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AREA INTEGRALS FOR RIESZ MEASURES ON THE SIEGEL UPPER HALF SPACE OF TYPE II

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Abstract. The area integrals of harmonic functions on the Siegel upper half space of type II are important tools for studying Hardy spaces and boundary behavior of harmonic functions. In this paper we will extend the area integrals to subharmonic functions on the Siegel upper half space of type II, and prove their L^p-estimates by the admissible maximal functions for all $0<p<\infty$. The extended area integrals are analogues of the area integrals which were introduced by T. McConell in the case of the Euclidean upper half space.

1. Introduction. The area integrals of harmonic functions on the Siegel upper half space of type II were studied extensively by Korányi and Vagi [7] and Geller [4] etc. In this paper we will extend the area integrals to subharmonic functions and prove their L^p-estimates by the admissible maximal functions. In the case of the Euclidean upper half space, such an extension of the area integrals was introduced by McConell [8] and L^p-estimates for the extended area integrals by nontangential maximal functions were proved by McConell [8], Uchiyama [10] and Kaneko [6].

Throughout this paper we denote by $\mathscr{U}^{n+1}$ the Siegel upper half space of type II, $\{(z_0, z)\in \boldsymbol{C}\times \boldsymbol{C}^n : \operatorname{Im} z_0-|z|^2>0\}$. Since one thinks of $\mathscr{U}^{n+1}$ as the product of the interval $(0, +\infty)$ and the Heisenberg group H^n under the coordinate change $(h, t, z)=(\operatorname{Im} z_0-|z|^2, \operatorname{Re} z_0, z)$ (cf. [7], [4]), we will be concerned in this paper with analysis on $(0, +\infty)\times H^n$ instead of $\mathscr{U}^{n+1}$. The Heisenberg group H^n is the Lie group whose underlying manifold is $\boldsymbol{R}\times \boldsymbol{C}^{n+1}$, with multiplication $(t, z)\cdot(t', z')=(t+t'+2\operatorname{Im}(\sum z_j\bar{z}'_j), z+z')$ and dilation $\delta_r(t, z)=(r^2t, rz)$ (see [3]).

In order to describe our analogue of McConell's area integral and the main result, we now mention the notation. Throughout this paper we denote by U^{n+1} the space $(0, \infty)\times H^n=\{(h, (t, z)) : h\in(0, \infty), (t, z)\in H^n=\boldsymbol{R}\times \boldsymbol{C}^{n+1}\}$. In the $(h, (t, z))$-coordinate, the following vector fields generate the tangent bundle $T(U^{n+1})$:

$$H=\frac{\partial}{\partial h}, \quad T=\frac{\partial}{\partial t}, \quad X_j=\frac{\partial}{\partial x_j}+2y_jT, \quad Y_j=\frac{\partial}{\partial y_j}-2x_jT,$$

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where $z=(z_1, \dots, z_n)$, $x_j=\mathrm{Re}\, z_j$ and $y_j=\mathrm{Im}\, z_j, j=1, \dots, n$.

We denote by g the Riemannian metric such that the frames $\{2hH, 2hT, \sqrt{h}X_j, \sqrt{h}Y_j\}$ are orthonormal. This metric is identified with a constant multiple of the Bergman metric of $\mathcal{U}^{n+1}$ under the coordinate change. By the definition of the Laplace-Beltrami operator L of the metric g we have $L=4h[hH^2-nH+hT^2-4L_0]$, where L_0 is the sub-Laplacian of the Heisenberg group, that is, $L_0=-\sum(X_j^2+Y_j^2)$. (For detail, see [3] and [4]).

We now recall the definitions of harmonic and subharmonic functions on U^{n+1}.

DEFINITION. (1) A C^2-function f on an open set Ω of U^{n+1} is harmonic if $Lf=0$ on Ω.

(2) A function u on Ω is subharmonic if (i) u satisfies $-\infty \le u < \infty$ on Ω and $u(z) > -\infty$ for some $z \in \Omega$; (ii) u is upper semicontinuous on Ω; (iii) for every smooth relatively compact domain D in Ω and for every continuous function h on the closure $\bar{D}$ of D which is harmonic on D and $u \le h$ on the boundary ∂D of D, we have $u \le h$ in D.

It is well known that if u is a subharmonic function on U^{n+1}, then there exists a unique nonnegative Radon measure μ_u on U^{n+1} such that for every compactly supported C^2-function φ on U^{n+1},

$$\int_{U^{n+1}} u(Z)L\varphi(Z)dV_g(Z) = \int_{U^{n+1}} \varphi(Z)d\mu_u(Z),$$

where dV_g is the Riemannian measure with respect to the metric g, that is, $dV_g(h, t, z)=(1/4h^{n+2})dV(h, t, z)$, where dV is the (Euclidean) Lebesgue measure on U^{n+1} (cf. [5, Theorems 2.3.2 and 2.3.3]). The measure μ_u is called the Riesz measure of u.

Now we are ready to define an analogue of McConell's area integral for a subharmonic function u on U^{n+1}. For $\alpha > 0$ and $x \in H^n$, our analogue $S_\alpha(\mu_u)(x)$ is defined as follows:

$$S_\alpha(\mu_u)(x) = \mu_u(A_\alpha(x)),$$

where $A_\alpha(x)$ is the admissible domain defined by

$$A_\alpha(x) = \{(h, y) \in (0, \infty) \times H^n : |y^{-1} \cdot x|^2 < \alpha h\},$$

where $|\ |$ is the usual homogeneous norm on H^n, that is, $|w|=(t^2+|z|^4)^{1/4}$ for $w=(t, z) \in H^n$ (see [4]). The function $S_\alpha(\mu_u)$ is regarded as an extension of the area integral, bacause if $u=|v|^2$ for some harmonic function v in U^{n+1}, then Green's formula with respect to the metric g yields $d\mu_u(Z)=Lu(Z)dV_g(Z)=|\nabla v(Z)|^2 dV_g(Z)$, and therefore

$$S_\alpha(\mu_u)(x) = \int_{A_\alpha(x)} |\nabla v(Z)|^2 dV_g(Z).$$

Here the right hand side is known as the square of the area integral of v.

For a subharmonic function u on U^{n+1}, we denote by u_β^* the admissible maximal function of u defined by

$$u_\alpha^*(x) = \| u\chi_{A_\alpha(x)} \|_{L^\infty(dV)},$$

where χ_E is the characteristic function of a set E.

Our main theorem is the following:

THEOREM. *Let u be a subharmonic function on U^{n+1}. Then for any $0<\alpha, \beta<\infty$ and $0<p<\infty$, there exists a constant C_p depending only on α, β, p and g such that*

$$\int_{H^n} \mu_u(A_\alpha(x))^p dx \le C_p \int_{H^n} u_\beta^*(x)^p dx, \qquad 0<p<\infty,$$

where dx is the Haar measure on H^n.

In the case of the Euclidean upper half space, the above inequality was obtained by McConell as a main result in [8] with a restriction on the range of p. However, the restriction was later removed by Uchiyama [10]. Recently, Kaneko [6] simplified Uchiyama's proof by using geometric features of sawtooth regions generated by nontangential cones. For the proof of Theorem, we will expand the idea of Kaneko into our non-Euclidean case.

As we have shown, if $u=|v|^2$ for some harmonic function v on U^{n+1}, the function $S_\alpha(\mu_u)$ is the square of the area integral of v. Consequently the Theorem is a generalization to subharmonic functions of Korányi and Vagi [7, Theorem 8.2] when $2^{-1}<p<\infty$, and of Geller [4, Theorem 5.1(a)] when $0<p\le 2^{-1}$.

After proving a preliminary lemma in Section 2, we will study in Section 3 sawtooth regions defined by admissible domains. Then in Section 4 we will prove Theorem.

2. Preliminaries. In this section we will generalize to homogeneous groups of a lemma of Murai-Uchiyama for the Euclidean space ([9]). To avoid wasting of space, we refer to the book [3] for all terminology and notation about homogeneous groups and functions of bounded mean oscillation, which will be used in this section. Throughout this section, G is a homogeneous group endowed with the quasi-metric $d(\cdot, \cdot)$ defined by $d(x, y)=|y^{-1}\cdot x|$, where $|\ |$ is the homogeneous norm of G. We denote by μ the bi-invariant Haar measure on G. The Heisenberg group is a typical example of homogeneous groups.

Let $\mathrm{BMO}(G)$ be the space of all functions of bounded mean oscillation on G, and let $\|\ \|_{\mathrm{BMO}}$ be the BMO-norm. The following lemma was proved by Murai and Uchiyama by using certain maximal dyadic cubes when G is the Euclidean space. Here we will generalize it to homogeneous groups by a proof in which no maximal dyadic cubes are used.

LEMMA 1. *Suppose $\mu(G)=+\infty$. If $f\in\mathrm{BMO}(G)$ and $\|f\|_{\mathrm{BMO}}\le 1$, then there exist* *positive constants c_1 and c_2 depending only on G, d and μ such that*

$$\mu(\{x\in G:|f(x)|>\gamma\})\leq c_1\exp(-c_2\gamma)\mu(\{x\in G:|f(x)|>1\}), \qquad \gamma>1 .$$

PROOF. Let $E=\{x\in G:|f(x)|>1\}$. We may assume that $\mu(E)<\infty$. For $x\in G$ and $r>0$, let $B(x, r)=\{y\in G:d(y, x)<r\}$. Take a family of balls, say $\{B(\bar{x}, N)\}_{N=1,2,\dots}$, which is a covering of G, and let $E_N=E\cap B(\bar{x}, N)$ for every $N=1, 2, \dots$. Let N be a large number so that $E_N\neq\varnothing$. Denote by L_N the set of points of density of E_N. Then for every $x\in L_N$, we can define

$$r'(x)=\sup\{r>0:\mu(E\cap B(x, r))>\frac{1}{2}\mu(B(x, r))\}$$

and obrain that $0<r'(x)$. Moreover, the assumptions that $\mu(G)=+\infty$ and $\mu(E)<\infty$ yield $r'(x)<+\infty$. Therefore by the Wiener lemma (cf. [3, (1.66)]), we can take points $x_1, x_2, \dots\in L_N$ in such a way that the balls $B(x_j, r'(x_j))$ are disjoint and $L_N\subset\bigcup_j B(x_j, Cr'(x_j))$, where C is a positive constant depending only on G, d and μ. Hence by the definition of $r'(x)$ there exists $r(x_j)\in(0, r'(x_j)]$ satisfying

$$\mu(E\cap B(x_j, r(x_j)))>\frac{1}{2}\mu(B(x_j, r(x_j))) , \tag{1}$$

$$\mu(E\cap B(x_j, 2Cr(x_j)))\leq\frac{1}{2}\mu(B(x_j, 2Cr(x_j))) , \tag{2}$$

$$L_N\subset\bigcup_j B(x_j, 2Cr(x_j)) . \tag{3}$$

Let $B_j=B(x_j, r(x_j))$ and $CB_j=B(x_j, 2Cr(x_j))$. Then

$$1\geq\|f\|_{\mathrm{BMO}}\geq\frac{1}{\mu(CB_i)}\int_{CB_i\cap\{|f|>1\}}|f(x)-f_{CB_i}|d\mu(x) ,$$

where $f_{CB_i}=\int_{CB_i}fdx/\mu(CB_i)$. Hence

$$1\geq\frac{1}{\mu(CB_i)}\int_{CB_i\cap\{|f|>1\}}|f|d\mu-|f|_{CB_i}\frac{\mu(CB_i\cap\{|f|>1\})}{\mu(CB_i)} . \tag{4}$$

By (2) and (4) we have

$$\frac{1}{\mu(CB_i)}\int_{CB_i\cap\{|f|>1\}}|f|d\mu\leq 1+\frac{1}{2}|f|_{CB_i} .$$

Therefore

$$|f|_{CB_i}\leq\frac{1}{\mu(CB_i)}\int_{CB_i\cap\{|f|\leq 1\}}|f|d\mu+1+\frac{1}{2}|f|_{CB_i}\leq 2+\frac{1}{2}|f|_{CB_i} .$$

Accordingly $|f|_{CB_i}\leq 4$. Since $B(\bar{x}, N)\cap\{|f|>\gamma\}\subset B(\bar{x}, N)\cap\{|f|>1\}\subset_{\text{a.e.}}\bigcup_j CB_j$ and $|f-f_{CB_i}|\geq|f|-|f|_{CB_i}>\gamma-4$ on $\{|f|>\gamma\}$, we get by [3, Theorem 5.15] that

$$\begin{aligned}\mu(B(\bar{x}, N)\cap\{|f|>\gamma\}) &\le \sum_j \mu(CB_i\cap\{|f|>\gamma\})\\ &\le \sum_j \mu(CB_i\cap\{|f-f_{CB_i}|>\gamma-4\})\\ &\le C_1\exp(-c_2(\gamma-4))\sum_j \mu(CB_j)\\ &\le C_2\exp(-c_2\gamma)\sum_j \mu(B_j)\\ &\le 2C_2\exp(-c_2\gamma)\sum_j \mu(B_j\cap E) \quad \text{(by (1))}\\ &\le c_1\exp(-c_2\gamma)\mu(E)\,.\end{aligned}$$

(We used [3, Theorem 5.15] for the third inequality.) Here C_1 and C_2 are positive constants depending only on G. Thus by letting $N\to\infty$ we obtain the lemma. ■

3. Carleson measures and admissible regions. A main step of Uchiyama's proof of the extended McConell theorem is to verify that a measure defined by a sawtooth region in the Euclidean upper half space is a Carleson measure. In this section we will prove a non-Euclidean analogue of his lemma by enlarging an idea of Kaneko [6] into our setting. For $E\subset H^n$, let $W_\alpha(E)=\bigcup_{x\in E} A_\alpha(x)$. As is well known, the set is a Heisenberg analogue of the sawtooth region. From now on we use the following notation:

$$\varrho(x, y)=|y^{-1}\cdot x|^2\,, \qquad \text{for} \quad x, y\in H^n \quad \text{and}$$

$$B_2(x, r)=\{y\in H^n: \varrho(x, y)<r\} \qquad \text{for} \quad x\in H^n,\ r>0\,.$$

Let us recall that a nonnegative measure ν on U^{n+1} is called a Carleson measure if

$$\|\nu\|_c:=\sup\left\{\frac{\nu((0, r)\times B_2(y, r))}{|B_2(y, r)|}: y\in H^n,\ r>0\right\}<\infty\,,$$

where we denote the Haar measure of a measurable set F in H^n by $|F|$.

PROPOSITION 1. *Suppose $0<\alpha<\beta<\infty$. Let u be as in Theorem, and let $E=\{x\in H^n: u_\beta^*(x)\le 1\}$. If a measure ν is defined by*

$$\nu(F)=\int_{W_\alpha(E)\cap F} h^{n+1}d\mu_u(h, x)\,, \qquad F\subset U^{n+1}\,,$$

then $\|\nu\|_c<C_3$, where C_3 is a constant depending only on α, β and the metric g.

In order to prove this proposition, we will construct a smooth function χ which behaves like the characteristic function of $W_\alpha(E)\cap W_\alpha(B)$ for a ball B in H^n. Using such a smooth function one can compute the measure $\nu((0, r)\times B)$, where r is the radius of

B, because it behaves very much like the integral $\int u(h,x)L(\chi(h,x)h^{n+1})dV_g(h,x)$. We will begin with providing such a function:

LEMMA 2. *Let E be a closed set in H^n, and let B be a ball $B_2(\bar{x}, r_0)$. Suppose $0<h_0<r_0$ and $0<\alpha<\beta<\infty$. Then there exist a nonnegative C^∞ function χ on U^{n+1} and two constants $c_3>1$, $c_4>1$ depending only on α, β and H^n, which satisfy*

(1) $\chi=1$ *on* $W_\alpha(E)\cap W_\alpha(B)\cap((h_0, r_0)\times H^n)$;

(2) $\chi=0$ *off* $W_\beta(E)\cap W_\beta(B)\cap((c_3^{-1}h_0, c_3 r_0)\times B(\bar{x}, c_4 r_0))$;

(3) $|L\chi|\leq C_4$; *and*

(4) $|(\nabla\chi, \nabla h^{n+1})|\leq C_5 h^{n+1}$, *where C_4 and C_5 are positive constants depending only on α, β and g, and where $(\nabla\chi, \nabla h^{n+1})$ is the inner product of the gradients of χ and h^{n+1} with respect to the metric g.*

PROOF. For any small number $s>0$ and any $(h,x)\in U^{n+1}=(0,\infty)\times H^n$, let

$$\mathscr{B}((h,x),s)=((1-s)h,(1+s)h)\times B_2(x, sh)\subset U^{n+1}.$$

Then it is easy to check that there exists a constant $\tau\in(0,1)$ depending on α, β and H^n such that for every $w\in H^n$,

(5) $\mathscr{B}((h,x),\tau)\subset A_{(\alpha+\beta)/2}(w)$ when $(h,x)\in A_\alpha(w)$, and

(6) $\mathscr{B}((h,x),\tau)\cap A_{(\alpha+\beta)/2}(w)=\emptyset$ when $(h,x)\notin A_\beta(w)$.

Now take a C^∞-function ρ on C^{n+1} such that $\rho\geq 0$, $\mathrm{supp}(\rho)\subset(-1,1)\times B_2(0,1)$ and $\int_{C^{n+1}}\rho(h,x)dhdx=1$. Let

$$W=W_{(\alpha+\beta)/2}(E)\cap W_{(\alpha+\beta)/2}(B)\cap\{((1-\tau)h_0,(1+\tau)r_0)\times H^n\},$$

and

$$\chi(h,x)=\frac{1}{(\tau h)^{n+2}}\int_W \rho\left(\frac{h-s}{\tau h}, \delta_{1/\sqrt{\tau h}}(y^{-1}\cdot x)\right)dsdy, \qquad (h,x)\in U^{n+1}.$$

From (5), (6) and the definition of χ is follows that χ satisfies the conditions (1) and (2) when we put $c_3=(1+\tau)/(1-\tau)$ and

$$c_4=K\left(\left(\frac{\tau(1+\tau)}{1-\tau}\right)^{1/2}+\left(\frac{\beta}{1-\tau}\right)^{1/2}(1+\tau)+1\right),$$

where K is the constant such that $|x\cdot y|\leq|x|+K|y|$ for every $x, y\in H^n$ (see [4] for this triangle inequality).

By easy calculation we get

$$|H\chi(h,x)|\leq c_\tau h^{-1}, \quad |H^2\chi(h,x)|\leq c_\tau h^{-2}, \quad |T^2\chi(h,x)|\leq c_\tau h^{-2}, \quad |L_0\chi(h,x)|\leq c_\tau h^{-1},$$

where c_τ is a constant depending only on α, β, g and τ, but τ is a constant determined by α, β and the Heisenberg group structure. Therefore we have

$$|L\chi(h,x)|=|4h[hH^2-nH+hT^2-4L_0]\chi(h,x)|\leq C_4,$$

and furthermore,

$$|(\nabla\chi(h, x), \nabla h^{n+1})| = 4\,|h^2 H\chi(h, x)(n+1)h^n| \leq C_5 h^{n+1} ,$$

where C_4 and C_5 are positive constants depending only on α, β and g. ■

By this lemma, Proposition 1 is proved as follws:

PROOF OF PROPOSITION 1. Let E be as in Proposition 1. Take a ball $B = B_2(\bar{x}, r_0)$. For E and B, let χ be as in the previous lemma, and let $F = \{(h, x) \in U^{n+1} : L(\chi h^{n+1}) \neq 0\}$. Then we have

$$\begin{aligned}
\nu((h_0, r_0) \times B) &\leq \int_{W_\alpha(E) \cap W_\alpha(B) \cap ((h_0, r_0) \times H^n)} h^{n+1} d\mu_u \leq \int_{U^{n+1}} \chi h^{n+1} d\mu_u \\
&= \int_{U^{n+1}} u(h, x) L(\chi(h, x) h^{n+1}) dV_g(h, x) \\
&= \int_F u(h, x)\{h^{n+1} L\chi(h, x) + (\nabla\chi(h, x), \nabla h^{n+1})\} dV_g(h, x) \\
&\leq C_6 \int_F |u(h, x)|\, h^{n+1} dV_g(h, x) \\
&\leq C_6 \int_F h^{n+1} dV_g(h, x) = 4^{-1} C_6 \int_F h^{-1} dh dx \\
&\leq 4^{-1} C_6 \int_{B_2(\bar{x}, c_4 r_0)} \left\{ \int_{\{h : (h, x) \in F\}} h^{-1} dh \right\} dx =: (\mathrm{I}) ,
\end{aligned}$$

where C_6 is a positive constant depending only on α, β and g, and c_4 is the constant as in Lemma 2. In the same way as in [6, p. 593], we obtain that $(\mathrm{I}) \leq C_7 |B|$, where C_7 is a positive constant depending only on α, β and g. Indeed, for $x \in B(\bar{x}, c_4 r_0)$, we have

$$\{h : (h, x) \in F\} \subset \bigcup_{j=1}^{3} A_j ,$$

where $A_1 = [\{(1-\tau)/(1+\tau)\} h_0, h_0]$, $A_2 = [\{\alpha/\beta\} r_0, \{(1+\tau)/(1-\tau)\} r_0]$ and $A_3 = [d(x)^2/\beta, d(x)^2/\alpha]$, with $d(x) = \max\{\inf\{|y^{-1} \cdot x| : y \in E\}, \inf\{|y^{-1} \cdot x| : y \in B\}\}$. Thus

$$\nu((0, r_0) \times B) = \lim_{h_0 \to 0} \nu((h_0, r_0) \times B) \leq C_7 |B| .$$

■

4. A good λ inequality and proof of the Theorem. We will begin with proving a good λ inequality which implies the Theorem. For $W \subset U^{n+1}$, $\alpha > 0$ and the Riesz measure μ_u of a subharmonic function u on U^{n+1}, let

$$S_\alpha(W, \mu_u)(x) = \mu_u(W \cap A_\alpha(x)) , \qquad x \in H^n .$$

As a consequence of Proposition 1 and Lemma 1 we have the following:

PROPOSITION 2 (A good λ inequality). *Suppose* $0<\alpha<\beta<\infty$. *Let* u *and* μ_u *be as in the Theorem. For* $R>0$, *let* $T_R=(-R, R)\times B_2(0, r)$. *Then for all* $\gamma>1$ *and for all* $R>0$,

$$|\{x\in H^n: S_\alpha(T_R, \mu_u)(x)>\gamma, u_\beta^*(x)\le 1\}|\le c_5 \exp(-c_6\gamma)|\{x\in H^n: S_{2\alpha}(T_R, \mu_u)(x)>1\}|,$$

where c_5 *and* c_6 *are positive constants depending only on* α, β *and* g.

PROOF. Let Ψ be an infinitely differentiable function on H^n such that $\Psi=1$ on $B_2(0, \alpha)$, $\Psi=0$ off $B_2(0, 2\alpha)$ and $0\le\Psi\le 1$ on H^n. Let $\Psi_h(x)=h^{-n-1}\Psi(\delta_{1/\sqrt{h}}x)$, $x\in H^n$. Then we get:

ASSERTION 1. *If a measure* dm *on* U^{n+1} *is a Carleson measure, then the balayage* Ψ^*m *of* dm *defined by*

$$\Psi^*m(y)=\int_{U^{n+1}}\Psi_h(y^{-1}\cdot x)dm(h, x), \quad y\in H^n,$$

is a function of bounded mean oscillation. Indeed

$$\|\Psi^*m\|_{\mathrm{BMO}}\le C_8\|m\|_c,$$

where C_8 *is a positive constant depending only on* α *and* H^n.

REMARK. Note that since $B_2(x, r)=B(x, \sqrt{r})$, so for a locally integrable function f on H^n, we have

$$\|f\|_{\mathrm{BMO}}=\sup\left\{\frac{1}{|B_2(x, r)|}\int_{B_2(x,r)}\left|f(y)-\frac{1}{|B_2(x, r)|}\int_{B_2(x,r)}f(y')dy'\right|dy: x\in H^n, r>0\right\}.$$

PROOF OF ASSERTION 1. Since Ψ behaves very much like the characteristic function of $B(0, \alpha)$, we see by [1] that the operator $Mf(x)=\sup\{|\Psi_h*f(y)|: |y^{-1}\cdot x|^2<h\}$ is bounded from $L^q(H^n, dx)$ to $L^q(H^n, dx)$ for $1<q<\infty$. Moreover, by easy calculus we obtain

$$|\Psi_h(x)|\le C_9\frac{h^{n+1}}{[|x|+h]^{2n+2}}, \qquad x\in H^n,$$

and from the mean value theorem (cf. [3]) is follows that

$$|\Psi_h(y^{-1}\cdot x)-\Psi_h(x)|\le C_{10}\{|y|^2[|x|^2+h]^{-n-2}+|y|[|x|^2+h]^{-n-3/2}\}$$

when $h+|x|^2\ge c|y|^2$, where c is a sufficiently large constant depending only on H^n, and C_9 and C_{10} are positive constants depending only on H^n and Ψ. Therefore Assertion 1 is an immediate consequence of Theorem 4(i) of [1]. (The end of the proof of Assertion 1).

Now we return to the proof of Proposition 2. By what we obtained so far, we can apply the argument in [6, p. 594] to our case: The definition of Ψ implies that for $W \subset U^{n+1}$ and $R>0$,

$$S_\alpha(W_\alpha(E) \cap T_R, \mu_u)(x) \leq \Psi^* \nu_R(x) \leq S_{2\alpha}(T_R, \mu_u)(x) , \qquad x \in H^n ,$$

where ν_R is the measure defined by $\nu_R(F) = \nu(T_R \cap F)$ while ν is the measure defined in Proposition 1. Hence by Proposition 1 and Assertion 1 we have $\| \Psi^* \nu_R \|_{\mathrm{BMO}} \leq C_8 C_3$. Therefore Lemma 1 implies that

$$|\{x : \Psi^* \nu_R(x) > \gamma\}| \leq c_1 \exp(-c_2 \gamma) |\{x : \Psi^* \nu_R(x) > 1\}| ,$$

for $\gamma > 1$. Thus we can prove Proposition 2 by using the fact that $S_\alpha(W_\alpha(E) \cap T_R, \mu_u)(x) = S_\alpha(T_R, \mu_u)(x)$ when $x \in E$.

PROOF OF THE THEOREM. What we have seen so far allow us to apply the argument in [10, pp. 376–377] and [6, pp. 594–595] to our setting: Let $a = (\alpha + \beta)/2$ and $b = \max\{a, \beta\}$. By Proposition 2 we get

$$\begin{aligned} |\{S_a(T_R, \mu_u) > \gamma\lambda\}| &\leq |\{S_a(T_R, \mu_u) > \gamma\lambda, u_b^* \leq \lambda\}| + |\{u_b^* > \lambda\}| \\ &\leq c_7 \exp(-c_8 \gamma) |\{S_{2a}(T_R, \mu_u) > \lambda\}| + |\{u_b^* > \lambda\}| , \end{aligned}$$

where c_7 and c_8 are positive constants depending only on α, β and g. Therefore

$$\gamma^{-p} \| S_a(T_R, \mu_u) \|_p^p \leq c_7 \exp(-c_8 \gamma) \| S_{2a}(T_R, \mu_u) \|_p^p + \| u_b^* \|_p^p . \tag{1}$$

We need the following:

LEMMA 3. *Suppose* $0 < \alpha < \beta < \infty$, *and* $0 < p < \infty$. *Then for every compactly supported nonnegative Radon measure* ν,

$$\int_{H^n} \{\nu(A_\beta(x))\}^p dx \leq C_{11} \int_{H^n} \{\nu(A_\alpha(x))\}^p dx ,$$

where C_{11} *is a positive constant depending on* α, β, p *and* H^n *but is independent of* ν.

PROOF. Since ν is compactly supported, the set $\{\nu(A_\beta(x)) \leq \lambda\}$ is closed for $\lambda > 0$. Hence we get the lemma in the same way as in the proof of Proposition 4 in Coifman, Meyer and Stein [2]. (The end of proof of Lemma 3).

Now we are ready to complete the proof of the Theorem. By Lemma 3 and the inequality (1) we see that if γ is sufficient large, then

$$2^{-1} \gamma^{-p} \| S_\alpha(T_R, \mu_u) \|_p^p \leq C_{12} \| u_b^* \|_p^p ,$$

where C_{12} is a positive constant depending only on α, β and g. Thus we are done letting $R \to \infty$ and recalling the fact that the L^p-norm of u_b^* and that of u_β^* are equivalent.

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