+ \int_{t-A}^{t} \psi_{ij}(t,u)\left(\int_{\alpha_3}^{u-A} + \int_{u-A}^{u}\right) c_{ij}(u,v)\|y_{jv}\|_s dv du$$

$$\leq \int_{\alpha_2}^{t-A} \psi_{ij}(t,u)\int_{\alpha_3}^{u} c_{ij}(u,v)\Omega dv du + \int_{t-A}^{t} \psi_{ij}(t,u)\int_{\alpha_3}^{u-A} c_{ij}(u,v)\Omega dv du$$

$$+ \int_{t-A}^{t} \psi_{ij}(t,u)\int_{u-A}^{u} c_{ij}(u,v)\|y_{jv}\|_s dv du$$

$$\leq \int_{t-A}^{t} \psi_{ij}(t,u)\int_{u-A}^{u} c_{ij}(u,v)\|y_{jv}\|_s dv du + \frac{\beta}{3m},$$

for $t \geq t_0 + 2A$. By (3), (13), (14) and the first inequality in (12) it follows that

$$y_i(t) \leq \sum_{j=1}^{m} \left[a_{ij}(t)\|y_{jt}\|_s + \int_{t-A}^{t} b_{ij}(t,u)\|y_{ju}\|_s du + \int_{t-A}^{t} \psi_{ij}(t,u)\int_{u-A}^{u} c_{ij}(u,v)\|y_{jv}\|_s dv du \right] + \beta. \tag{15}$$

From $s = r(t) \to \infty$ $(t \to \infty)$, there must be $t_1 \geq t_0$ such that

$$\Lambda(t) \equiv \min\{r(t), t-A\} \geq t_0, \qquad \forall t \geq t_1. \tag{16}$$

In the same way, there exist $t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ such that

$$\Lambda(t) \geq t_k, \qquad \forall t \geq t_{k+1}, \quad k = 1, 2, \ldots. \tag{17}$$

From the boundedness of $y_i(t)$, we can let

$$\eta_k = \max_i \left\{ \sup_{t_k \leq \theta < \infty} \{w_i y_i(\theta)\} \right\}. \tag{18}$$

For a given k, if

$$\eta_k = \max_i \left\{ \overline{\lim_{\theta \to \infty}} \{w_i y_i(\theta)\} \right\}, \tag{19}$$

then by (15)

(20) $$\eta_k \le \Delta \eta_{k-3} + \beta \max_i \{w_i\} .$$

If (19) is not true, by (15) there is a $\bar{t} \ge t_k$ such that

(21) $$\eta_k \le \max_i \left\{ w_i \sum_{j=1}^{m} \left[a_{ij}(\bar{t}) w_j^{-1} \eta_{k-1} + \int_{\bar{t}-A}^{\bar{t}} b_{ij}(\bar{t}, u) w_j^{-1} \eta_{k-2} du + \int_{\bar{t}-A}^{\bar{t}} \psi_{ij}(\bar{t}, u) \int_{u-A}^{u} c_{ij}(u, v) w_j^{-1} \eta_{k-3} dv du \right] + w_i \beta \right\} .$$

Noting that $\eta_k \ge \eta_{k+1} \ge 0$ for all $k = 0, 1, \ldots$, we also see that

(22) $$\eta_k \le \Delta \eta_{k-3} + \beta \max_i \{w_i\} .$$

Let $\bar{\beta} = \beta \max_i \{w_i\}$. From the definition of β,

(23) $$\bar{\beta}/[1 - \Delta] \le \varepsilon \min_i \{w_i\}/2 \le w_i \varepsilon/2 .$$

Thus,

(24) $$\begin{aligned} \eta_{3k} &\le \Delta \eta_{3(k-1)} + \bar{\beta} \\ &\le \Delta[\Delta \eta_{3(k-2)} + \bar{\beta}] + \bar{\beta} \\ &\cdots\cdots\cdots\cdots\cdots \\ &\le \Delta^k \eta_0 + (1 + \Delta + \cdots + \Delta^{k-1}) \bar{\beta} \\ &\le \Delta^k \eta_0 + \bar{\beta}/[1 - \Delta] \le \Delta^k \eta_0 + w_i \varepsilon/2 . \end{aligned}$$

Since $\Delta < 1$, there is an integer p such that

$$\Delta^p < \varepsilon \min_i \{w_i/2\}/\eta_0 .$$

Thus

(25) $$\eta_{3p} \le w_i \varepsilon .$$

Taking $T = 3pA$, we get

(26) $$\sup_{t_{3p} \le \theta < \infty} \{w_i y_i(\theta)\} \le w_i \varepsilon \quad \text{or} \quad y_i(t) \le \varepsilon , \qquad \forall t \ge t_0 + T ,$$

which proves $\lim_{t \to \infty} y_i(t) = 0$.

(ii) From (8) and $h_i(t) \equiv 0$, for any $N > 0$ we may take

(27) $$\delta = N \min_i \{(1 - \Delta_i)/w_i b, w_i^{-1}\} .$$

When $\|\phi\|_\alpha<\delta$, (6) may be proved in a fashion completely analogous to that of (9).

Finally, we prove that (7) is true. Since $g_i(t, t_0)$ is uniformly bounded, $g_i(t+t_0, t_0)\to 0$ as $t\to\infty$ uniformly with respect t_0, and since Ω defined through (9) corresponding to N in (6) is independent of t_0, we can find a positive number $A\geq r\ (\geq t-r(t))$, independent of t_0, such that all the inequalities in (12) hold. Then t_k in (17) can be given by

$$t_k=t_0+kA, \qquad k=1,2,\dots. \tag{28}$$

Using techniques similar to that before and taking $T=3pA$, independent of t_0, we have

$$y_i(t)\leq\varepsilon, \qquad \forall t\geq t_0+T. \tag{29}$$

The proof of this theorem is complete.

REMARK 1. If (3) holds under the condition $y_i(t)<\bar{\delta}$ for some $\bar{\delta}$, then (6) is true for any $0<N<\bar{\delta}\min_i\{w_i\}$, provided δ is defined by (27).

THEOREM 2. *Let $y_i\in C[R, R^+]$ $(i=1,\dots,m)$ and*

$$y_i(t)\leq\sum_{j=1}^{m}\left\{a_{ij}(t)\|y_{jt}\|_s+\int_{\alpha_1}^{t}\psi_{ij}(t,u)\|y_{ju}\|_s du\right. \tag{30}$$
$$\left.+\int_{\alpha_2}^{t}\xi_{ij}(t,u)\int_{\alpha_3}^{u}\zeta_{ij}(u,v)\|y_{jv}\|_s dvdu+b_{ij}\|\phi\|_\alpha e^{-\delta_j(t-t_0)}\right\},$$

where $\alpha_i\in(-\infty,0]$ $(i=1,2,3)$, $s=t-r$, r, $b_{ij}\geq 0$, $\delta_j>0$ are constants, $a_{ij}(t)$, $\psi_{ij}(t,u)$, $\xi_{ij}(t,u)$, $\zeta_{ij}(t,u)$ are continuous for $-\infty<u\leq t<\infty$.

Suppose there are nonnegative numbers μ_{ij} and a positive number σ such that

$$a_{ij}(t)+\int_{\alpha_1}^{t}\psi_{ij}(t,u)e^{\sigma(t-u)}du+\int_{\alpha_2}^{t}\xi_{ij}(t,u)\int_{\alpha_3}^{u}\zeta_{ij}(u,v)e^{\sigma(t-v)}dvdu\leq\mu_{ij}, \qquad \forall t\geq t_0, \tag{31}$$

and the spectral radius of the matrix (μ_{ij}) satisfies

$$\rho(\mu_{ij})<1. \tag{32}$$

Then there is a constant $\lambda>0$ and for every $N>0$ there exist positive numbers w_i and $\delta<\min\{w_i^{-1}N\}$ such that for $i=1,\dots,m$,

$$w_iy_i(t)<Ne^{-\lambda(t-t_0)}\equiv z(t), \qquad \forall t\geq t_0, \quad \|\phi\|_\alpha<\delta. \tag{33}$$

PROOF. From $\rho(\mu_{ij})<1$, for any given r there are $w_i>0$ and sufficiently small positive number $\lambda<\min\{\sigma,\delta_i\}$ such that

$$\sum_{j=1}^{m}w_iw_j^{-1}\mu_{ij}e^{\lambda r}<1. \tag{34}$$

For any given $N>0$, there is a sufficiently small $\delta>0$ such that

$$\Delta_i(\lambda)\equiv w_i\sum_{j=1}^{m}\{w_j^{-1}\mu_{ij}e^{\lambda r}+b_{ij}\delta/N\}<1\ . \tag{35}$$

If (33) is not the case, there is a number $c>t_0$ and some i such that

$$w_iy_i(c)=z(c)\ ,\quad w_jy_j(t)\le z(t)\ ,\qquad \forall t\in(-\infty,c]\ ,\quad j=1,\ldots,m\ . \tag{36}$$

By (30), (36) and (33), when $\|\phi\|_\alpha<\delta$ we have

$$\begin{aligned}w_iy_i(c)\le{}& w_i\sum_{j=1}^{m}\left\{a_{ij}(c)w_j^{-1}Ne^{-\lambda(c-t_0)}e^{\lambda r}+\int_{\alpha_1}^{c}\psi_{ij}(c,u)w_j^{-1}Ne^{-\lambda(u-t_0)}e^{\lambda r}du\right.\\&\left.+\int_{\alpha_2}^{c}\xi_{ij}(c,u)\int_{\alpha_3}^{u}\zeta_{ij}(u,v)w_j^{-1}Ne^{-\lambda(v-t_0)}e^{\lambda r}dvdu+b_{ij}\|\phi\|_\alpha e^{-\delta_j(c-t_0)}\right\}\\ \le{}& Ne^{-\lambda(c-t_0)}w_i\sum_{j=1}^{m}\left\{w_j^{-1}e^{\lambda r}\left[a_{ij}(c)+\int_{\alpha_1}^{c}\psi_{ij}(c,u)e^{\lambda(c-u)}du\right.\right.\\&\left.\left.+\int_{\alpha_2}^{c}\xi_{ij}(c,u)\int_{\alpha_3}^{u}\zeta_{ij}(u,v)e^{\lambda(c-v)}dvdu\right]+b_{ij}\delta/N\right\}\\ \le{}& N\Delta_i(\lambda)e^{-\lambda(c-t_0)}<Ne^{-\lambda(c-t_0)}=z(c)\ ,\end{aligned} \tag{37}$$

which is a contradiction. Theorem is now proved.

3. Stability of integro-differential equation. To derive conditions for the stability of the systems with large dimensionality, let us assume that the system (1) can be decomposed into m subsystems described by the equations

$$\dot{x}_i(t)=A_i(t)x_i(t)+F_i(t,x(\cdot))+\int_{\alpha}^{t}G_i(t,s,x(r_2(s)))ds\ , \tag{38}$$

where $x_i\in R^{n_i}$, $\sum_{i=1}^{m}n_i=n$, $F_i(t,x(\cdot))=\sum_{j=1\,j\ne i}^{m}A_{ij}(t)x_j(t)+f_i(t,x(r_1(t)))$.

We make the following assumptions

$$\text{(A)}\quad\begin{cases}|F_i(t,x(\cdot))|\le\sum_{j=1}^{m}[b_{ij}^{(1)}(t)\|x_{jt}\|_s+b_{ij}^{(2)}(t)\circ(\|x_{jt}\|_s)]+g_i(t)\\ |G_i(t,u,x(r_2(u)))|\le\sum_{j=1}^{m}[d_{ij}^{(1)}(t)c_{ij}^{(1)}(t,u)\|x_{ju}\|_s+d_{ij}^{(2)}(t)c_{ij}^{(2)}(t,u)\circ(\|x_{ju}\|_s)]\ ,\end{cases}$$

where $b_{ij}^{(k)}(t)$, $d_{ij}^{(k)}(t)\in C[R^+,R^+]$, $c_{ij}^{(k)}(t,u)\in C[R^+\times R,R^+]$ $(k=1,2)$, $\circ(v)/v\to0$ as $v\to0$.

Let $\Phi_i(t,t_0)$ be a fundamental matrix [7, p. 82] of the linear equation $\dot{x}_i(t)=A_i(t)x_i(t)$ and assume that

$$\text{(B)}\quad\begin{cases}|\Phi_i(t,u)|d_{ij}^{(k)}(u)\in UC_t\ ,\quad |\Phi_i(t,t_0)|\to0\ \ (t\to\infty)\\ |\Phi_i(t,u)|b_{ij}^{(k)}(u)\in UC_t\ ,\quad c_{ij}^{(k)}(t,u)\in UC_t\ \ (k=1,2)\ .\end{cases}$$

THEOREM 3. *Let the assumptions* (A) *and* (B) *hold and suppose there are non-negative numbers* $\pi_{ij}^{(k)}$ $(k=1,2)$ *such that for any* $t\ge\tau\in R^+$

$$\int_{t_0}^{t} |\Phi_i(t, u)|[b_{ij}^{(k)}(u) + d_{ij}^{(k)}(u)\int_{\alpha}^{u} c_{ij}^{(k)}(u, v)dv]du \le \pi_{ij}^{(k)} . \tag{39}$$

If the spectral radius $\rho(\Pi^{(1)})$ of the matrix $\Pi^{(1)} = (\pi_{ij}^{(1)})$ is less than one, then the following hold:

(i) $\circ(\cdot) \equiv 0$ *and* $\int_{t_0}^{t} |\Phi_i(t, u)| g_i(u)du \to 0$ $(t \to \infty)$ *imply that*

$$|x_i(t, t_0, \phi)| \to 0 , \quad \forall \phi \in C . \tag{40}$$

In the following (ii) *and* (iii) *we let $g_i(t) \equiv 0$ so that $x \equiv 0$ is a solution of* (38) *and is called the zero solution.*

(ii) *The zero solution is* AS.

(iii) *The zero solution is* UAS *if $|\Phi_i(t + t_0, t_0)| \to 0$ as $t \to \infty$ uniformly in $t_0 \ge \tau \in R^+$ and there are $r > 0$ and $b > 0$ such that $t - r(t) \le r$ (for any $t \ge \tau$) and $|\Phi_i(t, u)| \le b$ for $\tau \le u \le t < \infty$.*

(iv) *If in* (ii) *and* (iii) $\circ(\cdot) \equiv 0$ *and* (A) *holds for all $x \in R^n$, the zero solution is globally* AS *and globally* UAS, *respectively.*

PROOF. By the assumptions (A), (B) and the variation of parameters formula, we have

$$\begin{aligned} |x_i(t)| \le \int_{t_0}^{t} |\Phi_i(t, u)| \Big\{ \sum_{j=1}^{m} [b_{ij}^{(1)}(u)\|x_{ju}\|_s + b_{ij}^{(2)}(u) \circ (\|x_{ju}\|_s)] \\ + \int_{\alpha}^{u} \sum_{j=1}^{m} [d_{ij}^{(1)}(u)c_{ij}^{(1)}(u, v)\|x_{jv}\|_s + d_{ij}^{(2)}(u)c_{ij}^{(2)}(u, v) \circ (\|x_{jv}\|_s)]dv \Big\} du \\ + \int_{t_0}^{t} |\Phi_i(t, u)| g_i(u)du + |\Phi_i(t, t_0)| |\phi_i(t_0)| . \end{aligned} \tag{41}$$

Under the conditions of (i), in (5) $\pi_{ij} = \pi_{ij}^{(1)}$ and $\rho(\pi_{ij}^{(1)}) < 1$. So (i) holds by Theorem 1.

We now prove (ii), (iii). Since $\rho(\Pi^{(1)}) < 1$, there is a positive number $\bar{\varepsilon} \ll 1$ such that

$$\rho(\Pi^{(1)} + \bar{\varepsilon}\Pi^{(2)}) < 1 \tag{42}$$

by the property that the spectral radius of a matrix vary continuously as the elements of the matrix vary continuously.

For the above $\bar{\varepsilon}$, there exists $\delta > 0$ such that $|x_{it}|_s < \delta$ implies $\circ(\|x_{it}\|_s) < \bar{\varepsilon}\|x_{it}\|_s$. By using (41) and $g_i(t) \equiv 0$, we have

$$\begin{aligned} |x_i(t)| \le \sum_{j=1}^{m} \int_{t_0}^{t} |\Phi_i(t, u)| \{ [b_{ij}^{(1)}(u) + \bar{\varepsilon} b_{ij}^{(2)}(u)] \|x_{ju}\|_s + \int_{\alpha}^{u} [d_{ij}^{(1)}(u)c_{ij}^{(1)}(u, v) \\ + \bar{\varepsilon} d_{ij}^{(2)}(u)c_{ij}^{(2)}(u, v)] \|x_{jv}\|_s dv \} du + |\Phi_i(t, t_0)| |\phi_i(t_0)| \end{aligned} \tag{43}$$

as long as $|x_i(s)| < \delta$ for all $s \in (-\infty, t]$.

By (39) and (43), in (5) we may take $\pi_{ij}=\pi_{ij}^{(1)}+\bar{\varepsilon}\pi_{ij}^{(2)}$. Thus (42) implies that $\rho(\pi_{ij})<1$ under the condition $|x_i(t)|<\delta$. By means of (ii) in Theorem 1 and Remark 1, for the above δ, there exists a positive number $\eta\le\delta$ such that $|x_i(t)|<\delta$ (for any $t\in R$) as $\|\phi\|_\alpha<\eta$. In (ii) we have the stability by taking an arbitrary small N in (6). In (iii) the uniform stability is implied by (27) and the uniform boundedness of $\Phi_i(t,t_0)$, which together with the other conditions assure that $x_i(t)\to 0$ as $t\to\infty$ uniformly by the last conclusion of Theorem 1.

Under the assumptions of (iv), the δ in (6) may be taken as $\delta=N\min_i\{(1-\Delta_i)/w_ib, w_i^{-1}\}$ (see (27)), where b is an upper bound of $|\Phi_i(t,t_0)|$. Applying Theorem 1, we get the global AS (or UAS) by taking an arbitrary large N so that the δ may be arbitrarily large. The proof is now complete.

COROLLARY 1. *If the assumption* (A) *holds and* (B) *is replaced by*

$$(\mathrm{B}')\quad \begin{cases} |\Phi_i(t,u)|\le M_i\exp\{-\int_u^t\alpha_i(v)dv\}, & c_{ij}^{(k)}(t,u)\in UC_t\\ b_{ij}^{(k)}(t)\le\bar{a}_{ij}^{(k)}\alpha_i(t), \quad d_{ij}^{(k)}(t)\le\bar{b}_{ij}^{(k)}\alpha_i(t) & (k=1,2), \end{cases}$$

where $\alpha_i(u)\ge 0$ *satisfies* $\int_{t-T}^t\alpha_i(u)du\rightrightarrows\infty$ *in* $t\ge\tau$, *that is,*

$$\int_{t-T}^t\alpha_i(u)du\to\infty \quad \text{as} \quad T\to\infty \text{ uniformly in } t\ge\tau\,, \tag{44}$$

and if the spectral radius $\rho\{M_i(\bar{a}_{ij}^{(1)}+\bar{b}_{ij}^{(1)}\bar{c}_{ij}^{(1)})\}<1$, *in which* $\bar{c}_{ij}^{(1)}\ge\int_\alpha^t c_{ij}^{(1)}(t,s)ds$ *for all* $t\ge\tau\in R^+$, *then all the conclusions of Theorem 3 are true under the conditions except the uniform boundedness of* $\Phi_i(t,u)$ *in* (iii) (*which is included in* (B′)).

PROOF. From the assumption (B′),

$$\begin{aligned}\int_{t_0}^t|\Phi_i(t,u)|b_{ij}^{(k)}(u)du&\le M_i\int_{t_0}^t\exp\left\{-\int_u^t\alpha_i(v)dv\right\}\bar{a}_{ij}^{(k)}\alpha_i(u)du\\ &\le M_i\bar{a}_{ij}^{(k)}\exp\left\{-\int_u^t\alpha_i(v)dv\right\}\Bigg|_{t_0}^t\le M_i\bar{a}_{ij}^{(k)}\,,\end{aligned} \tag{45}$$

which implies that $|\Phi_i(t,u)|b_{ij}^{(k)}(u)\in UC_t$ by $\int_{t-T}^t\alpha_i(u)du\rightrightarrows\infty$ in $t\ge\tau$. Using the same method, we can get

$$\int_{t_0}^t|\Phi_i(t,u)|d_{ij}^{(k)}(u)du\le M_i\bar{b}_{ij}^{(k)} \tag{46}$$

and $|\Phi_i(t,u)|d_{ij}^{(k)}(u)\in UC_t$. Hence $\pi_{ij}^{(k)}$ in (39) may be $M_i(\bar{a}_{ij}^{(k)}+\bar{b}_{ij}^{(k)}\bar{c}_{ij}^{(k)})$ and $\rho(\pi_{ij}^{(1)})<1$. Thus by Theorem 3 the proof is complete.

EXAMPLE 1. Consider the scalar equation

$$\dot{x}(t)=-a(t)x(t)+b(t)x(r(t))+c(t)\int_0^t k(t,s)x(s)ds\,, \tag{47}$$

where $a(t) \geq 0$, $r(t) \leq t$, $b(t)$, $c(t)$ and $k(t, s)$ are continuous functions and $r(t) \to \infty$ as $t \to \infty$. Let $t_+ = \max\{t, 0\}$. Suppose $\int_{t-T}^{t} a_i(u)du \rightrightarrows \infty$ in $t \geq 0$, $|b(t)| \leq ba(t)$, $c(t) \leq ca(t)$ $(a, c \geq 0)$ and

$$\sup_{t \geq 0} \int_0^{(t-T)_+} |k(t, s)| ds \to 0 \quad \text{as} \quad T \to \infty , \tag{48}$$

$$\sup_{t \geq 0} \left\{ b + c \int_0^t |k(t, s)| ds \right\} < 1 . \tag{49}$$

Then by Corollary 1 the zero solution of (47) is globally AS. Furthermore, if $t - r(t)$ is bounded, the zero solution is globally UAS.

REMARK 2. Example 1 is notable because $a(t)$, $b(t)$ and $c(t)$ may be unbounded and can vanish over the time interval sets with infinite measure; yet we conclude AS (or UAS if $t - r(t)$ is bounded). If $b(t) \equiv 0$, $c(t) \equiv 1$ and $\int_0^t k(t, s)ds \leq ca(t)$ $(c < 1)$, an analogue of Theorem 8 (II) in [8] can be obtained by Corollary 1.

COROLLARY 2. *Suppose that there exist numbers* $M_i > 0$, $\delta_i > 0$, $b_{ij} \geq 0$ *and continuous functions* $c_{ij}(v)$ *such that*

$$|\Phi_i(t, t_0)| \leq M_i e^{-\delta_i(t-t_0)} , \tag{50}$$

$$|F_i(t, x(\cdot))| \leq \sum_{j=1}^{m} b_{ij} \|x_{jt}\|_s + g_i(t) , \qquad g_i \in C(R^+, R^+) , \tag{51}$$

$$|G_i(t, u, x(r_2(u)))| \leq \sum_{j=1}^{m} c_{ij}(t-u) \|x_{ju}\|_s , \qquad \forall x \in R^n . \tag{52}$$

If

$$\rho(\pi_{ij}) \equiv \rho\left[(b_{ij} + \int_0^\infty c_{ij}(v)dv) M_i / \delta_i \right] < 1 , \tag{53}$$

then:

(i) *All solutions of* (38) *approach zero as* $t \to \infty$ *if* $g_i(t)$ *is bounded and* $\int_0^\infty g_i(v)dv < \infty$.

(ii) *The zero solution of* (38) *is globally* UAS *if* $g_i(t) \equiv 0$ *and* $t - r(t) \leq r$.

PROOF. From the conditions (50), (51) and (52), the assumptions (A) and (B′) are satisfied, and

$$\bar{a}_{ij}^{(1)} = \frac{b_{ij}}{\delta_i} , \quad \bar{b}_{ij}^{(1)} = \frac{1}{\delta_i} , \quad \bar{c}_{ij}^{(1)} = \int_0^\infty c_{ij}(v)dv . \tag{54}$$

Hence by (53) $\rho\{M_i(\bar{a}_{ij}^{(1)} + \bar{b}_{ij}^{(1)} \bar{c}_{ij}^{(1)})\} < 1$ and (ii) holds by Corollary 1.

In the following we will show that under the assumptions of (i)

(55) $$\int_{t_0}^{t} |\Phi_i(t, u)| g_i(u) du \to 0 .$$

Let $g_i(t) \leq A$, for any $\varepsilon > 0$ there is a constant $T \geq 0$ such that

(56) $$\int_{t_0}^{t-T} |\Phi_i(t, u)| du \leq \int_{t_0}^{t-T} M_i e^{-\delta_i(t-u)} du \leq \frac{M_i}{\delta_i} e^{-\delta_i T} < \varepsilon/2A$$

and for $t - T \gg 1$, $\int_{t-T}^{t} g_i(v) dv < \varepsilon/2M_i$. Thus

(57) $$\int_{t_0}^{t} |\Phi_i(t, u)| g_i(u) du = \left\{ \int_{t_0}^{t-T} + \int_{t-T}^{t} \right\} |\Phi_i(t, u)| g_i(u) du$$

$$\leq A \int_{t_0}^{t-T} |\Phi_i(t, u)| du + M_i \int_{t-T}^{t} g_i(u) du < \varepsilon .$$

This proves that (55) holds and (i) is now proved.

REMARK 3. The spectral radius is bounded by all norms. In particular, taking $\|(a_{ij})\| = \max_j \sum_{i=1}^{m} |a_{ij}|$, we have in (53)

(58) $$\rho(\pi_{ij}) \leq \left\{ \|(b_{ij})\| + \left\| \left(\int_0^{\infty} c_{ij}(v) dv \right) \right\| \right\} M_i/\delta_i .$$

Therefore, $\rho(\pi_{ij}) < 1$ if

(59) $$\|(b_{ij})\| + \left\| \left(\int_0^{\infty} c_{ij}(v) dv \right) \right\| < \delta_i/M_i ,$$

which shows that Corollary 2 includes Theorem 2.1 of Gopalsamy [5] as a special case when $x_i \in R$, $A_i(t) = a_{ii}$, $F_i(t, x(\cdot)) = \sum_{j=1\, j\neq i}^{m} a_{ij} x_j(t - \pi_{ij})$, $G_i(t, s, x(r_2(s))) = \sum_{j=1}^{m} k_{ij}(t-s) x_j(s)$ with $\int_0^{\infty} s |k_{ij}(s)| ds < \infty$ (which is not necessary in Corollary 2).

REMARK 4. When $m = 1$ (i.e. $x_1 \in R^n$), $F_1 \equiv 0$, the condition (53) becomes

(60) $$\int_0^{\infty} c_{11}(v) dv < \frac{\delta_1}{M_1} ,$$

which is the condition (5.2) of Theorem 5.1 in Hara et al. [6].

To drive conditions for EAS, we now make the following assumptions:

(A′) $$\begin{cases} |F_i(t, x(\cdot))| \leq \sum_{j=1}^{m} [b_{ij}^{(1)}(t) \|x_{jt}\|_s + b_{ij}^{(2)}(t) \circ (\|x_{jt}\|_s)] , & s = t - r \\ |G_i(t, u, x(r_2(u)))| \leq \sum_{j=1}^{m} [c_{ij}^{(1)}(t, u) \|x_{ju}\|_s + c_{ij}^{(2)}(t) \circ (\|x_{ju}\|_s)] , & \end{cases}$$

(B″) $|\Phi_i(t, t_0)| \leq M_i e^{-\delta_i(t - t_0)}$ ($M_i > 0$, $\delta_i > 0$ are constants).

THEOREM 4. *Suppose* (A′) *and* (B″) *hold. Suppose that there are nonnegative* *numbers* $\beta_{ij}^{(k)}$, $\gamma_{ij}^{(k)}$ $(k=1,2)$ *and a positive number* σ *such that*

$$\int_{t_0}^{t} |\Phi_{ij}(t,u)| b_{ij}^{(k)}(u) e^{\sigma(t-u)} du \leq \beta_{ij}^{(k)}, \tag{61}$$

$$\int_{t_0}^{t} |\Phi_{ij}(t,u)| \int_{\alpha}^{u} c_{ij}^{(k)}(u,v) e^{\sigma(t-v)} dv du \leq \gamma_{ij}^{(k)}, \qquad \forall t \geq \tau \in R^+. \tag{62}$$

Suppose further that the spectral radius

$$\rho(\beta_{ij}^{(1)} + \gamma_{ij}^{(1)}) < 1. \tag{63}$$

Then the zero solution of (38) *is* EAS. *Furthermore, if* $\circ(\|x_{jt}\|_s) \equiv 0$ *and* (A′) *holds for all* $x \in R^n$, *the zero solution is globally* EAS.

PROOF. By (63) there exists a sufficiently small positive number $\tilde{\delta}$ such that

$$\rho[\beta_{ij}^{(1)} + \gamma_{ij}^{(1)} + \tilde{\delta}(\beta_{ij}^{(2)} + \gamma_{ij}^{(2)})] \equiv \rho(\mu_{ij}) < 1. \tag{64}$$

For the above $\tilde{\delta}$ there exists a positive number $\bar{\delta} \leq \tilde{\delta}$ such that $\|x_{it}\|_s < \bar{\delta}$ implies $\circ(\|x_{it}\|_s) < \tilde{\delta}\|x_{it}\|_s$. By the assumptions (A′), (B″), and imitating the proof of (43), we have

$$|x_i(t)| \leq \|\phi\|_\alpha M_i e^{-\delta_i(t-t_0)} + \sum_{j=1}^{m} \int_{t_0}^{t} |\Phi_i(t,u)| \left\{ [b_{ij}^{(1)}(u) + \tilde{\delta} b_{ij}^{(2)}(u)] \|x_{iu}\|_s + \int_{\alpha}^{u} [c_{ij}^{(1)}(u,v) + \tilde{\delta} c_{ij}^{(2)}(u,v)] \|x_{iv}\|_s dv \right\} du. \tag{65}$$

Hence we can use Theorem 2 to prove that the zero solution of (38) is EAS.

When $\circ(\|x_{jt}\|_s) \equiv 0$, (63) implies that there exist $w_i > 0$ and $\lambda > 0$ such that

$$\sum_{j=1}^{m} w_i w_j^{-1} [\beta_{ij}^{(1)} + \gamma_{ij}^{(1)}] e^{\lambda r} < 1. \tag{66}$$

Thus for any $\delta > 0$, there exists $N(\delta) \gg 1$ such that

$$\sum_{j=1}^{m} w_i w_j^{-1} [\beta_{ij}^{(1)} + \gamma_{ij}^{(1)}] e^{\lambda r} + w_i \delta M_i / N < 1. \tag{67}$$

Letting $\mu_{ij} = \beta_{ij}^{(1)} + \gamma_{ij}^{(1)}$ and $b_{ij} = M_i$, (35) holds by (67). Following the remainder of the proof of Theorem 2, the zero solution is globally EAS. The proof is now complete.

REMARK 5. Theorem 4 includes Theorem 6.1 of Hara et al. [6] as a special case when $m=1$ (i.e. $x_1 \in R^n$), $F_1 \equiv 0$ and

$$\sup_{u \geq 0} \int_{0}^{u} c_{11}^{(1)}(u,v) e^{\sigma(u-v)} dv < \frac{\delta_1}{M_1}. \tag{68}$$

It is difficult to check the following example by (68) and the Liapunov method.

EXAMPLE 2. Let $a>0$, $\tau>0$, $\alpha>0$, b, c be constants, and consider the scalar equation

$$\dot{x}(t) = -atx(t) + btx(t-\tau) + ct \int_{-\infty}^{t} e^{-\alpha(t-s)} x(s) ds . \tag{69}$$

Then (A′) and (B″) hold, $|\Phi(t, t_0)| = \exp\{-\int_{t_0}^{t} avdv\} \leq e^{-a(t-t_0)}$ for $t_0 \geq 1$, and for any positive number $\sigma < \max\{a, \alpha\}$

$$\int_{t_0}^{t} \exp\left\{\int_{u}^{t} -avdv\right\} |b| ue^{\sigma(t-u)} du \leq \int_{t_0}^{t} \exp\left\{-(a-\sigma)\int_{u}^{t} vdv\right\} |b| udu \leq \frac{|b|}{a-\sigma} , \tag{70}$$

$$\int_{t_0}^{t} \exp\left\{\int_{u}^{t} -awdw\right\} |c| u \int_{-\infty}^{u} e^{-\alpha(u-v)} e^{\sigma(t-v)} dvdu \leq \frac{|c|}{(a-\sigma)(\alpha-\sigma)} . \tag{71}$$

(63) implies that there exists $\sigma>0$ such that

$$\frac{|b|}{a-\sigma} + \frac{|c|}{(a-\sigma)(\alpha-\sigma)} < 1 . \tag{72}$$

This is equivalent to

$$\frac{|b|}{a} + \frac{|c|}{a\alpha} < 1 . \tag{73}$$

Therefore if (73) holds, then by Theorem 4, the zero solution of (69) is globally EAS.

REMARK 6. It is worth noting that in Example 2 the coefficients at, bt and ct are unbounded and the delay is infinite. Similar examples for UAS (or AS) were given by Burton, Casal and Somolinos [2], Burton and Hatvani [3], Busenberg and Cook [4] and Xu [12]. However, the EAS cannot be implied by the UAS even if the equations with infinite delay are linear and autonomous (Murakami [10]).

REMARK 7. The methods obtained can be applied to the stability analysis of neutral functional differential equations [11].

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