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# ADMISSIBLE SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS

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**Abstract.** We treat second order differential equations which have admissible meromorphic solutions. With the aid of Nevanlinna theory, we obtain generalizations of the celebrated theorem of Malmquist-Yosida.

**1. Introduction.** We will treat differential equations of second order

$$w''=F(z, w, w'), \tag{1.1}$$

where $F$ is a polynomial in $w$ and $w'$ with meromorphic coefficients.

There are famous theorems due to Painlevé, Malmquist, Yosida and others for the analytic theory of ordinary differential equations.

Painlevé classified the equation (1.1) according to the nature of their singularities. Fixed singularities can arise at the locations of singularities of the coefficients. Singularities that are not fixed are said to be movable. Painlevé and his collaborators found six equations whose solutions do not have movable singularities except poles. They are known as the Painlevé transcendents and have a great variety of interesting properties (see [13, pp. 294–298] or [24, pp. 375–377]).

On the other hand, Malmquist investigated equations which possess meromorphic solutions. With the aid of Nevanlinna theory, Yosida [26] generalized the theorem of Malmquist, which is the starting point in this field.

THEOREM A (Malmquist-Yosida). *Let $R(z, w)$ be a rational function in $z$ and $w$. If the differential equation*

$$(w')^p=R(z, w) \tag{1.2}$$

*possesses a transcendental meromorphic solution, then $R(z, w)$ must be a polynomial in $w$ of degree at most $2p$.*

Then, several mathematicians treated the differential equations with the aid of Nevanlinna theory, and many generalizations of this theorem have been obtained, for example [7], [16]. In particular, equations of second order have been investigated in [18], [21]–[23], [25]. Steinmetz [21] treated the equation

$$w''=Q(z, w)w'+P(z, w), \tag{1.3}$$

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where $Q(z, w)$ and $P(z, w)$ are polynomials in $w$ with rational coefficients. He proved the following theorem of Malmquist-Yosida type.

THEOREM B (Steinmetz [21]). *If the equation* (1.3) *possesses a transcendental meromorphic solution* $w(z)$, *then*

(i) *either* $w(z)$ *satisfies an equation of Riccati type or*

(ii) $\deg_w[Q(z, w)] \leq 1$ *and* $\deg_w[P(z, w)] \leq 3$.

*We note that, in the case* (ii), *the equation* (1.3) *takes the form*

$$w'' = (q_1(z)w + q_0(z))w' + p_3(z)w^3 + p_2(z)w^2 + p_1(z)w + p_0(z) . \tag{1.4}$$

For binomial equation (1.2) of first order, possible types of the equations have been settled completely by Steinmetz [19], Bank and Kaufmann [2] and He and Laine [9]. As far as we know, there are few articles which determined the form of higher order differential equations with meromorphic solutions.

In this note, we treat the differential equation (1.4) with meromorphic (maybe transcendental) coefficients. We have two cases, according as $p_3(z) \not\equiv 0$ or $p_3(z) \equiv 0$. An example of the first case is the Painlevé equation II: $w'' = 2w^3 + zw + C$. As examples of the second case, we know the Painlevé equation I: $w'' = 6w^2 + z$ and the equation $w'' = 3w^2 + cw + c_1$, which is derived from the KdV equation.

We use standard notation in Nevanlinna theory [8], [14], [17]. Let $f(z)$ be a meromorphic function. As usual, $m(r,f)$, and $N(r,f)$, and $T(r,f)$ denote the proximity function, the counting function, and the characteristic function of $f(z)$, respectively. For $c \in \boldsymbol{C} \cup \{\infty\}$, $N(r, 1/(f-c))$ is written as $N(r, c; f)$. Sometimes we write $N(r,f)$ as $N(r, \infty; f)$.

DEFINITIONS. (i) A function $\varphi(r)$, $0 \leq r < \infty$, is said to be $S(r,f)$ if there is a set $E \subset \boldsymbol{R}^+$ of finite linear measure such that $\varphi(r) = o(T(r,f))$ *as* $r \to \infty$, with $r \notin E$ (see, e.g., [20, p. 40]).

(ii) A meromorphic function $a(z)$ is *small* with respect to $f(z)$, if $T(r, a) = S(r,f)$.

Below, $\mathcal{M} = \{a(z)\}$ denotes a given finite collection of meromorphic functions.

(iii) A transcendental meromorphic function $w(z)$ is *admissible* with respect to $\mathcal{M}$, if $T(r, a) = S(r, w)$ for any $a(z) \in \mathcal{M}$.

(iv) Let $\Omega(z, w, w', \ldots, w^{(n)})$ be a differential polynomial in $w$ with meromorphic coefficients and let $\mathcal{M}$ be the collection of the coefficients of $\Omega$. A meromorphic solution $w(z)$ of the equation

$$\Omega(z, w, w', \ldots, w^{(n)}) = 0 ,$$

is an *admissible solution* if $w(z)$ is admissible with respect to $\mathcal{M}$.

(v) Let $c \in \boldsymbol{C} \cup \{\infty\}$, $z_0$ is a $c$-point of $w(z)$ if $w(z_0) - c = 0$. Suppose a transcendental meromorphic function $w(z)$ is admissible with respect to $\mathcal{M}$. A $c$-point $z_0$ of $w(z)$ is an *admissible c-point* with respect to $\mathcal{M}$, if $a(z_0) \neq 0, \infty$ for any $a(z) \in \mathcal{M}$. Clearly, there are admissible $c$-points of $w(z)$, provided that $\bar{N}(r, c; w) \neq S(r, w)$.

(vi) Suppose $N(r, c; w) \neq S(r, w)$, for a $c \in \boldsymbol{C} \cup \{\infty\}$. Let C1 be a property. We denote by $n^*_{C1}(r, c; w)$, the number of $c$-points in $|z| \leqq r$ which admit the property C1. $N^*_{C1}(r, c; w)$ is defined in the usual way. If

$$N(r, c; w) - N^*_{C1}(r, c; w) = S(r, w),$$

then we say that *almost all* $c$-point admit the property C1.

REMARK 1.1. Suppose a transcendental meromorphic function $w(z)$ is admissible with respect to $\mathcal{M}$. Let $\eta(z)$ be a rational function in members of $\mathcal{M}$ and their derivatives. Then we have $T(r, \eta) \leq K \sum_{a_\nu \in \mathcal{M}} T(r, a_\nu) + S(r, w)$, for some $K > 0$. Thus $\eta(z)$ is a small function with respect to $w(z)$. We denote by $n^*_\eta(r, c; w)$, the number of $c$-points $z_0$ of $w(z)$ in $|z| \leqq r$ such that $\eta(z_0) = 0$. $N^*_\eta(r, c; w)$ is defined in the usual way. If $\bar{N}^*_\eta(r, c; w) \neq S(r, w)$, then $\eta(z) \equiv 0$. Further for an admissible solution $w(z)$, we may assume that $N_{(M}(r, w) = S(r, w)$ for some $M > 0$. If we suppose the contrary, then $w(z)$ satisfies a linear differential equation of first order (see [12, Lemma 3]).

Now we turn to the equation (1.4) and consider the case $p_3(z) \not\equiv 0$.

THEOREM 1.1. *In the differential equation*

$$w'' = (q_1(z)w + q_0(z))w' + p_3(z)w^3 + p_2(z)w^2 + p_1(z)w + p_0(z), \tag{1.4}$$

*suppose that the coefficients* $q_1(z)$, $q_0(z)$, $p_3(z)$, $p_2(z)$, $p_1(z)$ *and* $p_0(z)$ *are meromorphic and* $p_3(z) \not\equiv 0$. *Further, suppose that* (1.4) *possesses an admissible solution* $w(z)$.

*When* $q_1(z) \not\equiv 0$, *we have the following two possibilities*:

(i) *either* $w(z)$ *satisfies the equation of first order*

$$c(z)w'^2 + B(z, w)w' + A(z, w) = 0, \tag{1.5}$$

*where* $c(z)$ *is a small (with respect to* $w(z)$*) function and* $B(z, w)$, $A(z, w)$ *are polynomials of* $w$ *with small (with respect to* $w(z)$*) coefficients such that* $\deg_w[B(z, w)] \leq 2$, $\deg_w[A(z, w)] \leq 4$,

(ii) *or, by putting* $w = \lambda_1(z)u + \lambda_0(z)$, *with small functions* $\lambda_j(z)$, $j = 0, 1$, *we can transform* (1.4) *into one of the equations of the following two types: either*

$$u'' + 3uu' + u^3 = \tilde{p}_1(z)u + \tilde{p}_0(z), \quad or \tag{1.6}$$

$$u'' + uu' - u^3 = \tilde{p}(z)(u^2 + 3u') + H(z)u + S(z), \tag{1.7}$$

*in which the coefficients* $\tilde{p}(z)$, $H(z)$, $S(z)$ *satisfy the following relation*:

$$\Delta(z) := 2H(z)\tilde{p}(z) - H'(z) + 4\tilde{p}(z)^3 - 6\tilde{p}(z)\tilde{p}'(z) + \tilde{p}''(z) - S(z) = 0. \tag{1.7'}$$

*When* $q_1(z) \equiv 0$, *we have the following three possibilities*:

(i′) *either* $w(z)$ *satisfies an equation of the type* (1.5),

(iii) *or, by putting* $w = \lambda_1(z)u + \lambda_0(z)$ *with small functions* $\lambda_j(z)$, $j = 0, 1$, *we can transform* (1.4) *into*

$$u'' = \tilde{q}_0(z)u' + \tilde{p}_3(z)u^3 + \tilde{p}_1(z)u + C, \tag{1.8}$$

*where C is a non-zero constant, and the coefficients satisfy the following relations* (1.9) *and* (1.10).

$$2\tilde{q}_0(z)+\frac{\tilde{p}_3'(z)}{\tilde{p}_3(z)}=0\,, \tag{1.9}$$

$$\tilde{p}_1''(z)-\frac{5}{6}\left(\frac{\tilde{p}_3'(z)}{\tilde{p}_3(z)}\right)\tilde{p}_1'(z)-\left(\frac{\tilde{p}_3''(z)}{3\tilde{p}_3(z)}-\frac{1}{2}\left(\frac{\tilde{p}_3'(z)}{\tilde{p}_3(z)}\right)^2\right)\tilde{p}_1(z)-\left(\frac{\tilde{p}_3''(z)}{\tilde{p}_3(z)}\right)^2 \tag{1.10}$$
$$+\frac{40}{9}\left(\frac{\tilde{p}_3'(z)}{\tilde{p}_3(z)}\right)^2\frac{\tilde{p}_3''(z)}{\tilde{p}_3(z)}-\frac{245}{108}\left(\frac{\tilde{p}_3'(z)}{\tilde{p}_3(z)}\right)^4+\frac{\tilde{p}_3^{(4)}(z)}{3\tilde{p}_3(z)}-\frac{3\tilde{p}_3'(z)\tilde{p}_3'''(z)}{2\tilde{p}_3(z)^2}=0\,,$$

(iv) *or, by putting* $\lambda_1(z)w'/w+\lambda_0(z)=u$ *with small functions* $\lambda_j(z)$, $j=0, 1$, (*and reiterating the transformation, if necessary*) *we obtain* (1.8) *with* (1.9) *and* (1.10), *or* $u(z)$ *satisfies an equation of the form* (1.5).

Theorem 1.1 follows from Lemmas 3.1 and 3.2 below.

The equation (1.5) was investigated by Steinmetz [20] when coefficients are polynomials and $C(z)\equiv 1$. He showed that if (1.5) possesses an admissible solution $w(z)$, then by a suitable transformation $y=(a(z)w+b(z))/(c(z)w+d(z))$ with rational coefficients, (1.5) is transformed into either

$$(y')^2=\tilde{a}(z)(y-e_1)(y-e_2)(y-e_3)\,,$$

or

$$(y'+\tilde{b}(z)y)^2=\tilde{a}(z)y(1+\tilde{c}(z)y)^2\,,$$

where $\tilde{a}(z)$, $\tilde{b}(z)$, $\tilde{c}(z)$ are rational functions and $e_1$, $e_2$, $e_3$ are constants.

For the equations (1.6), (1.7) and (1.8), we give the following remarks, (see [11, pp. 317–355]).

REMARK 1.2. $w(z)=e^{\cos z}-z$ is a solution of the equation

$$w'''-a(z)w'-b(z)w=0\,,$$

where $b(z)=(\sin z+3\cos z\sin z-\sin^3 z)/(1+z\sin z)$ and $a(z)=-zb(z)$.

$u(z)=w'(z)/w(z)=(-1-e^{\cos z}\sin z)/(e^{\cos z}-z)$ is an admissible solution of the equation

$$u''+3u'u+u^3=a(z)u+b(z)\,.$$

REMARK 1.3. Put $\tilde{p}(z)\equiv 0$, $H(z)=-12q(z)$, and $S(z)=12q'(z)$, with $q(z)=1/(z+c)^2$ ($c$ constant) in (1.7$'$). Then $\Delta(z)\equiv 0$ and $q(z)$ is a solution of the following differential equation

$$f''=6f^2\,. \tag{1.11}$$

The Weierstrass $\wp(z)$ function which is a solution of $w'^2=4w^3+C$ ($C$ nonzero constant) satisfies the equation (1.11). $U(z):=(\wp'(z)-q'(z))/(\wp(z)-q(z))$ satisfies an equation of the type (1.7)

$$u''+uu'-u^3=-12q(z)u+12q'(z)\,. \tag{1.12}$$

Since we have $T(r,q)=S(r,U)$, the equation (1.12) possesses an admissible solution $U(z)$.

REMARK 1.4. If $\tilde{q}_0(z)\equiv 0$ in (1.8), then by (1.9) and (1.10), $\tilde{p}_3(z)$ is constant and $p_1(z)$ is linear. Thus by a suitable transformation $w=au$ and $z=a_1t+a_0$, (1.8) is transformed into the Painlevé equation II, where $a$, $a_1$, $a_0$ are constants.

Secondly we consider the case $p_3(z)\equiv 0$ in (1.4).

THEOREM 1.2. *Suppose $p_3(z)\equiv 0$ in* (1.4) *and that the differential equation* (1.4) *possesses an admissible solution $w(z)$. Then we have the following three possibilities*:

(i) *either $w(z)$ satisfies the first order differential equation* (1.5),

(ii) *or, $u(z)=\lambda_1(z)w(z)+\lambda_0(z)$, with small functions $\lambda_j(z)$, $j=0,1$, satisfies the following type of equation*

$$u''=q(z)u'+6u^2+p(z)\,, \tag{1.13}$$

*where the coefficients satisfy the following relations* (1.14) *and* (1.15)

$$T'+q(z)T=0\,,\qquad T(z)\not\equiv 0\,, \tag{1.14}$$

$$\begin{aligned} T(z)=\;&15000p(z)q(z)-18750p'(z)+36q(z)^5-900q(z)^3q'(z)+2000q(z)^2q''(z)\\ &+2500q(z)q'(z)^2-1875q(z)q'''(z)-3125q'(z)q''(z)+625q^{(4)}(z)\,, \end{aligned} \tag{1.15}$$

(iii) *or, $u(z)=\eta(z)w'+\eta_2(z)w^2+\eta_1(z)w+\eta_0(z)$, with $\eta(z)$, $\eta_2(z)$, $\eta_1(z)$, $\eta_0(z)$ are small* (*with respect to $w(z)$*) *functions, satisfies a first order linear equation.*

Theorem 1.2 follows from Lemmas 3.3 and 3.4 below.

REMARK 1.5. In (1.13), suppose $q(z)$ is entire and there exists a positive number $K$ such that

$$T(r,p)\leqq KT(r,q)+S(r,q)\,. \tag{1.16}$$

Then both of the conditions (1.14) and (1.15) do not hold. Hence the inequality (1.16) does not hold for the case (ii) when $q(z)$ is entire.

If $q(z)\equiv 0$ in (1.13), then by (1.14) $T(z)$ is constant. By (1.15) $p'(z)$ is also constant, which implies that (1.13) is the Painlevé equation I.

## 2. Dominant behavior.

To investigate the dominant behavior of an admissible solution in a sufficiently small neighbourhood of the pole, we use the basic *Test-Power* test (see [10, pp. 87–96]). This is very effective in the case of movable singularities but can be used also for fixed singularities, at least for the purpose of orientation. The simple idea is that, if the differential equation, for example (1.4), has an admissible solution $w(z)$ which has an admissible pole $z_0$, and at $z_0$

$$w(z)=\frac{R_\mu}{(z-z_0)^\mu}+O(z-z_0)^{-(\mu-1)}, \qquad R_\mu\neq 0,$$

then for special values of $\mu$, two or more terms in (1.4) may balance (the number of the balancing terms depends on the values of $\mu$ and $R_\mu$). The balancing terms are called *leading terms* (see [1, pp. 717–718]).

We look for the next highest order term in the Laurent series of an admissible solution. From the given differential equation, the coefficients of the Laurent series in a neighbourhood of an admissible pole $z_0$ may not be represented by small functions. In some cases, the Laurent series contains arbitrary coefficients called *resonances*. The series containing resonances are called *resonant series* (see [1, pp. 718–720] or [15, pp. 334–340]). For example, the expansion of the transcendent of the Painlevé equation II: $w''=2w^3+zw+\alpha$

$$w(z)=\frac{1}{z-z_0}-\frac{z_0}{6}(z-z_0)-\frac{\alpha+1}{4}(z-z_0)^2+h(z-z_0)^3+\cdots,$$

at an admissible pole $z_0$ has an arbitrary constant $h$.

Theorem 1.1 follows from the ideas contained in the following Lemma C. This kind of ideas is used in many papers, for example [6].

LEMMA C (cf. [21], [22]). *Let $w=w(z)$ be a transcendental meromorphic function such that $m(r, w)+N_1(r, w)=S(r, w)$. Suppose that for almost all poles $z_0$, there exist small (with respect to $w(z)$) functions $R(z)$ and $\alpha(z)$ such that $w(z)$ is written near $z_0$ as*

$$w(z)=\frac{R(z_0)}{z-z_0}+\alpha(z_0)+O(z-z_0).$$

*Then $w(z)$ satisfies an equation of Riccati type*

$$w'=a(z)w^2+b(z)w+c(z), \qquad a(z)\not\equiv 0,$$

*where $a(z)$, $b(z)$ and $c(z)$ are small functions with respect to $w(z)$.*

Before stating our lemmas, we fix notation and recall some propositions. Let $f(z)$ be a transcendental meromorphic function and let $R(z)$ and $\alpha(z)$ be small functions with respect to $f(z)$. Let $z_0$ be a simple pole of $f(z)$. We say that $z_0$ is *representable in the first sense by $R(z)$ and $\alpha(z)$*, if

$$f(z)=\frac{R(z_0)}{z-z_0}+\alpha(z_0)+O(z-z_0),$$

in a neighbourhood of $z_0$. For the sake of simplicity, we call such a simple pole an S1-*pole*. Lemma C means that if almost all poles of $w(z)$ are S1-poles and $m(r, w)=S(r, w)$, then $w(z)$ satisfies an equation of Riccati type.

For the definition of S2-pole, we introduce the following further material. Let $\lambda_1$, $\lambda_0$ be complex constants and let $\boldsymbol{L}$ be a set of linear transformations of a quantity $R$,

$$(2.1)\qquad \boldsymbol{L}=\boldsymbol{L}_{(\lambda_1,\lambda_0)}=\left\{L=\frac{l_1R+l_2}{l_3R+l_4}\,\middle|\, l_4^2-\lambda_1 l_3 l_4+\lambda_0 l_3^2\neq 0\ ,\ l_j\in \boldsymbol{C}, j=1,2,3,4\right\}.$$

We define an equivalence relation $\sim$ in $\boldsymbol{L}$ by

$$L=(a_1R+a_2)/(a_3R+a_4)\sim M=(b_1R+b_2)/(b_3R+b_4)\in \boldsymbol{L}\ ,$$

if

$$(2.2)\qquad \begin{cases}\lambda_0(a_1b_3-b_1a_3)=a_2b_4-b_2a_4\ ,\\ \lambda_1(a_1b_3-b_1a_3)=a_1b_4-b_1a_4+a_2b_3-a_3b_2\ .\end{cases}$$

Proposition D. (i) *If* $L=(a_1R+a_2)/(a_3R+a_4)\in\boldsymbol{L}$, *then* $L\sim L^*=A_1R+A_2$, *where*

$$A_1=\frac{-a_2a_3+a_1a_4}{\lambda_0a_3^2-\lambda_1a_3a_4+a_4^2}\ ,\quad A_2=\frac{\lambda_0a_1a_3-\lambda_1a_2a_3+a_2a_4}{\lambda_0a_3^2-\lambda_1a_3a_4+a_4^2}\ .$$

(ii) *If* $L=a_1R+a_2\sim M=b_1R+b_2$, *then* $a_1=b_1$ *and* $a_2=b_2$.

By Proposition D, we can take, for each equivalent class in $\boldsymbol{L}$, a unique representative which is an entire linear transformation. We denote by $\boldsymbol{L}^*=\boldsymbol{L}^*(\lambda_1,\lambda_0)$ the set of all such representatives. We define $aL+bM$ and $LM$ as follows: For $a,b\in\boldsymbol{C}$, $L=a_1R+a_2$, $M=b_1R+b_2\in\boldsymbol{L}^*$,

$$(2.3)\qquad aL+bM=(aa_1+bb_1)R+aa_2+bb_2\ ,$$

$$(2.4)\qquad LM=(a_1b_2+a_2b_1-\lambda_1a_1b_1)R+(a_2b_2-\lambda_0a_1b_1)\ .$$

Let $L=a_1R+a_2$, $M=b_1R+b_2$ be two elements of $\boldsymbol{L}^*$. We say that $L$ and $M$ are independent, if $a_1b_2-a_2b_1\neq 0$.

We can easily obtain the following propositions:

Proposition E. *Let $L$ and $M$ be elements of $\boldsymbol{L}^*$. If $L$ and $M$ are independent, then for any $N\in\boldsymbol{L}^*$, there exist $\tau_1$, $\tau_2$ such that $N=\tau_1L+\tau_2M$.*

Proposition F. *Let $L$ and $M$ be elements of $\boldsymbol{L}^*$. If $L$ and $M$ are independent, then for any $N=aR+b\in\boldsymbol{L}^*$ with $\lambda_0a^2-\lambda_1ab+b^2\neq 0$, $NL$ and $NM$ are also independent.*

Let $f(z)$ be a transcendental meromorphic function. Let all functions $\alpha_1(z),\dots,\alpha_4(z)$, $\beta_1(z),\dots,\beta_4(z)$, $\gamma_1(z),\dots,\gamma_4(z)$, $\lambda_1(z)$, $\lambda_0(z)$ be small functions with respect to $f(z)$ satisfying

$$(2.5)\qquad \begin{aligned} &\Lambda(z):=\lambda_1(z)^2-4\lambda_0(z)\not\equiv 0\,,\\ &\tilde{\alpha}(z):=\alpha_4(z)^2-\lambda_1(z)\alpha_3(z)\alpha_4(z)+\lambda_0(z)\alpha_3(z)^2\not\equiv 0\,,\\ &\tilde{\beta}(z):=\beta_4(z)^2-\lambda_1(z)\beta_3(z)\beta_4(z)+\lambda_0(z)\beta_3(z)^2\not\equiv 0\,,\\ &\tilde{\gamma}(z):=\gamma_4(z)^2-\lambda_1(z)\gamma_3(z)\gamma_4(z)+\lambda_0(z)\gamma_3(z)^2\not\equiv 0\,. \end{aligned}$$

Let $z_0$ be a simple pole of $f(z)$. We say that $z_0$ is *representable in the second sense* by $\alpha_1(z), \dots, \alpha_4(z)$, $\beta_1(z), \dots, \beta_4(z)$, $\gamma_1(z), \dots, \gamma_4(z)$, $\lambda_1(z)$ and $\lambda_0(z)$, if

$$(2.6)\qquad f(z)=\frac{R}{z-z_0}+\alpha+\beta(z-z_0)+\gamma(z-z_0)^2+\delta(z-z_0)^3+O(z-z_0)^4$$

in a neighbourhood of $z_0$, and

$$(2.7)\qquad R^2+\lambda_1(z_0)R+\lambda_0(z_0)=0\,,\qquad \Lambda(z_0)\neq 0\,,$$

$$(2.8)\qquad \alpha=\frac{\alpha_1(z_0)R+\alpha_2(z_0)}{\alpha_3(z_0)R+\alpha_4(z_0)}\,,\quad \beta=\frac{\beta_1(z_0)R+\beta_2(z_0)}{\beta_3(z_0)R+\beta_4(z_0)}\,,\quad \gamma=\frac{\gamma_1(z_0)R+\gamma_2(z_0)}{\gamma_3(z_0)R+\gamma_4(z_0)}\,,$$
$$\tilde{\alpha}(z_0)\neq 0\,,\ \tilde{\beta}(z_0)\neq 0\,,\ \tilde{\gamma}(z_0)\neq 0\,.$$

For the sake of brevity, we call such a simple pole an S2-*pole*.

In addition to the condition (2.5), let $\delta_1(z), \dots, \delta_4(z)$ be small functions with respect to $w(z)$ so that

$$(2.9)\qquad \tilde{\delta}(z):=\delta_4(z)^2-\lambda_1(z)\delta_3(z)\delta_4(z)+\lambda_0(z)\delta_3(z)^2\not\equiv 0\,.$$

Let $z_0$ be a simple pole of $f(z)$. We say that $z_0$ is *strongly representable in the second sense* by $\alpha_1(z), \dots, \alpha_4(z)$, $\beta_1(z), \dots, \beta_4(z)$, $\gamma_1(z), \dots, \gamma_4(z)$, $\delta_1(z), \dots, \delta_4(z)$, $\lambda_1(z)$ and $\lambda_0(z)$, if $f(z)$ is written as in (2.6), $R$ satisfies (2.7), and $\alpha$, $\beta$, $\gamma$, are represented as in (2.8), and

$$(2.10)\qquad \delta=\frac{\delta_1(z_0)R+\delta_2(z_0)}{\delta_3(z_0)R+\delta_4(z_0)}\,,\qquad \tilde{\delta}(z_0)\neq 0\,.$$

For the sake of brevity, we call such a simple pole an SS2-*pole*.

Let $z_0$ be a pole of $f(z)$ such that $\Lambda(z_0)\neq 0$. We denote by $\boldsymbol{L}(z_0)$ the set of linear transformations of $R$ as in (2.1):

$$(2.1')\qquad \boldsymbol{L}(z_0)=\boldsymbol{L}_{(\lambda_1(z),\,\lambda_0(z))}(z_0)=\left\{L=\frac{l_1(z_0)R+l_2(z_0)}{l_3(z_0)R+l_4(z_0)}\;\middle|\; l_j(z), j=1, 2, 3, 4\right.$$
$$\left.\text{small for } f(z), \text{ with } l_4(z_0)^2-\lambda_1(z_0)l_3(z_0)l_4(z_0)+\lambda_0(z_0)l_3(z_0)^2\neq 0\right\}.$$

Let $R_1$ and $R_2$ be the roots of (2.7) for a fixed $z_0$. Since $\Lambda(z_0)\neq 0$, we have $R_1\neq R_2$. By simple calculation, $L=(a_1(z_0)R+a_2(z_0))/(a_3(z_0)R+a_4(z_0))$, $M=(b_1(z_0)R+b_2(z_0))/(b_3(z_0)R+b_4(z_0))\in \boldsymbol{L}(z_0)$, satisfying $L_{|R=R_j}=M_{|R=R_j}$, $j=1, 2$ if and only if

$$(2.2')\qquad \begin{cases} \lambda_0(z_0)(a_1(z_0)b_3(z_0)-b_1(z_0)a_3(z_0))=a_2(z_0)b_4(z_0)-b_2(z_0)a_4(z_0)\,, \\ \lambda_1(z_0)(a_1(z_0)b_3(z_0)-b_1(z_0)a_3(z_0))=a_1(z_0)b_4(z_0)-b_1(z_0)a_4(z_0) \\ \qquad\qquad +a_2(z_0)b_3(z_0)-a_3(z_0)b_2(z_0)\,. \end{cases}$$

Hence, the following (A) and (B) are equivalent to each other:

(A) $L, M \in \boldsymbol{L}(z_0)$, $L \sim M$,

(B) $L, M \in \boldsymbol{L}(z_0)$, $L = M$ under the condition (2.7).

The conditions in (2.5) imply $\alpha, \beta, \gamma \in \boldsymbol{L}(z_0)$, while (2.9) implies $\delta \in \boldsymbol{L}(z_0)$. In other words, the conditions (2.5) and (2.9) are the criteria for $\alpha$, $\beta$, $\gamma$ and $\delta$ to be resonances or not.

By Proposition D, for any $L \in \boldsymbol{L}(z_0)$, we have a unique entire form $L^* \in \boldsymbol{L}^*(z_0)$ such that $L_{|R=R_j} = L^*_{|R=R_j}$, $j=1,2$. From now on, under the condition (2.7), we write $L=(a_1(z_0)R+a_2(z_0))/(a_3(z_0)R+a_4(z_0))$, in the form $A_1(z_0)R+A_2(z_0)$, where $A_1(z)$ and $A_2(z)$ are defined as in Proposition D, (i).

We can ascertain that the operations (2.3) and (2.4) in $\boldsymbol{L}^*(z_0)$ are well defined under the condition (2.7). Hence Propositions E and F hold for the elements of $\boldsymbol{L}^*(z_0)$.

Let $[R]$ be a root of (2.7) for a fixed $z_0$, where $\lambda_1(z_0)^2-4\lambda_0(z_0)\neq 0$. We denote by $[\boldsymbol{L}]^*(z_0)$ the set of values of the elements of $\boldsymbol{L}^*(z_0)$ for $R=[R]$.

The following lemma will be proved in Section 4.

LEMMA 2.1. *Let $w(z)$ be a transcendental meromorphic function and let $\alpha_1(z), \ldots, \alpha_4(z)$, $\beta_1(z), \ldots, \beta_4(z)$, $\gamma_1(z), \ldots, \gamma_4(z)$, $\delta_1(z), \ldots, \delta_4(z)$, $\lambda_1(z)$ and $\lambda_0(z)$ be small functions with respect to $w(z)$. We denote by $n_{\langle S2 \rangle}(r, w)$ and $n_{\langle SS2 \rangle}(r, w)$ the numbers of the* S2*-poles of $w(z)$ and the* SS2*-poles in $|z| \leq r$, respectively. $N_{\langle S2 \rangle}(r, w)$ and $N_{\langle SS2 \rangle}(r, w)$ are defined in terms of $n_{\langle S2 \rangle}(r, w)$ and $n_{\langle SS2 \rangle}(r, w)$ in the usual way, respectively. If*

$$(2.11)\qquad m(r, w)+(N(r, w)-N_{\langle S2 \rangle}(r, w))=S(r, w)\,,$$

*then either $w(z)$ satisfies an equation of the form* (1.4), *or $w(z)$ satisfies an equation of the form* (1.5).

*Further if*

$$(2.12)\qquad m(r, w)+(N(r, w)-N_{\langle SS2 \rangle}(r, w))=S(r, w)\,,$$

*then $w(z)$ satisfies an equation of the form* (1.5).

## 3. Some lemmas.

We make use of the following results. Lemma G, due to Clunie, and Lemma H are applicable to the estimation of the proximity function of differential polynomials.

LEMMA G (cf. [3], [8]). *Let $Q(z,f)$ and $Q^*(z,f)$ be differential polynomials of a transcendental meromorphic function $f(z)$, having coefficients $a_j(z)$ and $a_k^*(z)$. Suppose that $m(r, a_j)=S(r,f)$ and $m(r, a_k^*)=S(r,f)$. If $\deg_f[Q(z,f)] \leq n$ and $f^n(z)Q^*(z,f(z))=Q(z,f(z))$,* *then*

$$m(r, Q^*) = S(r, f),$$

*where* $Q^*(z) = Q^*(z, f(z))$.

LEMMA H (cf. [4]). *Let* $w(z)$ *be a transcendental meromorphic function and* $\Omega(z, w, w', \dots, w^{(i)})$ *be a differential polynomial with small (with respect to* $w(z)$*) meromorphic coefficients. Then*

$$m(r, \Omega) \leq \deg_{w, w', \dots, w^{(i)}}[\Omega(z, w, w', \dots, w^{(i)})]m(r, w) + S(r, w),$$

*where* $\Omega(z) = \Omega(z, w(z), w'(z), \dots, w^{(i)}(z))$.

Eremenko proved the following result that is a generalization of the Malmquist-Yosida theorem for a first order algebraic differential equation.

LEMMA I (cf. [5]). *Suppose the following differential equation possesses an admissible solution* $w(z)$

$$Q_k(z, w)w'^k + Q_{k-1}(z, w)w'^{k-1} + \cdots + Q_0(z, w) = 0, \qquad k \geqq 1, \tag{3.1}$$

*where* $Q_j(z, w), j = 0, 1, \dots, k$ *are polynomials in* $w$ *with meromorphic coefficients. If* (3.1) *is an irreducible polynomial in* $w$ *and* $w'$, *then*

$$\deg_w[Q_k(z, w)] = 0 \quad and \quad \deg_w[Q_j(z, w)] \leq 2(k-j), \qquad j = 0, 1, \dots, k-1. \tag{3.2}$$

For the case $p_3(z) \not\equiv 0$ in (1.4), we show the following Lemmas 3.1 and 3.2.

LEMMA 3.1. *Suppose* $p_3(z) \not\equiv 0$ *in* (1.4) *and that the equation* (1.4) *possesses an admissible solution* $w(z)$. *Suppose further that*

$$9p_3(z) + q_1(z)^2 \not\equiv 0 \quad and \quad p_3(z) - q_1(z)^2 \not\equiv 0. \tag{3.3}$$

*If* $q_1(z) \not\equiv 0$, *then* $w(z)$ *satisfies an equation of the form* (1.5).

*If* $q_1(z) \equiv 0$, *then either*:

(i) *by a suitable transformation* $w = \lambda_1(z)u + \lambda_0(z)$, (1.4) *is transformed into the equation* (1.8) *with* (1.9), (1.10), *or*

(ii) *by a suitable transformation* $\lambda_1(z)w'/w + \lambda_0(z) = u$, (*and repetition if necessary*) (1.4) *is transformed into the equation* (1.8) *with* (1.9), (1.10), *or* $u(z)$ *satisfies an equation of the form* (1.5).

LEMMA 3.2. *Suppose* $p_3(z) \not\equiv 0$ *in* (1.4) *and that the equation* (1.4) *possesses an admissible solution* $w(z)$. *Suppose further that*

$$9p_3(z) + q_1(z)^2 \equiv 0 \quad or \quad p_3(z) - q_1(z)^2 \equiv 0. \tag{3.4}$$

*Then by a suitable transformation* $w = \lambda_1(z)u + \lambda_0(z)$, $u(z)$ *satisfies a Riccati equation or* (1.4) *is transformed into the equation* (1.6) *or* (1.7), *respectively.*

Lemmas 3.1 and 3.2 together imply Theorem 1.1. For the case $p_3(z)\equiv 0$ in (1.4), we show the following Lemmas 3.3 and 3.4.

LEMMA 3.3. *Suppose* $p_3(z)\equiv 0$, $q_1(z)\not\equiv 0$ *in* (1.4) *and the equation* (1.4) *possesses an admissible solution* $w(z)$. *Then by* $u=\eta(z)w'+\eta_2(z)w^2+\eta_1(z)w+\eta_0(z)$, (1.4) *is transformed into a linear equation of first order.*

LEMMA 3.4. *Suppose* $p_3(z)\equiv 0$, $q_1(z)\equiv 0$ *in* (1.4) *and the equation* (1.4) *possesses an admissible solution* $w(z)$. *Then by* $w=\lambda_1(z)u+\lambda_0(z)$, (1.4) *is transformed into the equations* (1.13) *with* (1.14), (1.15), *or* $u(z)$ *satisfies an equation of the form* (1.5).

Lemmas 3.3 and 3.4 together imply Theorem 1.2.

**4. Proof of Lemma 2.1.** Let $z_0$ be an admissible pole of $w(z)$ and write $w(z)$ in a neighbourhood of $z_0$ as

$$w(z)=\frac{R}{z-z_0}+\alpha+\beta(z-z_0)+\gamma(z-z_0)^2+\delta(z-z_0)^3+O(z-z_0)^4\,. \tag{4.1}$$

If $z_0$ is an S2-pole of $w(z)$, then by definition, we may suppose that $\alpha, \beta, \gamma\in[L]^*(z_0)$. Thus by simple calculation under the operations (2.3) and (2.4), the coefficients of the principle parts of the Laurent expansions of the functions $w(z)^2$, $w(z)^3$, $w(z)^4$, $w'(z)$, $w'(z)^2$, $w'(z)w(z)$, $w(z)^2w'(z)$ and $w''(z)$ belong to $[L]^*(z_0)$.

Further if $z_0$ is an SS2-pole of $w(z)$, then in addition to the above functions $w(z)^2$ etc., the coefficients of the principal parts of the Laurent expansions near $z_0$ of the functions $w(z)^5$, $w(z)^3w'(z)$, and $w(z)w'(z)^2$ also belong to $[L]^*(z_0)$.

If $\lambda_0(z)\equiv 0$, then $R=-\lambda_1(z_0)$ by (2.7). Thus from (2.8), $\alpha$ is written in terms of small functions. Hence by (2.11) (or (2.12)) and Lemma C, $w(z)$ satisfies a Riccati equation. Therefore we may suppose that $\lambda_0(z)\not\equiv 0$ and $\lambda_0(z_0)\neq 0$. Thus $R^2$ $(=-\lambda(z_0)R-\lambda_0(z_0))$ and $R$ are independent. Hence by Proposition F, $R^{n+1}$ and $R^n$ $(n=1, 2, 3, \ldots)$ are independent.

First we treat the case where $w(z)$ satisfies the condition (2.11). Let $F(z)$ be a meromorphic function which satisfies the following two conditions:

$$m(r, F)+(\bar{N}(r, F)-\bar{N}(r, w))=S(r, w)\,, \tag{4.2}$$

and in a neighbourhood of an admissible pole $z_0$ of $w(z)$,

$$F(z)=\frac{L_3}{(z-z_0)^3}+\frac{L_{32}}{(z-z_0)^2}+\frac{L_{31}}{z-z_0}+O(1)\,,\quad L_3, L_{32}, L_{31}\in[L]^*(z_0)\,. \tag{4.3}$$

By Proposition E, there exist small functions $\eta_1(z)$ and $\eta_2(z)$ with respect to $w(z)$, so that

$$D_{21}(z, w(z), w'(z))=\frac{L_{21}}{(z-z_0)^2}+\frac{L_{211}}{z-z_0}+O(1)\,,\quad L_{21}, L_{211}\in[L]^*(z_0)\,,$$

where

$$D_{21}(z, w, w') = F(z) + \eta_1(z)ww' + \eta_2(z)w^3 . \tag{4.4}$$

Put $D_{22}(z, w, w', w'') = w^3 - \lambda_1(z)ww' + (\lambda_0(z)/2)w''$. Then

$$D_{22}(z, w(z), w'(z)) = \frac{L_{22}}{(z-z_0)^2} + \frac{L_{221}}{z-z_0} + O(1) , \quad L_{22}, L_{221} \in [L]^*(z_0) .$$

By Proposition E, there exist $v_{11}(z)$, $v_{12}(z)$, $v_{21}(z)$ and $v_{22}(z)$, which are small functions with respect to $w(z)$ such that

$$D_{11}(z, w(z), w'(z)) = \frac{L_{11}}{z-z_0} + O(1) , \quad L_{11} \in [L]^*(z_0) ,$$

where

$$D_{11}(z, w, w') = D_{21}(z, w, w') + v_{11}(z)w' + v_{12}(z)w^2 , \tag{4.5}$$

and

$$D_{12}(z, w(z), w'(z), w''(z)) = \frac{L_{12}}{z-z_0} + O(1) , \quad L_{12} \in [L]^*(z_0) ,$$

where

$$D_{12}(z, w, w', w'') = D_{22}(z, w, w', w'') + v_{21}(z)w' + v_{22}(z)w^2 .$$

By Proposition E, there exist $\kappa_1(z)$, $\kappa_2(z)$ and $\kappa_3(z)$, with $|\kappa_1| + |\kappa_2| \neq 0$, which are small functions with respect to $w(z)$, so that if we put

$$\Phi(z, w, w', w'') = \kappa_1(z)D_{11}(z, w, w') + \kappa_2(z)D_{12}(z, w, w', w'') + \kappa_3(z)w ,$$

then $\Phi(z) = \Phi(z, w(z), w'(z), w''(z))$ is regular at $z_0$. Thus by (2.11) and (4.2), we have $N(r, \Phi) = S(r, w)$. By (2.11), (4.2) and Lemma H, we have $m(r, \Phi) = S(r, w)$. Hence $\Phi(z)$ is a small function with respect to $w(z)$.

Put $F_1(z, w, w') = w^4 - \lambda_1(z)w'w^2 + \lambda_0(z)w'^2$. Then $F_1(z) = F_1(z, w(z), w'(z))$ satisfies the conditions (4.2) and (4.3), which imply that $w(z)$ satisfies a differential equation of the form

$$\mu(z)w'' = c(z)w'^2 + B(z, w)w' + A(z, w) , \tag{4.6}$$

where $B(z, w)$ and $A(z, w)$ are polynomials in $w$, and their coefficients and $c(z)$, $\mu(z)$ are small functions with respect to $w(z)$.

Put $F_2(z, w, w') = 2w'^2 - w''w$. Then $F_2(z) = F_2(z, w(z), w'(z))$ also satisfies the conditions (4.2) and (4.3), which imply that $w(z)$ satisfies a differential equation of the form

$$(\sigma(z)w + \tau(z))w'' = \sigma(z)w'^2 + \tilde{B}(z, w)w' + \tilde{A}(z, w) , \tag{4.7}$$

where $\tilde{B}(z, w)$ and $\tilde{A}(z, w)$ are polynomials in $w$, and their coefficients and $\sigma(z)$, $\tau(z)$ are small functions with respect to $w(z)$.

From (4.6) and (4.7), if $\sigma(z)\not\equiv 0$, then $w(z)$ satisfies a first order differential equation of the form

$$P_2(z, w)w'^2+P_1(z, w)w'+P_0(z, w)=0\,, \tag{4.8}$$

where $P_j(z, w)$, $j=1, 2, 3$, are polynomials in $w$, and their coefficients are small functions with respect to $w(z)$. Thus by Lemma I, $w(z)$ satisfies a differential equation of the form (1.5).

If $\sigma(z)\equiv 0$, then from (4.7), $w(z)$ satisfies an equation of the form (1.4).

Secondly we treat the case where $w(z)$ satisfies the condition (2.12). Put $G_4(z, w, w')=w^5-\lambda_1(z)w'w^3+\lambda_0(z)w'^2w(z)$. Then

$$G_4(z, w(z), w'(z))=\frac{L_4}{(z-z_0)^4}+\frac{L_{43}}{(z-z_0)^3}+\frac{L_{42}}{(z-z_0)^2}+\frac{L_{41}}{z-z_0}+O(1),$$
$$L_4, L_{43}, L_{42}, L_{41}\in[L]^*(z_0)\,.$$

By Proposition E, there exist $p_1(z)$ and $p_2(z)$, which are small functions with respect to $w(z)$, such that

$$G_3(z, w(z), w'(z))=\frac{\tilde{L}_3}{(z-z_0)^3}+\frac{\tilde{L}_{32}}{(z-z_0)^2}+\frac{\tilde{L}_{31}}{z-z_0}+O(1), \qquad \tilde{L}_3, \tilde{L}_{32}, \tilde{L}_{31}\in[L]^*(z_0)\,,$$

where

$$G_3(z, w, w')=G_4(z, w, w')+p_1(z)w^2w'+p_2(z)w^4\,.$$

Since $G_3(z, w(z), w'(z))$ satisfies the condition (4.2) and (4.3), put $F(z)=G_3(z, w(z), w'(z))$ in (4.4) and (4.5). Then

$$\tilde{D}_{11}(z, w(z), w'(z))=\frac{\tilde{L}_{11}}{z-z_0}+O(1)\,, \quad \tilde{L}_{11}\in[L]^*(z_0)\,,$$

where

$$\tilde{D}_{11}(z, w, w')=G_3(z, w, w')+\tilde{\eta}_{21}(z)ww'+\tilde{\eta}_{22}(z)w^3+\tilde{\nu}_{11}(z)w'+\tilde{\nu}_{12}(z)w^2\,.$$

Put $F(z)=F_1(z, w(z), w'(z))$ in (4.4) and (4.5). Then

$$\hat{D}_{11}(z, w(z), w'(z))=\frac{\hat{L}_{11}}{z-z_0}+O(1)\,, \quad \hat{L}_{11}\in[L]^*(z_0)\,,$$

where

$$\hat{D}_{11}(z, w, w')=F_1(z, w, w')+\hat{\eta}_{21}(z)ww'+\hat{\eta}_{22}(z)w^3+\hat{\nu}_{11}(z)w'+\hat{\nu}_{12}(z)w^2\,.$$

By Proposition E, there exist $\tilde{\kappa}_1(z)$, $\tilde{\kappa}_2(z)$ and $\tilde{\kappa}_3(z)$, with $|\tilde{\kappa}_1|+|\tilde{\kappa}_2|\neq 0$, which are small functions with respect to $w(z)$, such that if we put

$$\tilde{\Phi}(z, w, w') = \tilde{\kappa}_1(z)\tilde{D}_{11}(z, w, w') + \tilde{\kappa}_2(z)\hat{D}_{11}(z, w, w') + \tilde{\kappa}_3(z)w ,$$

then $\tilde{\Phi}(z) = \tilde{\Phi}(z, w(z), w'(z))$ is regular at $z_0$. Thus by (4.2) and (2.12), we have $N(r, \tilde{\Phi}) = S(r, w)$. By (4.2), (2.12) and Lemma H, we have $m(r, \tilde{\Phi}) = S(r, w)$. Hence $\tilde{\Phi}(z)$ is a small function with respect to $w(z)$. Hence $w(z)$ satisfies an equation of first order of the form (4.8). Thus by Lemma I, $w(z)$ satisfies a differential equation of the form (1.5). ■

**5. Proof of Lemma 3.1.** Without loss of generality, we may assume that $p_2(z) \equiv 0$ and $p_0(z)$ is constant in (1.4) (if necessary put $w = v_1(z)u + v_0(z)$, where $v_0 = -p_2/3p_3$, $v_1 = -v_0'' + (q_1 v_0 + q_0)v_0 + p_3 v_0^3 + p_2 v_0^2 + p_1 v_0 + p_0$). Since $p_3(z) \not\equiv 0$, by Lemma G, we have $m(r, w) = S(r, w)$. Hence there exist infinitely many admissible poles. For a meromorphic function $g(z)$, we define $\omega(z_0, g)$ as follows: if $z_0$ is a pole of order $\mu$ $(\geq 1)$ for $g(z)$, then $\omega(z_0, g) = \mu$; if $g(z_0) \neq \infty$, then $\omega(z_0, g) = 0$. We look at the leading terms of (1.4) using the Test-Power test. Let $z_0$ be an admissible pole and put $\omega(z_0, w) = \mu$. Then $\omega(z_0, w'') = \mu + 2$, $\omega(z_0, ww') = 2\mu + 1$, and $\omega(z_0, w^3) = 3\mu$. If $\mu \geq 2$, then $\mu + 2 < 2\mu + 1 < 3\mu$, hence no terms balance for $\mu \geqq 2$. When $\mu = 1$, we have $\mu + 2 = 2\mu + 1 = 3\mu$. Thus every admissible pole must be a simple pole and the leading terms are $w''$, $ww'$ and $w^3$. Hence we get

$$m(r, w) + N_1(r, w) = S(r, w) . \tag{5.1}$$

Write $w(z)$ near an admissible pole $z_0$ as

$$w(z) = \frac{R}{z - z_0} + \alpha + \beta(z - z_0) + \gamma(z - z_0)^2 + \delta(z - z_0)^3 + O(z - z_0)^4 , \quad R \neq 0 . \tag{5.2}$$

We investigate whether $\alpha$, $\beta$, $\gamma$ and $\delta$ are written in terms of linear transformations of $R$ with small (with respect to $w(z)$) functions as coefficients, that is, whether almost all admissible poles are S2-poles or SS2-poles. From (1.4) and (5.2),

$$p_3(z_0)R^2 - q_1(z_0)R - 2 = 0 , \tag{5.3}$$

$$(3p_3(z_0)R - q_1(z_0))\alpha = P_1(R; z_0) , \tag{5.4}$$

$$6p_3(z_0)R\beta = P_2(R, \alpha; z_0) , \tag{5.5}$$

$$6(3p_3(z_0)R^2 + q_1(z_0)R - 2)\gamma = P_3(R, \alpha, \beta; z_0) , \tag{5.6}$$

$$(3p_3(z_0)R^2 - 2q_1(z_0)R - 6)\delta = P_4(R, \alpha, \beta, \gamma; z_0) , \tag{5.7}$$

where $P_j(\cdot\,; z_0)$ $(j = 1, 2, 3, 4)$ are polynomials in the corresponding arguments and the coefficients are values at $z_0$ of small (with respect to $w(z)$) functions.

If $\Lambda(z) := (q_1(z)/p_3(z))^2 + 8/p_3(z) \equiv 0$, then $q_1(z) \not\equiv 0$ and $R = -4/q_1(z_0)$ by (5.3). Thus with (5.4), $z_0$ is an S1-pole of $w(z)$. Thus by (5.1) and Lemma C, $w(z)$ satisfies a Riccati equation.

Hence, we have to consider merely the case $\Lambda(z)\not\equiv 0$. Since $\Lambda(z)$ is a small function with respect to $w(z)$, we may suppose that $\Lambda(z_0)\neq 0$.

We show that for almost all admissible poles $z_0$ of $w(z)$, $\alpha\in[L]^*(z_0)$ under the condition (5.3), (5.4) and $9p_3(z)+q_1(z)^2\not\equiv 0$. We denote by $\bar{n}^*(r,f)$ the number of the admissible simple poles $z_1$ of $w(z)$ in $|z|\leqq r$ each counted only once, so that

$$3p_3(z_1)R_{z=z_1}-q_1(z_1)=0\,.$$

$\bar{N}^*(r, w)$ is defined in terms of $\bar{n}^*(r, w)$ in the usual way. We have $\bar{N}^*(r, w)=S(r, w)$, which is shown as follows: Put $\varphi_1(z)=9p_3(z)+q_1(z)^2$ ($\not\equiv 0$). If $3p_3(z_1)R_{z=z_1}-q_1(z_1)=0$, then by (5.3) $\varphi_1(z_1)=9p_3(z_1)+q_1(z_1)^2=0$. Thus

$$N^*(r, w)\leqq N(r, 0, \varphi_1)\leqq T(r, \varphi_1)+S(r, w)\leqq S(r, w)\,.$$

Hence for almost all admissible poles $z_0$ of $w(z)$, by (5.4), (5.3), Remark 1.1 and Proposition D,

$$\alpha=P_1(R\,;z_0)/(3p_3(z_0)R-q_1(z_0))=A_1(z_0)R+A_2(z_0)\in[L]^*(z_0)\,,$$

where $A_j(z)$, $j=1, 2$, are small functions with respect to $w(z)$.

Since $R\neq 0$ and $p_3(z_0)\neq 0$, from (5.3)–(5.5), $\beta$ is written by means of linear transformation of $R$ with small (with respect to $w(z)$) functions as coefficients. Hence, for almost all admissible poles $z_0$ of $w(z)$, $\beta=B_1(z_0)R+B_2(z_0)\in[L]^*(z_0)$, where $B_j(z)$, $j=1, 2$, are small functions with respect to $w(z)$.

Similarly to the proof of $\alpha\in[L]^*(z_0)$, for almost all admissible poles $z_0$ of $w(z)$, we have $\gamma=C_1(z_0)R+C_2(z_0)\in[L]^*(z_0)$ by the conditions (5.3), (5.6) and $p_3(z)-q_1(z)^2\not\equiv 0$, where $C_j(z)$, $j=1, 2$, are small functions with respect to $w(z)$.

By (5.3), the left-hand side of (5.7) is $3q_1(z_0)\delta$. Thus if $q_1(z)\not\equiv 0$, then from (5.3)–(5.7), almost all poles of $w(z)$ are SS2-poles, hence

$$N(r, w)-N_{\langle SS2\rangle}(r, w)=S(r, w)\,. \tag{5.8}$$

Thus by (5.1), (5.8) and Lemma 2.1, $w(z)$ satisfies an equation of the form (1.5).

It remains to consider the case $q_1(z)\equiv 0$. From (5.7), we have $P_4(R, \alpha, \beta, \gamma\,; z_0)=0$ for almost all admissible poles $z_0$. From (5.3)–(5.7), eliminating $\alpha$, $\beta$, $\gamma$, and $R^n$ ($n\geqq 2$), we obtain

$$\delta_1(z_0)R+\delta_0(z_0)=0\,, \tag{5.9}$$

where $\delta_1(z)$ and $\delta_0(z)$ are small functions with respect to $w(z)$. In fact,

$$\delta_1(z)=-27p_0p_3(z)^4(2p_3(z)q_0(z)+p_3'(z))\,. \tag{5.10}$$

We denote by $\bar{n}^*_{\delta_1}(r,f)$ the number of admissible poles $z_0$ of $w(z)$ in $|z|\leqq r$ each counted only once so that $z_0$ satisfies $\delta_1(z_0)=0$. $\bar{N}^*_{\delta_1}(r,f)$ is defined in the usual way.

I. When $\bar{N}^*_{\delta_1}(r, w)=S(r, w)$, by Remark 1.1 for all admissible poles $z_0$, $R_{z=z_0}$ is written in terms of small (with respect to $w(z)$) functions directly, and by (5.4), $\alpha$ is also written in terms of small (with respect to $w(z)$) functions directly. Hence almost all admissible poles of $w(z)$ are S1-poles. Thus by Lemma C, $w(z)$ satisfies a Riccati equation.

II. When $\bar{N}^*_{\delta_1}(r, w) \neq S(r, w)$, we have $\delta_1(z) \equiv 0$ by Remark 1.1. Thus by (5.9) $\delta_0(z_0) = 0$ for almost all admissible pole $z_0$. Hence, by Remark 1.1 $\delta_0(z) \equiv 0$.

(i) First we treat the case $p_0 \neq 0$ in (1.4). From (5.10) we obtain (1.9).

From (5.3)–(5.7), we can calculate $\delta_0(z)$ as

$$
\begin{aligned}
\delta_0(z) = {} & 36p_1(z)p_3(z)^4q_0(z)^2 - 36p_1(z)p_3(z)^4q_0'(z) + 72p_1(z)p_3(z)^3p_3'(z)q_0(z) \\
& - 36p_1(z)p_3(z)^3p_3''(z) + 72p_1(z)p_3(z)^2p_3'(z)^2 - 90p_1'(z)p_3(z)^4q_0(z) \\
& - 90p_1'(z)p_3(z)^3p_3'(z) + 54p_1''(z)p_3(z)^4 + 8p_3(z)^4q_0(z)^4 \\
& - 60p_3(z)^4q_0(z)^2q_0'(z) + 54p_3(z)^4q_0(z)q_0''(z) + 36p_3(z)^4q_0'(z)^2 \\
& - 18p_3(z)^4q_0'''(z) + 14p_3(z)^3p_3'(z)q_0(z)^3 - 57p_3(z)^3p_3'(z)q_0(z)q_0'(z) \\
& + 27p_3(z)^3p_3'(z)q_0''(z) + 3p_3(z)^3p_3''(z)q_0(z)^2 - 18p_3(z)^3p_3'''(z)q_0(z) \\
& + 9p_3(z)^3p_3^{(4)}(z) - 6p_3(z)^2p_3'(z)^2q_0'(z) + 78p_3(z)^2p_3'(z)p_3''(z)q_0(z) \\
& - 54p_3(z)^2p_3'(z)p_3'''(z) - 36p_3(z)^2p_3''(z)^2 - 64p_3(z)p_3'(z)^3q_0(z) \\
& + 192p_3(z)p_3'(z)^2p_3''(z) - 112p_3'(z)^5 .
\end{aligned} \tag{5.11}
$$

From (1.9) and $\delta_0(z) \equiv 0$, and by elementary but tedious calculation, we obtain (1.10).

(ii) Secondly we treat the case $p_0 = 0$ in (1.4).

$$w'' = q_0(z)w' + p_3(z)w^3 + p_1(z)w . \tag{5.12}$$

By a suitable transformation $u = a(z)w'/w + b(z)$ in (5.12) we have

$$u'' = q(z)u' + 2u^3 + p(z)u + p_0^*(z) , \tag{5.13}$$

where $a(z)$, $b(z)$, $q(z)$, $p(z)$ and $p_0^*(z)$ are small functions with respect to $w(z)$.

If $p_0^*(z) \not\equiv 0$, then this case reduces to the case (i). Here we assume that $p_0^*(z) \equiv 0$ in (5.13). Put $u_1 = u'/u - q(z)/3$ in (5.13). Then

$$v_1'' = q(z)v_1' + 2v_1^3 + P_1(z)v_1 + D_1(z) , \tag{5.14}$$

where

$$
\begin{aligned}
P_1(z) &= q'(z) - \frac{2}{3}q(z)^2 - 2p(z) , \\
D_1(z) &= \frac{1}{27}(9q''(z) - 18q(z)q'(z) - 27p'(z) + 4q(z)^3 + 18p(z)q(z)) .
\end{aligned} \tag{5.15}
$$

Put $q(z) = -3f(z)/2$ in (5.15). Then

$$D_1(z) = f''(z) + 3f'(z)f(z) + f(z)^3 + 2p(z)f(z) + 2p'(z) . \tag{5.16}$$

When $D_1(z) \not\equiv 0$, the case reduces to the case (i). When $D_1(z) \equiv 0$, putting $v_2 = v_1'/v_1 - q(z)/3$

in (1.14) we have

$$v_2'' = q(z)v_2' + 2v_2^3 + P_2(z)v_2 + D_2(z), \tag{5.17}$$

where

$$D_2(z) = 2f''(z) + 12f'(z)f(z) + 5f(z)^3 + 2p(z)f(z) + 2p'(z). \tag{5.18}$$

If $D_2(z) \not\equiv 0$, then the case reduces to the case (i). Otherwise we iterate the transformation $v_3 = v_2'/v_2 - q(z)/3$.

$$v_3'' = q(z)v_3' + 2v_3^3 + P_3(z)v_3 + D_3(z), \tag{5.19}$$

where

$$D_3(z) = f''(z) + \frac{9}{4}f'(z)f(z) + \frac{5}{8}f(z)^3 + 2p(z)f(z) + 2p'(z). \tag{5.20}$$

If $D_3(z) \not\equiv 0$, then it reduces to the case (i). Thus we have to treat the case $D_1(z) = D_2(z) = D_3(z) \equiv 0$. From (5.16), (5.18) and (5.20), we have $f'(z) + f(z)^2/2 = 0$. Thus $f(z) = 0$ or $f(z) = 2/(z-c)$, where $c$ is a constant. If $f(z) = 0$, then $q(z) = 0$ and $p'(z) = 0$. Thus (5.13) reduces to the Painlevé equation II. If $f(z) = 2/(z-c)$, then by (5.16) $p(z) = d/(z-c)^2$ with $d$ constant. Hence (5.13) is of the form

$$u'' = \frac{-3}{z-c}u' + 2u^3 + \frac{d}{(z-c)^2}u. \tag{5.21}$$

Put $u = \tilde{u}/(z-c)$, $z = c + e^t$, $U(t) = \tilde{u}(c + e^t)$ in (5.21). Then

$$U'' = 2U^3 + (1+d)U. \tag{5.22}$$

Thus, integrating the equation (5.22), we see that $U(z)$ satisfies and equation of the form (1.5). ■

**6. Proof of Lemma 3.2.** First we consider the case $9p_3(z) + q_1(z)^2 \equiv 0$ in (1.4). Put $w = -3u/q_1(z) - 2q_1'(z)/q_1(z)^2 - q_0(z)/q_1(z)$. Then

$$m(r, u) + N_1(r, u) = S(r, u), \tag{6.1}$$

$$u'' + 3u'u + u^3 = \tilde{p}_2(z)u^2 + \tilde{p}_1(z)u + \tilde{p}_0(z), \tag{6.2}$$

where $\tilde{p}_2(z)$, $\tilde{p}_1(z)$, $\tilde{p}_0(z)$ are rational functions in the coefficients of (1.4) and their derivatives.

Let $z_0$ be an admissible pole of $u(z)$. Write it in a neighbourhood of $z_0$ as

$$u(z) = \frac{R}{z-z_0} + \alpha + O(z-z_0).$$

From (6.2) we get

$$(6.3)\qquad R^2-3R+2=0\,, \quad \text{hence} \quad R=1 \text{ or } 2\,.$$

$$(6.4)\qquad 3(R-1)\alpha=\tilde{p}_2(z_0)R\,.$$

We denote by $\bar{n}_\alpha(r, u)$ the number of poles $z_1$ of $u(z)$ each counted only once such that $R_{z=z_1}=1$, and $\bar{N}_\alpha(r, u)$ is defined in the usual way.

If $\bar{N}_\alpha(r, u)\neq S(r, u)$, then by Remark 1.1, we have $\tilde{p}_2(z)\equiv 0$, which implies that (6.2) is of the form (1.6).

If $\bar{N}_\alpha(r, u)=S(r, u)$, then by (6.3), (6.4) and Remark 1.1, $R=2$ and $\alpha=2\tilde{p}_2(z_0)/3$, for almost all admissible poles $z_0$, which implies that almost all admissible poles are S1-poles. Thus by (6.1) and Lemma C, $u(z)$ satisfies a Riccati equation.

Secondly we treat the case $p_3(z)-q_1(z)^2\equiv 0$ in (1.4). Put $w=-q_1(z)v-(5q_1'(z)+p_2(z))/6q_1(z)^2-3q_0(z)/2q_1(z)$. Then

$$(6.5)\qquad m(r, v)+N_1(r, v)=S(r, v)\,,$$

$$(6.6)\qquad v''+v'v-v^3=\tilde{p}(z)(v^2+3v')+H(z)v+S(z)\,,$$

where $\tilde{p}(z)$, $H(z)$, $S(z)$ are rational functions in the coefficients of (1.4) and their derivatives.

Let $z_0$ be an admissible pole of $v(z)$. Write it in a neighbourhood of $z_0$ as

$$v(z)=\frac{R}{z-z_0}+\alpha+\beta(z-z_0)+\gamma(z-z_0)+O(z-z_0)^3\,.$$

From (6.6)

$$(6.7)\qquad R^2+R-2=0\,, \quad \text{hence} \quad R=1 \text{ or } -2\,,$$

$$(6.8)\qquad (3R+1)\alpha=-\tilde{p}(z_0)R+3\tilde{p}(z_0)\,,$$

$$(6.9)\qquad 6R\beta=-6\alpha^2-6\alpha\tilde{p}(z_0)-2H(z_0)-3\tilde{p}'(z_0)R-4\tilde{p}(z_0)\,,$$

$$(6.10)\qquad 6(3R^2-R-2)\gamma=P_5(R, \alpha, \beta\,; z_0)\,,$$

where $P_5(R, \alpha, \beta\,; z_0)$ is a polynomial in $R$, $\alpha$, $\beta$ with small coefficients. We denote by $\bar{n}_\gamma(r, v)$ the number of poles $z_1$ of $v(z)$ each counted only once such that $R_{z=z_1}=1$ and $\bar{N}_\gamma(r, u)$ is defined in the usual way. If $R=1$ at $z=z_1$, then from (6.7)–(6.10) eliminating $R$, $\alpha$ and $\beta$ successively, we have $\Delta(z_1)=0$, where $\Delta(z)$ is defined as in Theorem 1.1 which is a small function with respect to $v(z)$.

When $\bar{N}_\gamma(r, u)\neq S(r, u)$, by Remark 1.1 $\Delta(z)\equiv 0$, which implies that (6.6) is of the type (1.7).

When $\bar{N}_\gamma(r, u)=S(r, u)$, by (6.7), (6.8) and Remark 1.1, for almost all admissible poles $z_0$, $R=-2$ and $\alpha=-\tilde{p}(z_0)$, which implies that almost all admissible poles are S1-poles. Thus by (6.1) and Lemma C, $v(z)$ satisfies a Riccati equation. ■

**7. Proof of Lemma 3.3.** In (1.4) put $w=2v/q_1(z)+(2/q_1(z))'-q_0(z)/q_1(z)$. Then

$$v''=2vv'+\tilde{p}_2(z)v^2+\tilde{p}_1(z)v+\tilde{p}_0(z)=(2v'+\tilde{p}_2(z)v)v+\tilde{p}_1(z)v+\tilde{p}_0(z)\,, \tag{7.1}$$

where $\tilde{p}_2(z)$, $\tilde{p}_1(z)$ and $\tilde{p}_0(z)$ are rational functions in the coefficients of (1.4) and their derivatives. Put $2v'+\tilde{p}_2(z)v=\varphi(z)$. Then by Lemma G, $m(r,\varphi)=S(r,v)$.

If $N(r,v)=S(r,v)$, then we have $T(r,\varphi)=S(r,v)$. Thus, $\varphi(z)$ is a small (with respect to $v(z)$) function, which implies that $v(z)$ satisfies a linear equation of first order.

We treat the case $N(r,v)\neq S(r,v)$. We may assume that there exists an admissible pole $z_0$ of $v$ by Remark 1.1. Put $\omega(z_0,v)=\mu$. The leading terms of (7.1) are $v''$ and $2vv'$, and $\mu=1$. Hence we may write $v(z)$ near $z_0$ as

$$v(z)=\frac{R}{z-z_0}+\alpha+\beta(z-z_0)+O(z-z_0)^2\,,\qquad R\neq 0\,. \tag{7.2}$$

From (7.1) and (7.2)

$$R+1=0\,,\quad 2\alpha-\tilde{p}_2(z_0)R=0\,,\quad 2\alpha\tilde{p}_2(z_0)+\tilde{p}'_2(z_0)R+\tilde{p}_1(z_0)=0\,. \tag{7.3}$$

Thus, from (7.3), $\tilde{p}_2(z_0)^2+\tilde{p}'_2(z_0)-\tilde{p}_1(z_0)=0$. By Remark 1.1, we have

$$\tilde{p}_2(z)^2+\tilde{p}'_2(z)-\tilde{p}_1(z)\equiv 0\,. \tag{7.4}$$

Put in (7.1) $u=(v-\tilde{p}_2(z)/2)'-(v-\tilde{p}_2(z)/2)^2$. Then by (7.4)

$$u'+\tilde{p}_2(z)u=\tilde{p}_0(z)+\frac{\tilde{p}''_2(z)}{2}-\frac{\tilde{p}_2(z)^3}{4}\,,$$

which implies that $u(z)$ satisfies a linear equation of first order. ■

**8. Proof of Lemma 3.4.** Since $q_1(z)\equiv 0$, if $p_2(z)\equiv 0$, then (1.4) is a linear differential equation. Thus we may assume that $p_2(z)\not\equiv 0$. Put in (1.4) $w=6u/p_2(z)+\{(1/p_2(z))''-q_0(z)(1/p_2(z))'-p_1(z)/p_2(z)\}$. Then

$$u''=q(z)u'+6u^2+p(z)\,, \tag{8.1}$$

where $q(z)$ and $p(z)$ are rational functions in the coefficients of (1.4) and their derivatives. By Lemma G, $m(r,u)=S(r,u)$. Thus by Remark 1.1 $u(z)$ has infinitely many admissible poles. Let $z_0$ be an admissible pole and put $\omega(z_0,u)=\mu$. In (8.1), the leading terms are $u''$ and $6u^2$, and $\mu=2$. Hence we write $u(z)$ near $z_0$ as

$$\begin{aligned} u(z)=&\frac{R_2}{(z-z_0)^2}+\frac{R_1}{z-z_0}+\alpha_0+\alpha_1(z-z_0)+\alpha_2(z-z_0)^2\\ &+\alpha_3(z-z_0)^3+\alpha_4(z-z_0)^4+O(z-z_0)^5\,,\qquad R_2\neq 0\,. \end{aligned} \tag{8.2}$$

By the Test-Power test of (8.1), the series (8.2) is a resonant series. In fact, $\alpha_4$ is an arbitrary constant, if the following condition holds:

$$(8.3)\qquad 1440\alpha_0\alpha_2+720\alpha_1^2+60\alpha_1 q''(z_0)+240\alpha_2 q'(z_0)+360\alpha_3 q(z_0) +1440\alpha_3 R_1+60p''(z_0)-5q^{(4)}(z_0)R_1-2q^{(5)}(z_0)R_2=0\,.$$

On the other hand, $R_1$, $R_2$, $\alpha_0$, $\alpha_1$, $\alpha_2$ and $\alpha_3$ are written directly in terms of small (with respect to $u(z)$) functions. Thus there are small (with respect to $u(z)$) functions $\sigma_6(z)$, $\sigma_5(z)$, $\sigma_4(z)$, $\sigma_3(z)$ and $\sigma_2(z)$ such that if we put

$$(8.4)\qquad D(z, u, u')=u'^2+\sigma_6(z)u^3+\sigma_5(z)u'u+\sigma_4(z)u^2+\sigma_3(z)u'+\sigma_2(z)u\,,$$

then $D(z)=D(z, u(z), u'(z))$ has at most a simple pole at $z_0$ and the residue is written in terms of small (with respect to $u(z)$) function as

$$(8.5)\qquad D(z)=\frac{\kappa(z_0)}{z-z_0}+O(1)\,,$$

where $\kappa(z)$ is a small function with respect to $u(z)$.

If $\kappa(z)\equiv 0$, then $D(z)$ is regular at $z_0$, which implies $N(r, D)=S(r, u)$. By Lemma H, $m(r, D)\leqq 3m(r, u)\leqq S(r, u)$. Hence $D(z)$ is a small function with respect to $u(z)$. Therefore $u(z)$ satisfies a differential equation of the form (1.5).

We consider the case $\kappa(z)\not\equiv 0$. We write $R_2$, $R_1$, $\alpha_0$, $\alpha_1$, $\alpha_2$, $\alpha_3$ in terms of small (with respect to $u(z)$) functions successively. Hence $\sigma_6(z)$, $\sigma_5(z)$, $\sigma_4(z)$, $\sigma_3(z)$, $\sigma_2(z)$ are also written in terms of small (with respect to $u(z)$) functions successively. Thus by elementary but tedious calculation we obtain

$$\kappa(z)=-(15000p(z)q(z)-18750p'(z)+36q(z)^5-900q(z)^3q'(z)+2000q(z)^2q''(z) +2500q(z)q'(z)^2-1875q(z)q'''(z)-3125q'(z)q''(z)+625q^{(4)}(z))/9375\,.$$

Further elimination $R_2$, $R_1$, $\alpha_0$, $\alpha_1$, $\alpha_2$ and $\alpha_3$ in (8.3), we obtain $V(z_0)=0$, where $V(z)$ is a differential polynomials in $p(z)$ and $q(z)$. Thus $V(z)$ is a small function with respect to $u(z)$, which implies $V(z)\equiv 0$ by Remark 1.1. By elementary but tedious calculations, putting $T(z)=-9375\kappa(z)$, in the differential equation for $p(z)$ and $q(z)$ ($V(z)\equiv 0$), we obtain a linear differential equation for $T(z)$

$$T'+q(z)T=0\,,\quad T(z)\not\equiv 0\,.$$

■

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