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POLARIZED SURFACES OF Δ-GENUS 3 AND DEGREE 5

MASANORI YOSHIOKA

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Abstract. In this paper we try to classify those polarized surfaces (M, L) of Δ-genus 3 and degree 5, for which the linear system $|L|$ has finite base locus and defines a non-birational rational map. Then a surface obtained by the blowing up at a point of M is a double cover of a desigularization of a quadric surface. Moreover, we divide these surfaces into four types according to the shape of the fiber containing the exceptional curve. Three of them are fiber spaces over the projective line and the other is an irrational ruled surface. Conversely, we show the existence of polarized surfaces in each of the four types.

1. Introduction. Fujita [3], [4] classified polarized surfaces of Δ-genus 1 or 2, but those of Δ-genus 3 are not well-understood up to now. Our aim is to classify the polarized surfaces (M, L) of Δ-genus 3 and degree 5 over the complex number field $\boldsymbol{C}$. In this paper, we only treat the case where the base locus of the linear system $|L|$ is finite and the rational map Φ_L is not birational.

In Fujita's theory of polarized varieties, regular rungs play an important role. We assume that $\Delta(M, L) \leq g(M, L)$ and that $\mathrm{Bs}|L|$ is finite. By Fujita's embedding theorem (Theorem 1 in §1 below), there exist nonsingular regular rungs for those surfaces which satisfy $L^2 \geq 2\Delta(M, L) - 1$. We are interested in the case $L^2 = 2\Delta(M, L) - 1$. The surfaces with $\Delta(M, L) = 1$ and $L^2 = 1$ or with $\Delta(M, L) = 2$ and $L^2 = 3$ are contained in the classification by Fujita. Hence we try to classify the next class of the surfaces which satisfy the above equality. For the surfaces with $\Delta(M, L) = 3$ and $L^2 = 5$, $\deg \Phi_L$ is one or two. In this paper we classify the case $\deg \Phi_L = 2$ in a method similar to that in [4].

In this case, the base locus of $|L|$ is a point p, and by the blowing up at p, we obtain a surface $\tilde{M}$ and a degree 2 morphism $\Phi_{\tilde{L}}: \tilde{M} \to \boldsymbol{P}^3$. Moreover, we lift it to a morphism f_0 to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ or Σ_2. We carry out the classification by dividing the surfaces to four classes by the type of a divisor $f_0^* \Gamma \subset \tilde{M}$ (cf. Theorem 2 in §2). We lift f_0 to a finite double covering to a surface obtained from either $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ or Σ_2 by the blowing up at one or two points (cf. Theorems 3, 4, 5, 6, 7, in §5, §6, §7, §7, §8). We describe the branch locus of the double covering. Conversely, we show the existence of polarized surfaces in each of the four types.

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valuable advice.

2. Fujita's theory of polarized varieties. In this section, we recall necessary results on polarized varieties which we need later. A polarized variety (M, L) is a pair of an n-dimensional complete algebraic variety M and an ample divisor L on M. By using the Hilbert polynomial $\chi(tL)$, we define integers $\chi_j(M, L)$ so that $\chi(tL)=\sum\chi_j(M, L)t^{[j]}/j!$, where $t^{[0]}:=1$ and $t^{[j]}:=t(t+1)(t+2)\cdots(t+j-1)$ for $j>0$. The sectional genus of (M, L) is defined as

$$g(M, L):=1-\chi_{n-1}(M, L).$$

If $n=1$, the sectional genus of (M, L) is equal to the genus of M. If $n=2$ and M is non-singular, the sectional genus of (M, L) is equal to the virtual genus of L. The Δ-genus of (M, L) is defined as

$$\Delta(M, L):=n+L^n-h^0(M, \mathcal{O}(L)).$$

If a divisor $R\in|L|$ is prime, R is called a rung of (M, L). If $r_R: H^0(M, \mathcal{O}(L))\to H^0(R, \mathcal{O}(L_{|R}))$ is surjective, $(\{|L|\}_{|R}=|L_{|R}|$, for short), then R is said to be regular. If R is regular, then we have $g(M, L)=g(R, L_{|R})$. A rung R is regular if and only if $\Delta(M, L)=\Delta(R, L_{|R})$.

LEMMA 1. *If (M, L) has a regular rung, then* $\mathrm{Bs}\,|L|=\mathrm{Bs}\,|L_{|R}|$.

PROOF. By definition, we have $\{|L|\}_{|R}=|L_{|R}|$. For all $R'\in|L_{|R}|$, there exists $L'\in|L|$ such that $L'\cap R=R'$. Hence we have $\mathrm{Bs}\,|L_{|R}|\supset\mathrm{Bs}\,|L|$. Consequently, we have $\mathrm{Bs}\,|L|=\mathrm{Bs}\,|L_{|R}|$.

As to the existence of a regular rung, Fujita gave the following theorem:

THEOREM 1 (Fujita [1], [5]). *Let (M, L) be a non-singular polarized variety. Assume that* $\mathrm{Bs}\,|L|$ *is finite and that $\Delta(M, L)\le g(M, L)$. Then we have*:
(1) *(M, L) has a nonsingular regular rung, if $L^n\ge 2\Delta(M, L)-1$.*
(2) $\mathrm{Bs}\,|L|=\varnothing$, *if $L^n\ge 2\Delta(M, L)$.*
(3) *$\Delta(M, L)=g(M, L)$ and L is very ample, if $L^n\ge 2\Delta(M, L)+1$.*

3. Liftings of Φ_L. In this paper from this section on we assume following two assumption.

ASSUMPTION 1. $\Delta(M, L)=3$, $L^2=5$, $g(M, L)\ge 3$ and $\mathrm{Bs}\,|L|\ne\varnothing$.

This assumption implies $h^0(M, \mathcal{O}(L))=4$. We now eliminate the base points in $\mathrm{Bs}\,|L|$ of the rational map $\Phi_L: M\to \boldsymbol{P}^3$ defined by the complete linear system $|L|$. Let p be a base point of $|L|$.

ASSUMPTION 2. Φ_L is not birational to its image.

PROPOSITION 1. $\mathrm{Bs}|L|$ *consists of the single point p. Moreover, any two general members of* $|L|$ *intersect each other transversely at p.*

PROOF. Since $\Delta(M, L)=3$, $L^2=5$ and $g(M, L)\geq 3$, the surface (M, L) has a nonsingular regular rung R by Theorem 1, (1). By Lemma 1, it is sufficient to show that $\mathrm{Bs}|L_{|R}|=\{p\}$. We set L_R be a divisor on the curve R which satisfies $|L_{|R}|=|L_R|+p$. We have $\deg(L_{|R})=\deg L_R+1$. Hence we obtain $\Delta(R, L_R)=\Delta(R, L_{|R})-1=2$, since $\Delta(M, L)=\Delta(R, L_{|R})$ and $\deg L_R=\deg(L_{|R})-1=L^2-1=4$. Consequently, we have $\deg L_R=2\Delta(R, L_R)$. Moreover, we have $g(R, L_R)\geq\Delta(R, L_R)$ by the Riemann-Roch theorem applied to the algebraic curve R. It follows that $\mathrm{Bs}|L_R|=\varnothing$ by Theorem 1, (2). Hence we have $\mathrm{Bs}|L_{|R}|=\{p\}$, and the coefficient for p of $L'_{|R}-p$ is equal to 0 for any general member L' of $|L|$. Therefore two general members of $|L|$ intersect each other at p with the local intersection number one. q.e.d.

Let $\pi: \tilde{M}\to M$ be the blowing up at p, and denote by E the exceptional curve over p. We denote by $\tilde{L}$ the proper transform of a general member of $|L|$. Two general members of $|L|$ intersect each other at p transversely by Proposition 1. Thus we have $\pi^*L=\tilde{L}+E$, and two general members of $|\tilde{L}|$ do not intersect each other on E. Hence $|\tilde{L}|$ has no base point because $|L|$ has only one base point p. Therefore the rational map $\Phi_{\tilde{L}}: \tilde{M}\to \boldsymbol{P}^3$ is a morphism such that $\Phi_{\tilde{L}}=\Phi_L\circ\pi$.

We set $W_0:=\Phi_{\tilde{L}}(\tilde{M})$. Since $\tilde{L}$ has no fixed components and $\tilde{L}^2=4>0$, we have $\dim W_0=2$. For a point $x\in W_0$, we denote by $\Lambda(x)$ the linear system of hyperplane sections on W_0. Then $\Phi_{\tilde{L}}^{-1}(x)$ is the base locus of $\Phi_{\tilde{L}}^*\Lambda(x)$. Since we assume that Φ_L is not birational, W_0 is a quadric surface in $\boldsymbol{P}^3$, and $\tilde{L}$ is the pullback of a hyperplane section of W_0. Then there are the following two cases.

Case (I) W_0 is a nonsingular quadric surface in $\boldsymbol{P}^3$.

Case (II) W_0 is a quadric cone in $\boldsymbol{P}^3$.

Now we study each of these cases.

Case (I) Let $W:=\boldsymbol{P}^1_1\times\boldsymbol{P}^1_2$, and denote $H=H_1+H_2$, where $H_1:=\{\mathrm{pt}\}\times\boldsymbol{P}^1_2$ and $H_2:=\boldsymbol{P}^1_1\times\{\mathrm{pt}\}$. There exists a morphism $f_0: \tilde{M}\to W$ such that $\Phi_{\tilde{L}}=\Phi_H\circ f_0$. Thus we have $\tilde{L}\sim f_0^*H$. By exchanging the indices if necessary, we may assume that $(f_0^*H_1)\cdot E=0$ and $(f_0^*H_2)\cdot E=1$ because $1=\tilde{L}\cdot E=(f_0^*H_1)\cdot E+(f_0^*H_2)\cdot E$, and $f_0^*H_i$ $(i=1, 2)$ are nef. Hence $\tilde{M}$ is a fiber space over $\boldsymbol{P}^1$ through $\Phi_{f_0^*H_1}: \tilde{M}\to W\to\boldsymbol{P}^1_1$. Moreover, $\Phi_{f_0^*H_1}(E)$ consists of a point because $f_0^*H_1\cdot E=0$. Thus E is a component of a fiber of $\Phi_{f_0^*H_1}$. $\Phi_{\tilde{L}}(E)$ is not a point since $\tilde{L}\cdot E>0$. Hence $f_0(E)$ is not a point. $\Gamma=f_0(E)$ is a fiber of $\Phi_{H_1}: W\to\boldsymbol{P}^1_1$. Consequently, we obtain $\Gamma\sim H_1$.

Case (II) We can lift $\Phi_{\tilde{L}}$ to a morphism to the Hirzebruch surface Σ_2 by the method in [7, p. 46] as follows. Let v be the singular point of W_0. We can choose a basis $\{\varphi_0, \varphi_1, \varphi_2, \varphi_3\}$ of $H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}))$ satisfying $\varphi_2^2=\varphi_0\cdot\varphi_1$. Let (φ_i) be the divisor defined by φ_i. Then we have $2(\varphi_2)=(\varphi_0)+(\varphi_1)$. Hence we have $0\sim(\varphi_2)-(\varphi_1)=(\varphi_0)-(\varphi_2)=\delta_0-\delta_1$, where δ_0 and δ_1 are mutually prime effective divisors. Let G be the common part of (φ_0) and (φ_1).

LEMMA 2. *In the above notation, we have*

$$(\varphi_0)=G+2\delta_0\,,\quad (\varphi_1)=G+2\delta_1\,,\quad (\varphi_2)=G+\delta_0+\delta_1\,.$$

PROOF. Let $(\varphi_0)=G+A_0$, and let $(\varphi_1)=G+A_1$. We have $A_0-A_1=(\varphi_0)-(\varphi_1)=2\delta_0-2\delta_1$. Hence $A_0+2\delta_1=A_1+2\delta_0$. It follows that $A_0=2\delta_0$ and $A_1=2\delta_1$, because A_0 and A_1 are mutually prime and so are δ_0 and δ_1. q.e.d.

Note that $\{\varphi_0, \varphi_1, \varphi_2\}$ is a basis of the linear subspace of $H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}))$ corresponding to $\Phi_{\tilde{L}}^* \Lambda(v)$, where $\Lambda(v)$ is the linear system consisting of the hyperplane sections of W_0 which contain v. Since $\mathrm{Bs}\,|\tilde{L}|$ is empty, so is $G\cap(\varphi_3)$. Hence we have $\tilde{L}\cdot G=0$. On the other hand, since δ_0 and δ_1 are linearly equivalent and mutually prime, we have $\delta_i\cdot E\geq 0$. Suppose that E is a component of G. Since $\tilde{L}$ is nef and $\tilde{L}\cdot E=1$, we have $\tilde{L}\cdot G>0$, a contradiction. Hence we have $G\cdot E\geq 0$. Consequently, we have $\delta_i\cdot E=0$ and $G\cdot E=1$, because $1=\tilde{L}\cdot E=2\delta_i\cdot E+G\cdot E$ and $\delta_i\cdot E$ and $G\cdot E$ are integers. In particular, G is not zero. G is irreducible and reduced, since $L\cdot G=(\tilde{L}+E)\cdot G=1$.

LEMMA 3. *The variable part* $|\tilde{L}-G|$ *of* $\Phi_{\tilde{L}}^*\Lambda(v)$ *has no base point.*

PROOF. Since $4=\tilde{L}^2=\tilde{L}\cdot(2\delta_i+G)=2\tilde{L}\cdot\delta_i$, we have $\tilde{L}\cdot\delta_i=2$. Hence we have $2\delta_i^2+G\cdot\delta_i=2$. Since $\delta_i^2=\delta_0\cdot\delta_1\geq 0$ and $\delta_i\cdot G\geq 0$, we have the following two cases: $\delta_i^2=1$, $\delta_i\cdot G=0$ and $\delta_i^2=0$, $\delta_i\cdot G=2$. However, the first case does not occur, since it implies $G=0$ by the Hodge index theorem. Hence we have $(\tilde{L}-G)^2=(2\delta_i)^2=0$, and then $|\tilde{L}-G|$ has no base point. q.e.d.

In Case (II), we denote by W the Hirzebruch surface Σ_2, by H_1 a fiber and H_2 the section with the self-intersection number -2, and we set $H=2H_1+H_2$. The morphism $\Phi_H\colon \Sigma_2\to W_0\subset \boldsymbol{P}^3$ is equal to the blowing up at $v\in W_0$. By Lemma 3, the inverse image of v by $\Phi_{\tilde{L}}$ is the effective divisor G. Thus there exists a morphism $f_0\colon \tilde{M}\to W$ such that $\Phi_{\tilde{L}}=\Phi_H\circ f_0$ by the universality of the blowing up. Then we get $f_0^*H_2=G$ and $\tilde{L}\sim f_0^*H$, and hence we have $f_0^*H_1\sim\delta_i$ by $\tilde{L}\sim 2\delta_i+G$. Thus $f_0^*H_1\cdot E=\delta_i\cdot E=0$ and $f_0^*H_2\cdot E=G\cdot E=1$. Hence it follows that $\tilde{M}$ is a fiber space over $\boldsymbol{P}^1$ through $\Phi_{f_0^*H_1}\colon \tilde{M}\to W\to\boldsymbol{P}^1$. Since $f_0^*H_1\cdot E=0$, the exceptional curve E is a component of a fiber. Since $\tilde{L}\cdot E=1$, $\Phi_{\tilde{L}}(E)$ is not a point. Hence $\Gamma=f_0(E)$ is a fiber of $\Phi_{H_1}\colon W\to\boldsymbol{P}_1^1$. Consequently $\Gamma\sim H_1$.

Note that we use common symbols in Cases (I) and (II) because of the similarity of the situation. We describe over result in this common notation when we do not need to distinguish the two cases.

4. Classification of the fiber containing E**.** The pull-back of Γ by the morphism $f_0\colon \tilde{M}\to W$ defined in Section 3 can be written as $f_0^*\Gamma=:\varepsilon E+E^*+D$, where ε is the multiplicity of E in $f_0^*\Gamma$, E^* is the sum of those other components which are not contracted by f_0, while D is the sum of the components which are contracted by f_0.

The following proposition is a special case of [4, Lemma 1.5].

PROPOSITION 2. *Let x be a smooth point of W_0 such that $X=\Phi_{\tilde{L}}^{-1}(x)$ is of positive dimension. Then X is irreducible and reduced with $E\cdot X=1$. Moreover, X is a component of D.*

PROOF. Let $\tilde{L}_x$ be a general member of $\Phi_{\tilde{L}}^*\Lambda(x)$. We can write $\tilde{L}_x=X+C$, where C is a divisor such that $|C|$ has no fixed component. Since $\pi^*(tL)-E$ is ample for sufficiently large t, it follows that $t\tilde{L}\cdot X'+(t-1)E\cdot X'=(\pi^*(tL)-E)\cdot X'>0$ for any divisor X' which is contracted by $\Phi_{\tilde{L}}$. Hence we have $E\cdot X'>0$ because $\tilde{L}\cdot X'=0$. Moreover, we have $X'\cdot E=1$ and $C\cdot E=0$ by $C\cdot E\geq 0$. If $X=X'+X''$, then $X\cdot E>1$, because $X'\cdot E>1$ and $X''\cdot E>1$, a contradiction. Consequently, X is irreducible and reduced. On the other hand, we have $0=f_0^*\Gamma\cdot X=\varepsilon+E^*\cdot X+D\cdot X$. Since $E^*\cdot X\geq 0$, we have $D\cdot X<0$. Consequently, X is a component of D. q.e.d.

By the above proposition, we can write $D=\sum X_i$, where X_i is an irreducible reduced curve. Let $\{\tilde{x}_i\}:=f_0(X_i)\in W$ and $\{x_i\}:=\Phi_{\tilde{L}}(X_i)\in W_0$. Let $\tilde{\Lambda}(\tilde{x}_i)$ be the linear system which consists of those divisors of the linear system $|(1+e)H_1+H_2|$ on W which contain $\tilde{x}_i$, where $e=0$ in Case (I) and $e=1$ in Case (II).

LEMMA 4. *Let C' and C'' be general members of the linear system $|\tilde{L}-X_i|$ which is the variable part of $\Phi_{\tilde{L}}^*\Lambda(x_i)=f_0^*\tilde{\Lambda}(\tilde{x}_i)$.*
(i) *If $(\tilde{L}-X_i)^2=2$, then C' and C'' do not intersect each other on $f_0^*\Gamma$. In particular, $\mathrm{Bs}\,|\tilde{L}-X_i|=\varnothing$.*
(ii) *If $(\tilde{L}-X_i)^2=3$, then C' and C'' intersect each other at one point on $f_0^*\Gamma$ transversely. Moreover, a base point of $|\tilde{L}-X_i|$, if it exists, is necessarily the point of intersection.*

PROOF. Case (I) Let S' and S'' be general members of $\Lambda(x_i)$. By $(\Phi_H^*S')\cdot(\Phi_H^*S'')=2$, Φ_H^*S' and Φ_H^*S'' intersect each other transversely at $\tilde{x}_i$ and another point $\tilde{x}_i'$. Moreover, $\tilde{x}_i'$ is not on Γ because $\Gamma\cdot(\Phi_H^*S')=\Gamma\cdot(\Phi_H^*S'')=1$. Let Γ' be the fiber which contains $\tilde{x}_i'$. Since the curves which are contracted by f_0 are components of $f_0^*\Gamma$ by Proposition 2, the morphism $f_0|_{\tilde{M}\setminus f_0^{-1}\Gamma}:\tilde{M}\setminus f_0^{-1}\Gamma\to W\setminus\Gamma$ is a finite double covering. Hence C' and C'' which are variable parts of $\Phi_{\tilde{L}}^*S'$ and $\Phi_{\tilde{L}}^*S''$, respectively, intersect each other transversely at two points on $f_0^{-1}\Gamma'$. Therefore if $(\tilde{L}-X_i)^2=2$, then C' and C'' do not intersect each other on $f_0^*\Gamma$. If $(\tilde{L}-X_i)^2=3$, then C' and C'' intersect each other at one point on $f_0^*\Gamma$ transversely. Since $\tilde{x}_i'$ moves according to the choice of S' and S'', the inverse image of $\tilde{x}_i'$ is not a base point of $f_0^*\tilde{\Lambda}(\tilde{x}_i)$.

Case (II) Since X_i is a component of $f_0^*\Gamma$, we have $X_i\neq G=f_0^*H_2$. Thus we have $\tilde{x}_i\notin H_2$. Let S' and S'' be general elements of $\Lambda(x_i)$. Since $(\Phi_H^*S')\cdot H_2=(\Phi_H^*S'')\cdot H_2=0$, we have $(\Phi_H^*S')\cap H_2=(\Phi_H S'')\cap H_2=\varnothing$. Thus H_2 is not a component of (Φ_H^*S') nor (Φ_H^*S''). The rest of the proof is similar to that in Case (I). q.e.d.

Now we classify the fiber containing E.

THEOREM 2. *In both Cases* (I) *and* (II), *we have the following four types, where* E^* *is an irreducible reduced curve in* (c) *and* (d):

(a) $f_0^*\Gamma = 2E + X_1 + X_2$.

(b) $f_0^*\Gamma = 2E + 2X_1$.

(c) $f_0^*\Gamma = E + E^* + X_1$, $\tilde{L}\cdot E^* = 1$, $E\cdot E^* = 0$.

(d) $f_0^*\Gamma = E + E^*$, $\tilde{L}\cdot E^* = 1$, $E\cdot E^* = 1$.

PROOF. Case (I) Since $2 = \tilde{L}\cdot f_0^*H_1 = \varepsilon + \tilde{L}\cdot E^*$, there are two cases (i) $\varepsilon = 2$, $\tilde{L}\cdot E^* = 0$ and (ii) $\varepsilon = 1$, $\tilde{L}\cdot E^* = 1$. We first treat the former case (i). If $E^* \neq 0$, then $\Phi_{\tilde{L}}(E^*)$ is a point by $\tilde{L}\cdot E^* = 0$, a contradiction. Hence $E^* = 0$. Since $0 = f_0^*H_1\cdot E = -2 + D\cdot E$, we have $D\cdot E = 2$. By Proposition 2, $D\cdot E$ is equal to the number of irreducible components of D. Therefore we have (a) $D = X_1 + X_2$ or (b) $D = 2X_1$. Let us treat the latter case (ii). If $E^* = E_1^* + E_2^*$, then we have $\tilde{L}\cdot E_1^* > 0$ and $\tilde{L}\cdot E_2^* > 0$, because any component of E^* is not contracted by f_0, a contradiction to $\tilde{L}\cdot E^* = 1$. Thus E^* is irreducible and reduced. On the other hand, we have $D\cdot E = 1 - E^*\cdot E$ by $0 = f_0^*H_1\cdot E = -1 + E^*\cdot E + D\cdot E$. Consequently, there are two cases (c) $E\cdot E^* = 0$, $D = X_1$ and (d) $E\cdot E^* = 1$, $D = 0$ by Proposition 2.

The proof in Case (II) is similar. q.e.d.

5. Classification of Type (a). From this section on, we use the same notation for a divisor and its total transform, when it does not cause confusion. In this section, we assume that $f_0^*\Gamma$ is of Type (a). Then the morphism $f_0: \tilde{M} \to W$ is not finite. Hence we lift it to a finite morphism. We first study the inverse image of $\tilde{x}_i$ by f_0.

LEMMA 5. *The variable part* $|\tilde{L} - X_i|$ *of* $f_0^*\tilde{\Lambda}(\tilde{x}_i)$ *has no base point.*

PROOF. The curves X_1 and X_2 are contracted to distinct points by f_0. Thus we have $X_1 \cap X_2 = \varnothing$, and we get $X_1\cdot X_2 = 0$. Therefore we have $0 = f_0^*\Gamma\cdot X_i = (2E + X_1 + X_2)\cdot X_i = 2 + X_i^2$, and hence $X_i^2 = -2$. Thus we get $(\tilde{L} - X_i)^2 = 2$. Then the linear system $|\tilde{L} - X_i|$ has no base point by Lemma 4. q.e.d.

Let $\sigma: \tilde{W} \to W$ be the blowing up at $\tilde{x}_1 = f_0(X_1)$ and $\tilde{x}_2 = f_0(X_2)$, and let Z_1 and Z_2 be the exceptional curves over $\tilde{x}_1$ and $\tilde{x}_2$, respectively. We denote by $\tilde{\Gamma}$ the proper transform of Γ. The inverse image of $\tilde{x}_i$ by f_0 is X_i. Hence by the universality of the blowing up, there exists a morphism $f: \tilde{M} \to \tilde{W}$ such that $f_0 = \sigma \circ f$ and $f^*Z_i = X_i$. Then f is a finite double covering. Since $\mathrm{Pic}(\tilde{W}) = \boldsymbol{Z}H_1 \oplus \boldsymbol{Z}H_2 \oplus \boldsymbol{Z}Z_1 \oplus \boldsymbol{Z}Z_2$, the branch locus is linearly equivalent to $2\alpha H_1 + 2\beta H_2 - 2\gamma_1 Z_1 - 2\gamma_2 Z_2$ for a unique quadruple $(\alpha, \beta, \gamma_1, \gamma_2)$ of integers.

THEOREM 3. *Let* $\sigma: \tilde{W} \to W$ *be the blowing up at the two points* $\tilde{x}_1 = f_0(X_1)$ *and* $\tilde{x}_2 = f_0(X_2)$ *with the exceptional curves* Z_1 *and* Z_2 *over* $\tilde{x}_1$ *and* $\tilde{x}_2$, *respectively. Then* $\tilde{M}$ *is a finite double covering of* $\tilde{W}$. *The branch locus* B *is linearly equivalent to* $2\alpha H_1 + 2\beta H_2 - 2\gamma_1 Z_1 - 2\gamma_2 Z_2$. *The integers* α, β, γ_1, γ_2, *satisfy the following conditions*:

Case (I) $\alpha\geq\gamma_i$, $\alpha+\beta\geq 4$, $\gamma_i\geq 1$, $\beta=\gamma_1+\gamma_2-1$.

Case (II) $\alpha>2\beta$, $\alpha\geq 4$, $\gamma_i\geq 1$, $\beta=\gamma_1+\gamma_2-1$.

Conversely, for each quadruple $(\alpha, \beta, \gamma_1, \gamma_2)$ *satisfying these conditions, there exists a polarized surface* (M, L) *giving rise to the quadruple.*

Case (I) Let $F=\alpha H_1+\beta H_2-\gamma_1 Z_1-\gamma_2 Z_2$. Then clearly $B\in|2F|$. Since $f^*\tilde{\Gamma}=f^*\Gamma-f^*Z_1-f^*Z_2=2E$, the curve $\tilde{\Gamma}$ is a component of B. Then we have $B=B'+\tilde{\Gamma}$, where B' is a nonsingular curve. Since the branch locus B is nonsingular, we have $B'\cap\tilde{\Gamma}=\varnothing$. Hence we have $B'\cdot\tilde{\Gamma}=(2F-\tilde{\Gamma})\cdot\tilde{\Gamma}=2(\beta-\gamma_1-\gamma_2+1)=0$, and obtain $\beta=\gamma_1+\gamma_2-1$. Let $\tilde{H}_2$ be the unique member of $|H_2-Z_1|$. If this is a component of B', then $B'\cdot\tilde{H}_2=\tilde{H}_2^2=-1$, since B' is nonsingular. Otherwise, $B'\cdot\tilde{H}_2\geq 0$, so in either case we have $-1\leq B'\cdot\tilde{H}_2=\{(2\alpha-1)\tilde{H}_1+(2\gamma_1-1)(H_2-Z_1)+(2\gamma_2-1)(H_2-Z_2)\}\cdot(H_2-Z_1)=2(\alpha-\gamma_1)$, hence $\alpha-\gamma_1\geq 0$. Similarly we have $\alpha\geq\gamma_2$. Since $Z_i\cdot\tilde{\Gamma}=1$, we have $H_2\cap Z_i\neq\varnothing$. Hence Z_i is not a component of B'. Thus we have $B'\cdot Z_i=2\gamma_i-1\geq 0$. Consequently, we obtain $\gamma_i\geq 1$ since γ_i is an integer. By the assumption $g(M, L)\geq 3$, we have $\alpha+\beta\geq 4$, since $g(M, L)=\alpha+\beta-1$ by the virtual genus formula.

Case (II) Similarly we have $\beta=\gamma_1+\gamma_2-1$ and $\gamma_i\geq 1$. By $H_1\cdot H_2=1$, H_2 is not a component of B', and so we have $B'\cdot H_2=2\alpha-4\beta-1\geq 0$. By the assumption $g(M, L)\geq 3$, we have $\alpha\geq 4$, since $g(M, L)=\alpha-1$ by the virtual genus formula.

In the rest of this section, we prove the existence part of this theorem. We take two distinct points $\tilde{x}_1$, $\tilde{x}_2$ on a fiber Γ of W. Let $\sigma: \tilde{W}\to W$ be the blowing up at these points with the exceptional curves Z_1 and Z_2 over $\tilde{x}_1$ and $\tilde{x}_2$, respectively. We denote by $\tilde{\Gamma}$ the proper transform of Γ.

PROPOSITION 3. *Let* α, β, γ_1, γ_2, *be the integers satisfying the conditions, and let* $F:=\alpha H_1+\beta H_2-\gamma_1 Z_1-\gamma_2 Z_2$. *The linear system* $|2F|$ *has a nonsingular member.*

PROOF. Case (I) The divisor $2F-\tilde{\Gamma}$ is linearly equivalent to $(2\alpha-1)H_1+(2\gamma_1-1)(H_2-Z_1)+(2\gamma_2-1)(H_2-Z_2)$. Since $\mathrm{Bs}\,|H_1+H_2-Z_i|=\varnothing$ and $\alpha\geq\gamma_i$, we have $\mathrm{Bs}\,|(2\alpha-1)H_1+(2\gamma_i-1)(H_2-Z_i)|=\varnothing$. Thus we get $\mathrm{Bs}\,|2F-\tilde{\Gamma}|\in(H_2-Z_i)$. It follows that $\mathrm{Bs}\,|2F-\tilde{\Gamma}|=\varnothing$ by $(H_2-Z_1)\cap(H_2-Z_2)=\varnothing$. Then the general member B' of $|2F-\tilde{\Gamma}|$ is nonsingular. Moreover, since $B'\cdot\tilde{\Gamma}=(2F-\tilde{\Gamma})\cdot\tilde{\Gamma}=0$, we get $B'\cap\tilde{\Gamma}=\varnothing$. Thus the divisor $B=B'+\tilde{\Gamma}$ is a nonsingular member of $|2F|$.

Case (II) The divisor $2F-\tilde{\Gamma}$ is linearly equivalent to $(2\alpha-4\beta-1)H_1+(2\gamma_1-1)(2H_1+H_2-Z_1)+(2\gamma_2-1)(2H_1+H_2-Z_2)$. $|2F-\tilde{\Gamma}|$ has no base point, since $2\alpha-4\beta-1\geq 0$ and $\mathrm{Bs}\,|H_1+H_2-Z_i|=\varnothing$. For a general member B' of $|2F-\tilde{\Gamma}|$, the divisor $B=B'+\tilde{\Gamma}$ is a nonsingular member of $|2F|$. q.e.d.

Hence there is a finite double covering $f: \tilde{M}\to\tilde{W}$ branched along B, where B is a nonsingular member of $|2F|$. Since $\tilde{\Gamma}$ is a component of the branch locus, we set $f^*\tilde{\Gamma}:=2E$. By $\tilde{\Gamma}^2=-2$, we get $E^2=-1$. Hence we obtain a surface M and a morphism $\pi: \tilde{M}\to M$ by the blowing down of $E\subset\tilde{M}$.

We now show the existence of an ample divisor L.

LEMMA 6. *Let M_1 be a nonsingular algebraic surface, and let L_1 be a divisor on M_1. Denote by $\pi: \tilde{M}_1 \to M_1$ the blowing up of M_1 at a point p_1 with the exceptional curve $E_1 \subset \tilde{M}_1$. Suppose that L_1 satisfies $\pi^*L_1 = \tilde{L}_1 + E_1$ and $\mathrm{Bs}|\tilde{L}_1| = \emptyset$. If any exceptional curve $\tilde{C}_1$ on $\tilde{M}_1$ for $\Phi_{\tilde{L}_1}$ satisfies $\tilde{C}_1 \cdot E_1 > 0$, then L_1 is ample.*

PROOF. Let $\tilde{C}_1$ be any curve on $\tilde{M}_1$ other than E_1. If $\tilde{C}_1$ is not contracted by $\Phi_{\tilde{L}_1}$, then $\tilde{L}_1 \cdot \tilde{C}_1 > 0$. On the other hand, we have $\tilde{C}_1 \cdot E_1 \geq 0$. If $\tilde{C}_1$ is contracted by $\Phi_{\tilde{L}_1}$, we have $\tilde{L}_1 \cdot \tilde{C}_1 = 0$. Let C_1 be any curve on M_1. We can write $\pi^*C_1 = \tilde{C}_1 + \mu E_1$ with $\tilde{C}_1 \neq E_1$. Hence it follows that $L_1 \cdot C_1 = \pi^*L_1 \cdot (\tilde{C}_1 + \mu E_1) = \pi^*L_1 \cdot \tilde{C}_1 = \tilde{L}_1 \cdot \tilde{C}_1 + \tilde{C}_1 \cdot E_1 > 0$. On the other hand, we have $(\tilde{L}_1 + E_1)^2 = (\tilde{L}_1 + E_1) \cdot \tilde{L}_1 = \tilde{L}_1^2 + 1 > 0$. Consequently, the divisor L_1 is ample by the Nakai criterion. q.e.d.

PROPOSITION 4. *Let $\tilde{L} := f^*H$ and $L := \pi_*\tilde{L}$. Then the divisor L is ample.*

PROOF. Case (I) From $2\tilde{L} \cdot E = (f^*H_1 + f^*H_2) \cdot f^*\tilde{\Gamma} = 1$, we have $\tilde{L} \cdot E = 1$. Hence $\pi^*L = \tilde{L} + E$. By the construction of $\tilde{M}$, the curves which are contracted by $\Phi_{\tilde{L}}$ are components of f^*Z_i. The curves Z_i and $\tilde{\Gamma}$ intersect each other transversely by $Z_i \cdot \tilde{\Gamma} = 1$. On the other hand, B' intersects Z_i transversely because B' is a general member of $|2F - \tilde{\Gamma}|$. Thus f^*Z_i is irreducible. Moreover, by $\tilde{\Gamma} \cdot Z_i = 1$, we have $(f^*Z_i) \cdot E = 1$. Consequently, L is ample by Lemma 6.

Case (II) By construction, the curves which are contracted by $\Phi_{\tilde{L}}$ are components of f^*H_2 or f^*Z_i. The rest of the proof is similar to that in Case (I). q.e.d.

We calculate the invariants. The canonical divisor $K_{\tilde{M}}$ of $\tilde{M}$ is linearly equivalent to $(\alpha-2)f^*H_1 + (\beta-2)f^*H_2 - (\gamma_1 - 1)X_1 - (\gamma_2 - 1)X_2$. Hence we have

$$K_M^2 = K_{\tilde{M}}^2 + 1 = 4(\alpha-2)(\beta-2) - 2(\gamma_1 - 1)^2 - 2(\gamma_2 - 1)^2 + 1$$

in Case (I), and

$$K_M^2 = K_{\tilde{M}}^2 + 1 = 4(\alpha-4)(\beta-2) - 4(\beta-2)^2 - 2(\gamma_1 - 1)^2 - 2(\gamma_2 - 1)^2 + 1$$

in Case (II).

LEMMA 7. *We have*

$$H^i(\tilde{W}, \mathcal{O}(-F)) = 0 \qquad \textit{for} \quad i < 2 .$$

PROOF. By Kawamata's vanishing theorem, it is sufficient to show that $F - (1/2)\tilde{\Gamma}$ is nef and big. The divisor $B' \sim 2F - \tilde{\Gamma}$ is nef since $|B'|$ has no base locus. Hence $F - (1/2)\tilde{\Gamma}$ is nef. Moreover, it is big because $(F - (1/2)\tilde{\Gamma})^2 = (\beta-2)^2 + 2\gamma_1\gamma_2 - 3/2 > 0$. q.e.d.

Since the morphism f is a finite double covering branched along $B \sim 2F$, we have

$$H^i(\tilde{M}, \mathcal{O}_{\tilde{M}}) = H^i(\tilde{W}, \mathcal{O}_{\tilde{W}}) \oplus H^i(\tilde{W}, \mathcal{O}_{\tilde{W}}(-F)) .$$

Since $\tilde{W}$ is a rational surface, we have by Lemma 7

$$p_g(M)=h^2(\tilde{W}, \mathcal{O}_{\tilde{W}}(-F)), \qquad q(M)=0.$$

By Lemma 7 and the Riemann-Roch theorem for algebraic surfaces, we have

$$p_g(M)=\frac{1}{2}(F^2+F\cdot K_{\tilde{W}})+1.$$

Hence we have

$$p_g(M)=(\alpha-2)(\beta-2)-\frac{1}{2}(\gamma_1-1)^2-\frac{1}{2}(\gamma_2-1)^2+\alpha+\frac{1}{2}\beta-\frac{5}{2}$$

in Case (I), and

$$p_g(M)=(\alpha-4)(\beta-2)+(\beta-2)^2-\frac{1}{2}(\gamma_1-1)^2-\frac{1}{2}(\gamma_2-1)^2+\alpha+\frac{1}{2}\beta-\frac{5}{2}$$

in Case (II).

6. Classification of Type (b). In this section, we treat Type (b) namely the case $f_0^*\Gamma=2E+2X_1$. We lift the morphism $f_0: \tilde{M}\to W$ to a finite covering. We first study the inverse image of $\tilde{x}_1$ by f_0.

LEMMA 8. *The variable part* $|\tilde{L}-X_1|$ *of* $f_0^*\tilde{\Lambda}(\tilde{x}_1)$ *has a base point, which is on* X_1. *Moreover, two general members of* $|\tilde{L}-X_1|$ *intersect each other transversely at this point.*

PROOF. Since $0=f_0^*H_1\cdot X_1=(2E+2X_1)\cdot X_1$, we have $X_1^2=-1$. Thus $(\tilde{L}-X_1)^2=3$. Let C_0, C_1 and C_2 be general members of $|\tilde{L}-X_1|$. By Lemma 4, C_i and C_j intersect each other transversely at a point on $f_0^{-1}\Gamma$. Since $C_i\cdot E=(\tilde{L}-X_1)\cdot E=0$, we have $C_i\cap E=\varnothing$. Thus the point of intersection of C_i and C_j on $f_0^{-1}\Gamma$ is on X_1. If the point of intersection of C_0 and C_1 is different from that of C_0 and C_2, then we have $C_0\cdot X_1\geq 2$, a contradiction to $C_0\cdot X_1=(\tilde{L}-X_1)\cdot X_1=1$. q.e.d.

Thus the inverse image of $\tilde{x}_1$ by f_0 has an isolated part. Let y be the base point of $|\tilde{L}-X_1|$. Denote by $\rho: \hat{M}\to\tilde{M}$ the blowing up at y with the exceptional curve Y over y. Let $\tilde{X}_1$ be the proper transform of X_1. By Lemma 8, the fixed part of $\rho^*\circ f_0^*\tilde{\Lambda}(\tilde{x}_1)$ is $\rho^*X_1+Y=\tilde{X}_1+2Y$, and the variable part is $|\rho^*\tilde{L}-\tilde{X}_1-2Y|$. Moreover, $|\rho^*\tilde{L}-\tilde{X}_1-2Y|$ has no base point. Thus the inverse image of $\tilde{x}_1$ by $f_0\circ\rho$ is the divisor $\tilde{X}_1+2Y$. Let $\sigma_1: \tilde{W}\to W$ be the blowing up at $\tilde{x}_1$, and let Z_1 be the exceptional curve over $\tilde{x}_1$ and $\tilde{\Gamma}$ the proper transform of Γ, respectively. By the universality of the blowing up, there exists a morphism $f_1: \hat{M}\to\tilde{W}$ satisfying $f_0\circ\rho=\sigma_1\circ f_1$ and $f_1^*Z_1=\tilde{X}_1+2Y$.

PROPOSITION 5. *The image of* $\tilde{X}_1$ *by* f_1 *is equal to the intersection* z *of* Z_1 *and* $\tilde{\Gamma}$, *and the morphism* $f_{1|Y}: Y\to Z_1$ *is an isomorphism.*

PROOF. Let q be a point on $Z_1 \simeq \boldsymbol{P}^1$. Since $\deg(f_{1|Y})^* q = \deg(f_{1|Y})^*(-Z_1)_{|Z_1} = -f_1^* Z_1 \cdot Y = 1$, we see that $f_{1|Y}: Y \to Z_1 \simeq \boldsymbol{P}^1$ is an isomorphism. On the other hand, since $\deg(f_{1|\tilde{X}_1})^* q = \deg(f_{1|\tilde{X}_1})^*(-Z_1)_{|Z_1} = -f_1^* Z_1 \cdot \tilde{X}_1 = 0$, the image of $\tilde{X}_1$ by $f_{1|\tilde{X}_1}: \tilde{X}_1 \to Z_1 \simeq \boldsymbol{P}^1$ is a point. Moreover, since $f_1^* \tilde{\Gamma} = f_1^* \Gamma - f_1^* Z_1 = 2E + \tilde{X}_1$, we have $f_1(\tilde{X}_1) \in \tilde{\Gamma}$. Hence $f_1(X_1) \in \tilde{\Gamma} \cap Z_1$. q.e.d.

Consequently, the morphism f_1 is not finite. Hence we carry out the same operation again. Let $\sigma_2: \hat{W} \to \tilde{W}$ be the blowing up at z, and let Z_2 be the exceptional curve over z. We denote by $\hat{\Gamma}$ and $\tilde{Z}_1$ the proper transforms of $\tilde{\Gamma}$ and Z_1, respectively. By Proposition 5, the inverse image of z is $\tilde{X}_1$. Thus by the universality of the blowing up, there exists a morphism $f: \hat{M} \to \hat{W}$ such that $f_1 = \sigma_2 \circ f$ and $f^* Z_2 = \tilde{X}_1$. Then f is a finite double covering.

Let $B \in \hat{W}$ be the branch locus of f. There exists a divisor $F = \alpha H_1 + \beta H_2 - \gamma_1 Z_1 - \gamma_2 Z_2$ such that $B \in |2F|$.

THEOREM 4. *Let $\rho: \hat{M} \to \tilde{M}$ be the blowing up at the base point y of the linear system $|\tilde{L} - X_1|$. Let $\rho: \tilde{W} \to W$ be the blowing up at $\tilde{x}_1 = f_0(X_1)$, and let Z_1 be the exceptional curve over $\tilde{x}_1$, and $\tilde{\Gamma}$ the proper transform of Γ, respectively. The intersection of $\tilde{\Gamma}$ and Z_1 is a point z. Let $\sigma_2: \hat{W} \to \tilde{W}$ be the blowing up at z, and let Z_2 be the exceptional curve over z. Then $\hat{M}$ is a finite double covering over $\hat{W}$. The branch locus $B \sim 2\alpha H_1 + 2\beta H_2 - 2\gamma_1 Z_1 - 2\gamma_2 Z_2$ satisfies the following conditions, respectively*:

Case (I) $2\alpha > \beta$, $\alpha + \beta \geq 4$, $\beta = 2\gamma_1 = 2\gamma_2 - 2 \geq 0$.

Case (II) $\alpha > 2\beta$, $\alpha \geq 4$, $\beta = 2\gamma_1 = 2\gamma_2 - 2 \geq 0$.

Conversely, for a quadruple $(\alpha, \beta, \gamma_1, \gamma_2)$ satisfying these conditions, there exists a polarized surface (M, L) giving rise to the quadruple.

Case (I) Since $f^* \hat{\Gamma} = f^* \Gamma - f^* Z_1 - f^* Z_2 = 2E$ and $f^* \tilde{Z}_1 = f^* Z_1 - f^* Z_2 = 2Y$, it follows that $\hat{\Gamma}$ and $\tilde{Z}_1$ are components of the branch locus. We can write $B = B' + \hat{\Gamma} + \tilde{Z}_1$, where B' is a nonsingular curve. Since the branch locus B is nonsingular, we have $B' \cap \hat{\Gamma} = \varnothing$ and $B' \cap \tilde{Z}_1 = \varnothing$. Hence we have $0 = B' \cdot \hat{\Gamma} = \{(2\alpha - 1)H_1 + 2\beta H_2 - 2\gamma_1 Z_1 - (2\gamma_2 - 2)Z_2\} \cdot \hat{\Gamma} = 2(\beta - \gamma_1 - \gamma_2 + 1)$ and $0 = B' \cdot \tilde{Z}_1 = \{(2\alpha - 1)H_1 + 2\beta H_2 - 2\gamma_1 Z_1 - (2\gamma_2 - 2)Z_2\} \cdot \tilde{Z}_1 = 2(\gamma_1 - \gamma_2 + 1)$. Thus we have $\beta = \gamma_1 + \gamma_2 - 1$ and $\gamma_1 = \gamma_2 - 1$. On the other hand, by $\tilde{Z}_1 \cdot Z_2 = 1$, Z_2 is not a component of B'. Hence we have $\leq B' \cdot Z_2 = \{(2\alpha - 1)H_1 + 2\beta H_2 - 2\gamma_1 Z_1 - (2\gamma_2 - 2)Z_2\} \cdot Z_2 = 2\gamma_2 - 2 \geq 0$. Thus $\gamma_2 \geq 1$ and $\gamma_1 \geq 0$. Similarly, we have $\alpha > \gamma_1$, since $\tilde{Z}_1 \cdot (H_2 - Z_1) = 1$ and since α and γ_1 are integers. We have $g(M, L) = \alpha + \beta - 1$. Hence $\alpha + \beta \geq 4$ by the assumption $g(M, L) \geq 3$.

Case (II) Similarly we have $\beta = \gamma_1 + \gamma_2 - 1$, $\gamma_1 = \gamma_2 - 1$ and $\gamma_2 \geq 1$. Since $H_2 \cdot \tilde{\Gamma} = 1$, and since H_2 is not a component of B', we have $0 \leq H_2 \cdot B' = 2\alpha - 4\beta - 1$. Since α and β are integers, we have $\alpha > 2\beta$. By the assumption $g(M, L) \geq 3$, we have $\alpha \geq 4$.

In the rest of this section, we prove the existence part of this theorem. We take a point of W and we obtain $\hat{W}$ by the blowing up of W at this point and an infinitely near point. We first show the existence of the branch locus B on $\hat{W}$.

PROPOSITION 6. *Let* $F=\alpha H_1+\beta H_2-\gamma_1 Z_1-\gamma_2 Z_2$. *Then the linear system* $|2F|$ *has a nonsingular member.*

PROOF. Case (I) The divisor $2F-\hat{\Gamma}-\tilde{Z}_1$ is linearly equivalent to $(2\alpha-\beta-1)H_1+\beta(H_1+2H_2-Z_1-Z_2)$. By $2\alpha>\beta$, it suffices to show that $|H_1+2H_2-Z_1-Z_2|$ has no base locus.

Let us first show that $|H_1+2H_2-Z_1-Z_2|$ has no fixed component. We take a divisor $H_1+2\tilde{H}_2+\tilde{Z}_1\in|H_1+2H_2-Z_1-Z_2|$, where $\tilde{H}_2\in|H_2-Z_1|$. It suffices to show that $\tilde{H}_2$ and $\tilde{Z}_1$ are not fixed components. By the exact sequences

$$0\to\mathcal{O}_{\tilde{W}}(H_1+2H_2-Z_1)\to\mathcal{O}_{\tilde{W}}(H_1+2H_2)\to\mathcal{O}_{Z_1}(H_1+2H_2)\to 0$$

and

$$0\to\mathcal{O}_{\hat{W}}(H_1+2H_2-Z_1-Z_2)\to\mathcal{O}_{\hat{W}}(H_1+2H_2-Z_1)\to\mathcal{O}_{Z_2}(H_1+2H_2-Z_1)\to 0\,,$$

we have $h^0(H_1+2H_2-Z_1-Z_2)\geq h^0(H_1+2H_2)-2=4$. On the other hand, by the exact sequence

$$0\to\mathcal{O}_{\hat{W}}(\tilde{H}_2+H_1)\to\mathcal{O}_{\hat{W}}(2\tilde{H}_2+H_1)\to\mathcal{O}_{\tilde{H}_2}(2\tilde{H}_2+H_1)\to 0$$

and $H^1(H_1+H_2-Z_1)=H^1(K_{\tilde{W}}+3H_1+3H_2-2Z_1)=0$, we have $h^0((H_1+2H_2-Z_1-Z_2)-\tilde{Z}_1)=3$. Hence $\tilde{Z}_1$ is not a fixed component. Similarly $\tilde{H}_2$ is not a fixed component.

If there exists a base point of the system $|H_1+2H_2-Z_1-Z_2|$, it is either on $\tilde{H}_2$ or on $\tilde{Z}_1$. However, the general members do not intersect $\tilde{H}_2$ and $\tilde{Z}_1$, since $(H_1+2H_2-Z_1-Z_2)\cdot(H_2-Z_1)$ and $(H_1+2H_2-Z_1-Z_2)\cdot\tilde{Z}_1$ are easily calculated to be zero. Hence the linear system $|H_1+2H_2-Z_1-Z_2|$ has no base point. Consequently, $B=B'+\hat{\Gamma}+\tilde{Z}_1$ is a nonsingular member of $|2F|$ for a general member B' of $|H_1+2H_2-Z_1-Z_2|$.

Case (II) The divisor $2F-\hat{\Gamma}-\tilde{Z}_1$ is linearly equivalent to $(2\alpha-4\beta-1)H_1+\beta(4H_1+2H_2-Z_1-Z_2)$. Since $\alpha>2\beta$ and $\hat{\Gamma}+H_2+(3H_1+H_2)\in|4H_1+2H_2-Z_1-Z_2|$, we can show that $B=B'+\hat{\Gamma}+\tilde{Z}_1$ is a nonsingular element of $|2F|$, as in Case (I).

q.e.d.

By this proposition, there exists a finite double covering $f\colon\hat{M}\to\hat{W}$ branched along B. Moreover, we can set $f^*\hat{\Gamma}=2E$ and $f^*\tilde{Z}_1=2Y$. Since $\hat{\Gamma}^2=-2$ and $\tilde{Z}_1^2=-2$, we have $E^2=-1$ and $Y^2=-1$. By $\hat{\Gamma}\cdot\tilde{Z}_1=0$, we have $E\cdot Y=0$. We set $f^*Z_2=\tilde{X}_1$. By blowing down Y and E successively, we obtain a surface M and morphisms $\rho\colon\hat{M}\to\tilde{M}$ and $\pi\colon\tilde{M}\to M$. Set $\rho_*\tilde{X}_1=X_1$.

We now show the existence of an ample divisor L on M. Since $(\sigma_1\circ\sigma_2\circ f)(Y)=\{\tilde{x}_1\}$, there exists a morphism $f_0\colon\tilde{M}\to W$ such that $f_0\circ\rho=\sigma_1\circ\sigma_2\circ f$.

PROPOSITION 7. *We choose a divisor* $\tilde{L}\sim f_0^*H$ *and set* $L\sim\pi_*\tilde{L}$. *Then the divisor* L *is ample.*

PROOF. Case (I) In a way similar to that in the case of Type (a), we have $\pi^*L=\tilde{L}+E$. By construction, the curve contracted by $\Phi_{\tilde{L}}$ is a component of X_1. Since $2\tilde{X}_1\cdot Y=f^*Z_2\cdot f^*\tilde{Z}_1=2$, we have $\tilde{X}_1\cdot Y=1$. Hence $\rho^*X_1=\tilde{X}_1+Y$. On the other hand, since B' is a general member of $|(2\alpha-1)H_1+2\beta H_2-\beta Z_1-\beta Z_2|$ and $Z_2\cdot\hat{\Gamma}=Z_2\cdot\tilde{Z}_1=1$, we obtain that Z_2 intersects B', $\hat{\Gamma}$ and $\tilde{Z}_1$ transversely, and that $\tilde{X}_1$ is irreducible and reduced. Hence X_1 is irreducible and reduced. Since $2\tilde{X}_1\cdot E=f^*Z_2\cdot f^*\hat{\Gamma}=2Z_2\cdot(H_1-Z_1-Z_2)=2$ and $4Y\cdot E=f^*\tilde{Z}_1\cdot f^*\hat{\Gamma}=2(Z_1-Z_2)\cdot(H_1-Z_1-Z_2)=0$, we have $X_1\cdot E=(\tilde{X}_1+Y)\cdot E>0$. Thus by Lemma 6, the divisor L is ample.

Case (II) By construction, the curves contracted by $\Phi_{\tilde{L}}$ are components of f^*H_2 or f^*Z_1. The rest of the proof is similar to that in Case (I). q.e.d.

We can calculate the invariants.

Case (I) $K_M^2=4(\alpha-2)(\beta-2)-4(\beta-2)^2-\beta^2+6\beta-8$,

$p_g(M)=(\alpha-2)(\beta-2)+\alpha-\dfrac{1}{4}\beta^2+2\beta-7$,

$q(M)=0$.

Case (II) $K_M^2=4(\alpha-4)(\beta-2)-4(\beta-2)^2-\beta^2+6\beta-8$,

$p_g(M)=(\alpha-4)(\beta-2)-(\beta-2)^2+\alpha-\dfrac{1}{4}\beta^2+2\beta-7$,

$q(M)=0$.

7. Classification of Type (c). In this section, we treat Type (c), namely the case $f_0^*\Gamma=E+E^*+X_1$. We divide surfaces of this type to two subtypes. We have $X_1\cdot E^*+C_1\cdot E^*=1$ for a general element X_1+C_1 of $f_0^*\tilde{\Lambda}(\tilde{x}_1)$, because $\tilde{L}\cdot E^*=1$. Since X_1 is not a component of E^*, we have $X_1\cdot E^*\geq 0$. On the other hand, since $|\tilde{L}-X_1|$ has no fixed component, we have $C_1\cdot E^*\geq 0$. Consequently, there are the following two subtypes.

(c-1) $X_1\cdot E^*=1,\quad C_1\cdot E^*=0$.

(c-2) $X_1\cdot E^*=0,\quad C_1\cdot E^*=1$.

7.1. Classification of Subtype (c-1). In this subsection, we treat Subtype (c-1) with $X_1\cdot E^*=1$, $C_1\cdot E^*=0$. The morphism $f_0\colon \tilde{M}\to W$ is not a finite morphism. We lift it to a finite double covering. Let $\rho\colon \tilde{W}\to W$ be the blowing up at $\tilde{x}_1$, and let Z_1 be the exceptional curve over $\tilde{x}_1$ and $\tilde{\Gamma}$ the proper transform of Γ, respectively. There exists a double covering $f\colon \tilde{M}\to\tilde{W}$ such that $f_0=\rho\circ f$ and $f^*Z_1=X_1$ by the universality of the blowing up. Moreover, f is a finite double covering.

THEOREM 5. *Let $\rho\colon \tilde{W}\to W$ be the blowing up at $\tilde{x}_1=f_0(X_1)$, and let Z_1 be the exceptional curve over $\tilde{x}_1$. Then $\tilde{M}$ is a finite double covering of $\tilde{W}$ branched along $B\sim 2\alpha H_1+2\beta H_2-2\beta Z_1$, where the coefficients satisfy the following conditions, respectively*:

Case (I) $\alpha\geq\beta,\quad \beta\geq 1,\quad \alpha+\beta\geq 4$.

Case (II) $\alpha \geq 2\beta$, $\beta \geq 1$, $\alpha \geq 4$.
Conversely, for any pair (α, β) *satisfying the conditions, there exists a polarized surface* (M, L) *giving rise to the pair.*

Case (I) Let B be the branch locus and let $F = \alpha H_1 + \beta H_2 - \gamma_1 Z_1$ be the divisor with $B \in |2F|$. Since $f^*\tilde{\Gamma} = E + E^*$ and $E \cdot E^* = 0$, f is not branched along $\tilde{\Gamma}$. Hence we have $B \cdot \tilde{\Gamma} = 2\beta - 2\gamma_1 = 0$. $\tilde{H}_2^2 = -1$ for the unique member $\tilde{H}_2$ of $|H_2 - Z_2|$. Hence $\tilde{H}_2$ is not a component of B and $0 \leq B \cdot (H_2 - Z_1) = (2\alpha H_1 + 2\beta H_2 - 2\beta Z_1) \cdot (H_2 - Z_1) = 2\alpha - 2\beta$. Since $\tilde{\Gamma} \cdot Z_1 = 1$, Z_1 is not a component of B. Thus we have $B \cdot Z_1 \geq 0$. If $B \cdot Z_1 = 0$, then we have $B \cap Z_1 = \varnothing$. Thus f is not branched along Z_1. Consequently, f^*Z_1 is a union of two curves, a contradiction to the fact that $X_1 = f^*Z_1$ is irreducible and reduced. Hence $B \cdot Z_1 \geq 0$, and so we have $B \cdot Z_1 = 2\gamma_1 > 0$. Moreover, by the assumption $g(M, L) \geq 3$, we have $\alpha + \beta \geq 4$.

Case (II) Similarly we have $\gamma_1 > 0$ and $\beta = \gamma_1$. Since $\tilde{\Gamma} \cdot H_2 = 1$, H_2 is not a component of B. Then we have $0 \leq H_2 \cdot B = 2\alpha - 4\beta$. It follows that $\alpha \geq 4$ by the assumption $g(M, L) \geq 3$.

In the rest of this section, we prove the existence part of this theorem. We obtain $\tilde{W}$ by the blowing up of W at an arbitrary point in Case (I) and an arbitrary point not on H_2 in Case (II). We first show the existence of the branch locus $B \in |2F|$.

Proposition 8. *Let* $F := \alpha H_1 + \beta H_2 - \beta Z_1$. *Then the linear system* $|2F|$ *has a nonsingular member.*

Proof. Case (I) We have $2F \sim (2\alpha - 2\beta)H_1 + 2\beta(H_1 + H_2 - Z_1)$. Since $\alpha \geq \beta$ and $\mathrm{Bs}|H_1 + H_2 - Z_1| = \varnothing$, $|2F|$ has no base point. Hence the general member B of $|2F|$ is nonsingular.

Case (II) We have $2F \sim (2\alpha - 4\beta)H_1 + 2\beta(2H_1 + H_2 - Z_1)$. Thus $|2F|$ has no base point, because $\alpha \geq 2\beta$ and $\mathrm{Bs}|2H_1 + H_2 - Z_1| = \varnothing$. q.e.d.

Hence there is a finite double covering $f: \tilde{M} \to \tilde{W}$ branched along B. Since $B \cdot \tilde{\Gamma} = 0$, we have $B \cap \tilde{\Gamma} = \varnothing$. Hence $f^*\tilde{\Gamma} = E + E^*$ and $E \cdot E^* = 0$. Moreover, we have $E^2 = E^{*2} = -1$ by $(f^*\tilde{\Gamma})^2 = -2$. Consequently, we obtain a surface M and a morphism $\pi: \tilde{M} \to M$ by blowing down $E \subset \tilde{M}$.

We can show the existence of an ample divisor L on M similarly as in Section 5. Moreover, we have

Case (I) $K_M^2 = 4(\alpha - 2)(\beta - 2) - 2(\beta - 1)^2 + 1$,

$$p_g(M) = (\alpha - 2)(\beta - 2) - \frac{1}{2}(\beta - 1)^2 + \alpha + \beta - \frac{5}{2},$$

$q(M) = 0$.

Case (II) $K_M^2 = 4(\alpha - 4)(\beta - 2) - 4(\beta - 2)^2 - 2(\beta - 1)^2 + 1$,

$$p_g(M) = (\alpha - 4)(\beta - 2) - (\beta - 2)^2 - \frac{1}{2}(\beta - 1)^2 + \alpha - \frac{5}{2},$$

$q(M)=0.$

7.2. Classification of Subtype (c-2). In this subsection, we treat Subtype (c-2), namely the case $X_1 \cdot E^*=0$, $C_1 \cdot E^*=1$. We now lift $f_0: \tilde{M} \to W$ to a finite double covering $f: \hat{M} \to \tilde{W}$.

LEMMA 9. *The variable part $|\tilde{L}-X_1|$ of the linear system $f_0^* \tilde{\Lambda}(\tilde{x}_1)$ has one base paint on E^*. Moreover, two general members of $|\tilde{L}-X_1|$ intersect each other at this point transversely.*

PROOF. Since $0=f_0^* H_1 \cdot X_1 = E \cdot X_1 + E^* \cdot X_1 + X_1^2$, we have $X_1^2=-1$, and so $(\tilde{L}-X_1)^2=3$. By Lemma 4, general members C' and C'' of $|\tilde{L}-X_1|$ intersect each other transversely at only one point on $f_0^{-1}\Gamma$. Because $f_{0|E^*}: E^* \to \Gamma$ is an isomorphism, we consider $y \in E^*$ corresponding to $\tilde{x}_1$. Obviously any member of $f_0^* \tilde{\Lambda}(\tilde{x}_1)$ contains y. Thus $y \in \mathrm{Supp}(X_1 + C')$. Hence we have $y \in C'$, since $X_1 \cap E^* = \emptyset$. Therefore $y \in \mathrm{Bs}\,|C'| = \mathrm{Bs}\,|\tilde{L}-X_1|$. q.e.d.

Let $\rho: \hat{M} \to \tilde{M}$ be the blowing up at the base point y of $|\tilde{L}-X_1|$, and let Y be the exceptional curve over y. We denote by $\tilde{E}^*$ the proper transform of E^*. Let $\sigma: \tilde{W} \to W$ be the blowing up at $\tilde{x}_1$, and let Z_1 be the exceptional curve over $\tilde{x}_1$. We denote by $\tilde{\Gamma}$ the proper transform of Γ. By an argument similar to that for Type (b), we get a morphism $f: \hat{M} \to \tilde{W}$ such that $f_0 \circ \rho = \sigma \circ f$ and $f^* Z_1 = X_1 + Y$. Let q be a point of $Z_1 \simeq \boldsymbol{P}^1$. We have $\deg(f_{|X_1})^* q = \deg(f_{|X_1})^*(-Z_1)_{|Z_1} = -f^* Z_1 \cdot X_1 = 1$. Thus $f_{|X_1}: X_1 \to Z_1$ is surjective. Similarly, $f_{|Y}: Y \to Z_1$ is surjective. Thus f is a finite double covering.

We study the branch locus $B \in |2F|$, where $F = \alpha H_1 + \beta H_2 - \gamma_1 Z_1$. Since $f^* \tilde{\Gamma} = E + E^*$ and $E \cdot E^* = 0$, f is not branched along $\tilde{\Gamma}$, and we have $B \cdot \tilde{\Gamma} = 2\beta - 2\gamma_1 = 0$. Moreover, since $f^* Z_1 = X_1 + Y$ and $X_1 \cdot Y = 0$, f is not branched along Z_1, either. Hence we have $B \cdot Z_1 = 2\gamma_1 = 0$. By the assumption $g(M, L) \geq 3$, we get $\alpha \geq 4$.

THEOREM 6. *Let $\rho: \hat{M} \to \tilde{M}$ be the blowing up at the base point y of the linear system $|\tilde{L}-X_1|$, which is the variable part of $f_0^* \tilde{\Lambda}(\tilde{x}_1)$, and let $\rho: \tilde{W} \to W$ be the blowing up at $\tilde{x}_1 = f(X_1)$. Then $\hat{M}$ is a finite double covering of $\tilde{W}$ branched along $B \sim \alpha H_1$ with an integer $\alpha \geq 4$. Conversely, for each integer $\alpha \geq 4$, there exist a polarized surface giving rise to the α.*

The existence of such a surface is proved in a way similar to that in the previous sections. We can also calculate the invariants:

Case (I) $K_M^2 = -4(\alpha-2)$.

Case (II) $K_M^2 = -4(\alpha-4)$.

General fibers of $\Phi_{H_1}: W \to \boldsymbol{P}^1$ do not intersect the branch locus $B \sim 2\alpha H_1$. Thus the general fiber of $\Phi_{f^* H_1}: \hat{M} \to \boldsymbol{P}^1$ consists of two components, which are isomorphic to $\boldsymbol{P}^1$. We consider the Stein factorization

$$\Phi_{f^* H_1}: \hat{M} \xrightarrow{g} C \xrightarrow{h} \boldsymbol{P}^1 .$$

Since any fiber of $g: \hat{M} \to C$ is equal to a connected component of the fiber of $\Phi_{f^*H_1}$, the general fiber of g is $\boldsymbol{P}^1$. Hence $\tilde{M}$ is a ruled surface, and we get

$$p_g(M)=0 .$$

Since the branch locus B is linearly equivalent to $2\alpha H_1$, $h: C \to \boldsymbol{P}^1$ is a finite double covering branched at 2α distinct points. Then by the Hurwitz formula, we have $g(C)=\alpha-1$, and

$$q(M)=\alpha-1 .$$

8. Classification of Type (d). In this section, we treat Type (d), namely the case $f_0^*\Gamma=E+E^*$. The morphism $f_0: \tilde{M} \to W$ is a finite double covering. Hence we study the branch locus $B \in |2F|$, where $F=\alpha H_1+\beta H_2$. Since $f_0^*\Gamma=E+E^*$ and $E \cdot E^*=1$, B and Γ intersect each other at $f_0(E \cap E^*)$ with multiplicity 2. Thus we have $2=B \cdot \Gamma=2\beta$, and so we have $\beta=1$. By the assumption $g(M, L) \geq 3$, we have $\alpha \geq 3+e$, where $e=0$ in Case (I) while $e=1$ in Case (II). Thus we have the following theorem:

THEOREM 7. *The surface $\tilde{M}$ is a finite double covering of W branched along $B \sim 2\alpha H_1+2H_2$ where $\alpha \geq 3+e$. Conversely, for α and e satisfying $\alpha \geq 3+e$, there exist such surfaces giving rise to α and e.*

The existence of such a surface is checked as follows: A general member B of $|2F|$ is non-singular and irreducible. By $B \cdot H_1=2$, we see that $(\Phi_{H_1})_{|B}: B \to \boldsymbol{P}^1$ is a finite double covering. Since B is irreducible, $(\Phi_{H_1})_{|B}$ has branch points. Let Γ be a fiber of Φ_{H_1} which contains one of the branch points. There exists a finite double covering $f: \tilde{M} \to W$ such that $f^*\Gamma=E+E^*$ and $E \cdot E^*=1$. Moreover, we have $E^2=-1$. Thus we obtain a surface M by blowing down $E \subset \tilde{M}$. It is easy to see that there exists an ample divisor L of M satisfying the conditions. Moreover, we have

Case (I) $K_M^2=-4\alpha+9$, $p_g(M)=0$, $q(M)=0$.

Case (II) $K_M^2=-4\alpha+13$, $p_g(M)=0$, $q(M)=0$.

The general fiber of $f_0: \tilde{M} \to W \to \boldsymbol{P}^1$ is a double covering of $\boldsymbol{P}^1$ branched at two points, thus is $\boldsymbol{P}^1$. Therefore $\tilde{M}$ is a rational surface.

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MATHEMATICAL INSTITUTE
FACULTY OF SCIENCE
TOHOKU UNIVERSITY
SENDAI, 980
JAPAN