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GLOBAL UNIQUENESS FOR OVALOIDS IN EUCLIDEAN AND AFFINE DIFFERENTIAL GEOMETRY

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Abstract. Ovaloids are uniquely determined by the connections induced from a relative normalization, and the volume form of the relative metric. While the equiaffine interpretations are new, the Euclidean specialization revisits results of Minkowski, Liebmann and Cohn-Vossen.

1. Introduction. From Bonnet's theorem two surfaces $x, x^{\#}: M \to E_3$ in Euclidean 3-space are equivalent modulo a Euclidean motion if the first and second fundamental forms I, II coincide on M:

$$\mathrm{I} = \mathrm{I}^{\#}, \quad \mathrm{II} = \mathrm{II}^{\#}.$$

If the Euclidean Gauss curvature K is non zero, one can state local analogues using two of the three fundamental forms I, II, III of the surfaces.

There are a series of well-known global uniqueness results for ovaloids. In (1.1)–(1.3) we recall three of them. We state the assumptions which imply the equivalence of $x, x^{\#}$ modulo Euclidean motions:

(1.1) MINKOWSKI 1903: $\mathrm{III} = \mathrm{III}^{\#}$, $K = K^{\#}$.

(1.2) COHN-VOSSEN 1927: $\mathrm{I} = \mathrm{I}^{\#}$. LIEBMANN proved in 1901 a corresponding result about infinitesimal rigidity.

(1.3) GROVE: $\mathrm{II} = \mathrm{II}^{\#}$, $K = K^{\#}$.

In [H et al] we collected different methods of proof for these results and generalizations due to various authors; references are included there.

The equiaffine analogue to Bonnet's local theorem is Radon's existence and uniqueness result, which similarly holds in relative differential geometry ([BLA, §60, 65]; [SCHI, Chap. IV, V, VIII]). Barthel proved a version of the fundamental theorem emphasizing the role of the induced first connection; this and some global uniqueness results for relative connections are considered in [SI-1].

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It is the aim of this paper to continue our global investigations from [SI-1] and to study the global role of the relative connections. These results illuminate in particular the common background for some of the Euclidean results recalled in the beginning.

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2. Global uniqueness for Riemannian metrics. We state a general uniqueness result for two-dimensional Riemannian manifolds.

2.1. THEOREM. *Let M be a closed 2-dimensional differentiable manifold of genus zero and let G, $G^\#$ be Riemannian metrics on M with the same Riemannian volume form $\omega(G)=\omega(G^\#)$. Let $\tilde{\nabla}$ be an affine connection without torsion on M. If G, $G^\#$ satisfy Codazzi equations with respect to $\tilde{\nabla}$, which means in local coordinates*

$$\text{(i)}\quad \tilde{\nabla}_k G_{ij}=\tilde{\nabla}_j G_{ik} \quad \textit{and} \quad \text{(ii)}\quad \tilde{\nabla}_k G^\#_{ij}=\tilde{\nabla}_j G^\#_{ik}\,,$$

then $G=G^\#$ on M.

PROOF. Define $g:=G-G^\#$. From (i) and (ii) g satisfies

$$\partial_j g_{ik}-\partial_k g_{ij}=\tilde{\Gamma}^r_{ij}g_{rk}-\tilde{\Gamma}^r_{ik}g_{rj}\,. \tag{2.1.1}$$

Define the symmetric (2.0)-tensor D by

$$D^{ik}:=G^{ik}+G^{\#ik}\,,$$

where G^{ik} and $G^{\#ik}$ are the components of G^{-1} and $(G^\#)^{-1}$, respectively; D is positive definite. We introduce isothermal coordinates for D [H et al, pp. 137–139]. As the volume forms coincide, we get

$$\text{trace}_D\, g=D^{ik}g_{ik}=0\,, \tag{2.1.2}$$

and thus we have $g_{11}=-g_{22}$ in isothermal coordinates of D. (2.1.2) implies $\det(g_{ij})\le 0$. Now (2.1.1) is a linear elliptic system in g_{11}, g_{12}

$$\partial_1 g_{12}-\partial_2 g_{11}=(\tilde{\Gamma}^1_{21}+\tilde{\Gamma}^2_{11})(-g_{11})+(\tilde{\Gamma}^1_{11}-\tilde{\Gamma}^2_{12})g_{12}\,,$$

$$\partial_2 g_{12}+\partial_1 g_{11}=(\tilde{\Gamma}^1_{22}+\tilde{\Gamma}^2_{21})g_{11}-(\tilde{\Gamma}^1_{21}-\tilde{\Gamma}^2_{22})g_{12}\,.$$

We use now the index method. The following is well-known [H et al, Chapter 2.2]: Either the solutions of this system are identically zero, or the points $p\in M$ with $g_{11}(p)=0=g_{12}(p)$ are isolated and the net defined globally by the equation $g(v,w)=0$ has negative index in p. But the index sum of M must be positive from the Euler-Poincaré formula, so the only global solution is $g=0$, that means $G=G^\#$.

2.2. PROBLEM. It would be of interest to solve the following open problem similar to (2.1): let M be defined as in (2.1), and let $\nabla, \nabla^{\#}$ be two affine connections without torsion; assume that a Riemannian metric g satisfies Codazzi equations with respect to both connections:

$$\nabla_k g_{ij} = \nabla_j g_{ik} \quad \text{and} \quad \nabla^{\#}_k g_{ij} = \nabla^{\#}_j g_{ik} .$$

Under which additional conditions do the connections $\nabla, \nabla^{\#}$ coincide?

In case that $\nabla, \nabla^{\#}$ are Levi-Civita connections of Riemannian metrics G, $G^{\#}$, respectively, and $\omega(G) = \omega(G^{\#})$, the answer is in the affirmative; see [S-W].

3. Relative geometry. To apply the foregoing results to relative geometry we recall some basic definitions and summarize the notation. [SCHI] gives in Chapter 8 a classical introduction, [S-S-V] in Chapters 3–4 a modernized approach.

Let A be a real affine space of dimension $n+1$, $n \geq 2$; denote by V, V^* the associated real vector space and its dual, and by $\langle\ ,\ \rangle : V^* \times V \to \boldsymbol{R}$ the canonical scalar product. Let Det denote an arbitrary determinant form and Det* its dual, and $\bar{\nabla}$ the canonical flat connection on A, and V, V^*, respectively.

Consider a connected, orientable C^{∞}-differentiable manifold of dimension n and a hypersurface immersion $x: M \to A$. With respect to a fixed origin $O \in A$ we denote its position vector by x again.

An arbitrary nowhere vanishing section X of the conormal bundle $C(M)$ is called a conormal field. Thus, for any tangent field v on M we have

$$\langle X, dx(v) \rangle = 0 .$$

We can interpret X as a differentiable mapping $X: M \to V^*$. Following Nomizu we call X centroaffine if X is an immersion and additionally, the field X itself is transversal along the hypersurface X. Obviously there exists a centroaffine conormal field for x if and only if any conormal field of x is centroaffine. We call the hypersurface non-degenerate (or regular) if it admits a centroaffine conormal field. A pair $\{X, y\}$ is called a relative normalization if X is a conormal field, $y: M \to V$ is a transversal field along x, and

$$\langle X, y \rangle = 1 , \quad \langle X, dy(v) \rangle = 0 .$$

Regular hypersurfaces admit (infinitely many different) relative normalizations [S-S-V; (3.6.1)].

For a nondegenerate hypersurface x, a relative normalization induces via the structure equations of Gauss and Weingarten

(W) $$dy(v) = -dx(S(v)) ,$$

(G-1) $$\bar{\nabla}_v dx(w) = dx({}^1\nabla_v w) + G(v, w) y ,$$

(G-2) $$\bar{\nabla}_v dX(w) = dX({}^2\nabla_v w) - \hat{S}(v, w) X ,$$

a so called relative geometry on M; here S is a linear operator, called the relative Weingarten operator; ${}^1\nabla$, ${}^2\nabla$ are affine connections without torsion; G and $\hat{S}$ are symmetric bilinear forms, and (M, G) is a semi-Riemannian space; G is called the relative metric. The difference tensor

$$C := \frac{1}{2}({}^1\nabla - {}^2\nabla)$$

defines the relative cubic form

$$\hat{C}(u, v, w) := G(C(u, v), w), \tag{3.1}$$

which is totally symmetric and satisfies

$$ {}^1\nabla G = -2\hat{C} = -{}^2\nabla G \tag{3.2}$$

(see [S-S-V; Chapter 4.1–4.3]); that means in particular that G satisfies Codazzi equations with respect to both connections. If ∇ denotes the Levi-Civita connection of G we have

$$\nabla + C = {}^1\nabla\,; \quad \nabla - C = {}^2\nabla\,; \quad \nabla = \frac{1}{2}({}^1\nabla + {}^2\nabla)\,. \tag{3.3}$$

Note that the uniqueness part of the relative fundamental theorem implies that two immersions $x, x^\#: M \to A$ are affinely equivalent if $G = G^\#$ and ${}^1\nabla = {}^1\nabla^\#$ (or $G = G^\#$ and ${}^2\nabla = {}^2\nabla^\#$, respectively); see [S-S-V; (4.12.3)].

4. Global uniqueness of ovaloids in relative geometry.

4.1. THEOREM. *Let $x, x^\#: M \to A$ be ovaloids in real affine 3-space with relative normalizations $\{X, y\}$ and $\{X^\#, y^\#\}$. Assume that on M the first connections and the Riemannian volume forms of the relative metrics coincide*:

$$ {}^1\nabla = {}^1\nabla^\#, \quad \omega(G) = \omega(G^\#)\,.$$

Then both triples $\{x, X, y\}$ and $\{x^\#, X^\#, y^\#\}$ are affinely equivalent.

PROOF. $G, G^\#$ satisfy Codazzi equations with respect to $\tilde{\nabla} := {}^1\nabla = {}^1\nabla^\#$ (Section 3). Theorem 2.1 gives $G = G^\#$, and the fundamental theorem of relative geometry (see Section 3) gives the equivalence.

4.2. THEOREM. *Let $x, x^\#$ be ovaloids with relative normalizations and assume*

$$ {}^2\nabla = {}^2\nabla^\#, \quad \omega(G) = \omega(G^\#)\,.$$

Then $\{x, X, y\}$ and $\{x^\#, X^\#, y^\#\}$ are affinely equivalent.

PROOF. $G, G^\#$ satisfy also Codazzi equations with respect to $\tilde{\nabla} := {}^2\nabla = {}^2\nabla^\#$.

4.3. Remark. We proved an analogue to (4.2) in any dimension in [SI-1, Theorem 4.3], but assumed additionally that the relative Weingarten operator of x has maximal rank on M. As far as we know a higher dimensional analogue of (4.1) is not known.

The following is a geometric version of Theorem (4.2). One can state it for hypersurfaces if one bears in mind the additional assumption in (4.3).

4.4. Theorem. *Let $x, x^{\#}$ be ovaloids in real affine 3-space with relative normalizations $y, y^{\#}$. Assume:*

(i) $\omega(G)=\omega(G^{\#})$;

(ii) *the tangent planes $dx(T_pM)$ and $dx^{\#}(T_pM)$ are parallel at each $p\in M$;*

(iii) *the relative normals $y, y^{\#}$ are parallel at each $p\in M$.*

Then $x, x^{\#}$ are affinely equivalent.

Proof. We consider the mappings $X, X^{\#}\colon M\to V^*$ and $y, y^{\#}\colon M\to V$. From the assumptions (ii)–(iii) there exist nowhere vanishing functions $\varphi, \psi\in C^\infty(M)$ such that on M:

$$X^{\#}=\varphi X \quad\text{and}\quad y^{\#}=\psi y.$$

From $\langle X^{\#}, y^{\#}\rangle=1=\langle X, y\rangle$ and $\langle X, dy\rangle=0=\langle X^{\#}, dy^{\#}\rangle$ we get $\varphi\psi=1$ and $d\psi=0$ on M. $G(v, w)=\langle X, (\text{Hess } x)(v, w)\rangle$, where Hess denotes the vector valued Hessian of x, implies $G^{\#}=\varphi G$ and $\omega(G^{\#})=\varphi^{n/2}\cdot\omega(G)$. But then $\varphi=1=\psi$ from (i). The assertion follows now from $X=X^{\#}$ (which implies ${}^2\nabla={}^2\nabla^{\#}$) and $\omega(G)=\omega(G^{\#})$; apply Theorem 2.1.

5. Equiaffine uniqueness results.

5.1. Theorem. *Let $x, x^{\#}$ be ovaloids in real affine 3-space with equiaffine normalizations $y, y^{\#}$, respectively; assume that the induced first connections coincide:*

$${}^1\nabla={}^1\nabla^{\#}.$$

Then $\{x, X, y\}$ and $\{x^{\#}, X^{\#}, y^{\#}\}$ are affinely equivalent.

Proof. Contraction of (3.2) and the apolarity of $C, C^{\#}$ give the following relation for the Christoffel symbols (see [LAU, p. 119] or [S-S-V; Sections 4.4.6–4.4.9]):

$$\frac{1}{2}\,\partial_i\log\det G=\Gamma^j_{ij}={}^1\Gamma^j_{ij}={}^1\Gamma^{\#j}_{ij}=\Gamma^{\#j}_{ij}=\frac{1}{2}\,\partial_i\log\det G^{\#},$$

so $\omega(G^{\#})=c\omega(G)$ for a constant $0<c\in\boldsymbol{R}$.

(i) $c=1$. Apply the relative result from (4.1); note that $x, x^{\#}$ are equiaffinely equivalent.

(ii) $c\neq 1$. Define $X^{\#\#}:=cX$ and $y^{\#\#}:=c^{-1}y$ as another relative normalization of x. The Tchebychev field of this normalization vanishes identically, so the normalization is equiaffine. We calculate $G^{\#\#}=cG$, ${}^1\nabla^{\#\#}={}^1\nabla$. Now the assertion follows from

$\omega(G^{\#})=\omega(G^{\#\#})$ and (i).

5.2. REMARK. Nomizu and Opozda [N-O] independently gave a proof of (5.1) under the additional assumption that $\operatorname{rank}(dy)=2$, using an equiaffine Herglotz integral formula. A proof in dimension $n>2$ is unknown to us.

5.3. THEOREM. *Let $x, x^{\#}: M\to A_3$ be ovaloids with equiaffine normalization $y, y^{\#}$ respectively; assume that the second induced connections coincide on M*:

$$^2\nabla={}^2\nabla^{\#}\,.$$

Then $\{x, X, y\}$ and $\{x^{\#}, X^{\#}, y^{\#}\}$ are affinely equivalent.

PROOF. The volume forms of the equiaffine metrics coincide; this follows from

$$\nabla-C={}^2\nabla={}^2\nabla^{\#}=\nabla^{\#}-C$$

as in (5.1).

5.4. GEOMETRIC VERSION OF 5.3 (see [SI-2, Theorem 2.11]). Let $x, x^{\#}: M\to A_3$ *be ovaloids such that for $p\in M$*

(i) *the tangent planes $dx(T_pM)$ and $dx^{\#}(T_pM)$ are parallel*;

(ii) *the directions of the affine normals $y(p)$ and $y^{\#}(p)$ are parallel.*

Then x, $x^{\#}$ are affinely equivalent.

For the proof see (4.4). We would like to point out that (5.4) generalizes a well-known characterization of the ellipsoid: An ovaloid with straight lines of gravity (Schwerelinien) is an ellipsoid (see [BLA, p. 213]).

5.5. REMARK. (i) Kurose [K] recently proved an existence result related to the uniqueness in (5.3).

(ii) In centroaffine geometry the corresponding uniqueness results are local; see [SI-1].

6. The uniqueness results of Liebmann, Cohn-Vossen and Minkowski in Euclidean geometry. The Euclidean interpretations of (4.1) show that Theorem 2.1 is the Riemannian generalization of the results of Liebmann, Cohn-Vossen and Minkowski which we recalled in the introduction. That (1.3) has in fact a weaker version for connections was already stated in [SI-1, p. 136]. We list the result for completeness.

6.1. THEOREM. *Let $x, x^{\#}: M\to E_3$ be ovaloids in Euclidean* 3*-space with Euclidean normalization. Assume that the Levi-Civita connections of the first fundamental forms* I, $\mathrm{I}^{\#}$ *coincide*

$$\nabla(\mathrm{I})=\nabla(\mathrm{I}^{\#})\,.$$

Then x, $x^{\#}$ are homothetic.

PROOF. The Riemannian manifolds (M, I), $(M, \mathrm{I}^{\#})$ are both irreducible. Moreover

$$\nabla(\mathrm{I})\mathrm{I}^{\#}=\nabla(\mathrm{I}^{\#})\mathrm{I}^{\#}=0\,,$$

that means $\mathrm{I}^{\#}$ is parallel with respect to $\nabla(\mathrm{I})$, so there exists $c>0$ such that $\mathrm{I}^{\#}=c\mathrm{I}$. Therefore the Euclidean Gauss-Kronecker curvatures satisfy $cK^{\#}=K$. Both relations imply that the volume elements of the second fundamental forms are related by $\omega(\mathrm{II}^{\#})=\sqrt{c}\,\omega(\mathrm{II})$. Recall that the second fundamental form is the relative metric induced by the Euclidean normalization. Apply now (4.1). It is obvious that the equiaffine equivalence must be a homothety.

6.2. MINKOWSKI'S THEOREM FOR CONNECTIONS. *Let x, $x^{\#}$ be ovaloids in E_3. Assume that the Levi-Civita connections of the third fundamental forms* III, III$^{\#}$ *and the volume forms of the second forms* II, II$^{\#}$ *coincide*:

$$\nabla(\mathrm{III})=\nabla(\mathrm{III}^{\#})\,,\quad \omega(\mathrm{II})=\omega(\mathrm{II}^{\#})\,.$$

Then x, $x^{\#}$ are equivalent modulo Euclidean motions.

That the above version is equivalent to Minkowski's result is obvious from the following facts:

(i) $(n-1)\,\mathrm{III}=\mathrm{Ric}(\mathrm{III})$, where Ric(III) is the Ricci tensor of the connection $\nabla(\mathrm{III})$; III determines the unit normal field up to Euclidean motions.

(ii) III and $\omega(\mathrm{II})$ determine the Gauss-Kronecker curvature K.

Conversely, $\nabla(\mathrm{III})$ and $\omega(\mathrm{II})$ can be determined from the unit normal field μ and K (given as functions of μ on the sphere).

The foregoing statements are true in any dimension $n\geqq 2$.

It was already known to Blaschke that Minkowski's uniqueness result has an equiaffine version (for details see [SI-1, §4]) which we generalized in (5.4).

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