

KOSTANT'S FORMULA FOR A CERTAIN CLASS OF GENERALIZED KAC-MOODY ALGEBRAS

SATOSHI NAITO

(Received November 7, 1991, revised March 10, 1992)

Abstract. Let $\mathfrak{g}(A)$ be a symmetrizable generalized Kac-Moody algebra with \mathfrak{h} its Cartan subalgebra and \mathfrak{n}_- the sum of all its negative root spaces. In this paper, we prove the generalization of Kostant's homology formula under a certain condition on the matrix A . This formula completely determines the \mathfrak{h} -module structure of the homology of \mathfrak{n}_- in the irreducible highest weight $\mathfrak{g}(A)$ -module $L(\lambda)$ with an arbitrary dominant integral weight λ .

Introduction. Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a real $n \times n$ matrix satisfying the following conditions:

- (C1) either $a_{ii} = 2$ or $a_{ii} \leq 0$;
- (C2) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbf{Z}$ if $a_{ii} = 2$;
- (C3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Let $\mathfrak{g}(A)$ be the generalized Kac-Moody algebra (GKM algebra), over the complex number field \mathbf{C} , associated to the above matrix A . When $a_{ii} = 2$ for all i , A is nothing but a generalized Cartan matrix and $\mathfrak{g}(A)$ a Kac-Moody algebra. GKM algebras were first introduced and studied by Borchers [2]. The present author studied them as *regular subalgebras* or *folding subalgebras* of a Kac-Moody algebra (cf. [11], [12]). Here, we present some homological feature of the class of GKM algebras.

We have the root space decomposition: $\mathfrak{g}(A) = \mathfrak{h} \oplus \sum_{\alpha \in \Delta}^{\oplus} \mathfrak{g}_{\alpha}$, where \mathfrak{h} is the Cartan subalgebra of $\mathfrak{g}(A)$, Δ the root system of $(\mathfrak{g}(A), \mathfrak{h})$, and \mathfrak{g}_{α} the root space attached to $\alpha \in \Delta$. Let Δ_+ be the set of all positive roots, $\Pi = \{\alpha_i\}_{i=1}^n \subset \mathfrak{h}^*$ all simple roots, and $\Pi^{\vee} = \{\alpha_i^{\vee}\}_{i=1}^n \subset \mathfrak{h}$ all simple coroots. Put $\mathfrak{n}_{\pm} := \sum_{\alpha \in \Delta_{\pm}} \mathfrak{g}_{\pm\alpha}$. Then, \mathfrak{n}_{\pm} are both subalgebras of $\mathfrak{g}(A)$, and $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

For i with $a_{ii} = 2$, we define the fundamental reflection r_i on the dual space \mathfrak{h}^* of \mathfrak{h} . Let W be the Weyl group generated by r_i 's with $a_{ii} = 2$. Now, for $\lambda \in \mathfrak{h}^*$, we denote by $L(\lambda)$ the irreducible highest weight $\mathfrak{g}(A)$ -module with highest weight λ , and by $C(\lambda)$ the irreducible (one-dimensional) \mathfrak{h} -module with weight λ . Then, for each $\lambda \in \mathfrak{h}^*$ and $j \in \mathbf{Z}_{\geq 0}$, $H_j(\mathfrak{n}_-, L(\lambda))$ (the j -th homology of \mathfrak{n}_- with coefficients in $L(\lambda)$) and $H_c^j(\mathfrak{n}_+, L(\lambda))$ (the j -th cohomology of \mathfrak{n}_+ with coefficients in $L(\lambda)$) become \mathfrak{h} -modules in the standard way. Here, we remark that our cohomology $H_c^j(\mathfrak{n}_+, L(\lambda))$ of \mathfrak{n}_+ is slightly different from the usual Lie algebra cohomology $H^j(\mathfrak{n}_+, L(\lambda))$, whereas the homology $H_j(\mathfrak{n}_-, L(\lambda))$ of \mathfrak{n}_- is the usual Lie algebra homology (see Section 2 for the

definition of $H_c^j(\mathfrak{n}_+, L(A))$). In this paper, we determine the structure of $H_j(\mathfrak{n}_-, L(A))$ and $H_c^j(\mathfrak{n}_+, L(A))$ as \mathfrak{h} -modules in the case where A is a dominant integral weight and A is a symmetrizable real matrix satisfying (C2), (C3), and the following ($\hat{C}1$):

$$(\hat{C}1) \quad \text{either } a_{ii} = 2 \text{ or } a_{ii} = 0.$$

Actually, we have the following theorem.

THEOREM. *Let $\Lambda \in \mathfrak{h}^*$ be dominant integral. Let \mathcal{S} be the set of all sums of distinct pairwise perpendicular elements from $\Pi^{\text{im}} := \{\alpha_i \in \Pi \mid a_{ii} \leq 0\}$ and $\mathcal{S}(A)$ the set of all elements of \mathcal{S} perpendicular to A . Then, as \mathfrak{h} -modules ($j \geq 0$),*

$$H_c^j(\mathfrak{n}_+, L(A)) \cong H_j(\mathfrak{n}_-, L(A)) \cong \sum_{\beta \in \mathcal{S}(A)}^{\oplus} \sum_{w \in W, \ell(w) = j - \text{ht}(\beta)}^{\oplus} C(w(\Lambda + \rho - \beta) - \rho).$$

Here, ρ is a fixed element of \mathfrak{h}^* such that $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$ ($1 \leq i \leq n$), $\ell(w)$ ($w \in W$) is the length of w , and, for $\beta \in \mathcal{S}$, $\text{ht}(\beta) = m$ if β is a sum of m elements from Π^{im} .

When $a_{ii} = 2$ for every i , the set $\mathcal{S}(A)$ consists of only one element $0 \in \mathfrak{h}^*$, and so the above theorem is ‘‘Kostant’s homology and cohomology formula’’ for symmetrizable Kac-Moody algebras, which was proved by Garland and Lepowsky in [3] and [7]. Note that when $\mathfrak{g}(A)$ is a finite-dimensional complex semi-simple Lie algebra, the above formula is nothing but Kostant’s classical result in [6].

We prove our theorem by imitating the method of Liu in [8] for Kac-Moody algebras, which is essentially similar to those in Aribaud [1] and Garland-Lepowsky [3].

This paper is organized as follows. In Section 1, we will review the theory of GKM algebras, rewriting some parts of [4] for Kac-Moody algebras. In Section 2, we recall the notion of homology and cohomology of Lie algebras \mathfrak{n}_\pm with coefficients in $L(A)$. In Section 3, we explain briefly some results of Liu [8] for Kac-Moody algebras, which is also true for GKM algebras with no modifications. In Section 4, we will establish our main result stated above in the principle of Aribaud and Liu, using the celebrated Weyl-Kac-Borcherds character formula. In Section 5, as an application of our main theorem, we give a simple proof of a presentation by generators and relations of GKM algebras, following the way of Mathieu [9].

I would like to express my heartfelt thanks to Professor Takeshi Hirai for his constant encouragement. I am also grateful to Professor Olivier Mathieu for the valuable discussions which made me aware of the importance of Kostant’s formula. I would also like to thank Professor Seok-Jin Kang for sending me some preprints.

1. Preliminaries for generalized Kac-Moody algebras. In this section, we fix notation and recall fundamental results about generalized Kac-Moody algebras which will be needed in the succeeding sections. For detailed accounts, see [2] and [4].

1.1 Generalized Kac-Moody algebras. Let $n \in \mathbb{Z}_{>0}$, and $A = (a_{ij})_{1 \leq i, j \leq n}$ be a real

$n \times n$ matrix satisfying the conditions (C1), (C2), and (C3) in the Introduction. Such a matrix is called a GGCM. A realization of a GGCM A is a triple $(\mathfrak{h}, \Pi = \{\alpha_i\}_{i=1}^n, \Pi^\vee = \{\alpha_i^\vee\}_{i=1}^n)$, where \mathfrak{h} is a complex vector space, satisfying the following:

- (R1) Π is a linearly independent subset of $\mathfrak{h}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$, and Π^\vee is a linearly independent subset of \mathfrak{h} ;
- (R2) $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ ($1 \leq i, j \leq n$), where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between \mathfrak{h} and \mathfrak{h}^* ;
- (R3) $\dim_{\mathbb{C}} \mathfrak{h} = 2n - \text{rank } A$.

We denote by $\tilde{\mathfrak{g}}(A)$ the Lie algebra generated by the above vector space \mathfrak{h} and $2n$ symbols e_i, f_i ($1 \leq i \leq n$) under the following relations:

- (F1) $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$ ($1 \leq i, j \leq n$),
- (F2) $[h, h'] = 0$ ($h, h' \in \mathfrak{h}$),
- (F3) $\begin{cases} [h, e_i] = \langle \alpha_i, h \rangle e_i \\ [h, f_i] = -\langle \alpha_i, h \rangle f_i \end{cases}$ ($1 \leq i \leq n, h \in \mathfrak{h}$).

Let $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}$, where \mathfrak{r} is the largest proper ideal of $\tilde{\mathfrak{g}}(A)$ intersecting \mathfrak{h} trivially. We call this Lie algebra $\mathfrak{g}(A)$ the *generalized Kac-Moody algebra* (GKM algebra for short) associated to the GGCM A . The subalgebra \mathfrak{h} of $\mathfrak{g}(A)$ is called the Cartan subalgebra. We call $\Pi = \{\alpha_i\}_{i=1}^n$ the simple root system and $\Pi^\vee = \{\alpha_i^\vee\}_{i=1}^n$ the simple coroot system. The elements e_i, f_i ($1 \leq i \leq n$) are called the Chevalley generators.

We have the root space decomposition of $\mathfrak{g}(A)$ with respect to \mathfrak{h} :

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \sum_{\alpha \in \mathfrak{h}^* \setminus \{0\}}^{\oplus} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha := \{x \in \mathfrak{g}(A) \mid [h, x] = \langle \alpha, h \rangle x \text{ for all } h \in \mathfrak{h}\}$ ($\alpha \in \mathfrak{h}^*$). Note that $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i, \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ ($1 \leq i \leq n$), and $\text{mult } \alpha := \dim_{\mathbb{C}} \mathfrak{g}_\alpha$ is finite ($\alpha \in \mathfrak{h}^*$). An element $\alpha \in \mathfrak{h}^* \setminus \{0\}$ is called a root if $\mathfrak{g}_\alpha \neq \{0\}$, and \mathfrak{g}_α is called the root space attached to α . A root α is said to be positive (resp. negative) if $\alpha \in Q_+ := \sum_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i$ (resp. $-\alpha \in Q_+$). We denote by Δ the root system of $(\mathfrak{g}(A), \mathfrak{h})$ and Δ_+ (resp. Δ_-) the set of all positive (resp. negative) roots. Then, we have $\Delta = \Delta_- \cup \Delta_+$ (disjoint union), $\Delta_- = -\Delta_+$. Therefore, we have the triangular decomposition: $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, with $\mathfrak{n}_\pm := \sum_{\alpha \in \Delta_\pm} \mathfrak{g}_{\pm\alpha}$.

Now, let Π^{re} (resp. Π^{im}) be the subset $\{\alpha_i \in \Pi \mid a_{ii} = 2 \text{ (resp. } a_{ii} \leq 0)\}$ of Π , and W ($\subset GL(\mathfrak{h}^*)$) the Weyl group generated by the fundamental reflections r_i defined by $\alpha_i \in \Pi^{\text{re}}: r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ ($\lambda \in \mathfrak{h}^*$). And let $C^\vee := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ if } a_{ii} = 2\}$ be the fundamental chamber, where $\mathfrak{h}_{\mathbb{R}}$ is a realization of A over the real number field \mathbb{R} such that $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$. Then, we have the following as in the case of Kac-Moody algebras.

PROPOSITION 1.1 (cf. [4]). (a) For $\lambda \in C^\vee$, the group $W_\lambda := \{w \in W \mid w(\lambda) = \lambda\}$ is generated by r_i 's such that $r_i \in W_\lambda$.

(b) Let $X := \bigcup_{w \in W} w(C^\vee)$ be the Tits cone. Then, C^\vee is a fundamental domain for the action of W on X .

From now on, we take and fix an element $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$

($1 \leq i \leq n$). For $w \in W$, define $\Phi_w := \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) \in \Delta_-\}$, and denote by $\ell(w)$ the smallest number m such that $w = r_{i_1} r_{i_2} \cdots r_{i_m}$ ($\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_m} \in \Pi^{re}$). Then, we have the following as in the case of Kac-Moody algebras.

PROPOSITION 1.2 (cf. [4]). (a) *The number of elements in Φ_w is equal to $\ell(w)$ for $w \in W$.*

(b) $\rho - w(\rho) = \sum_{\alpha \in \Phi_w} \alpha$, for $w \in W$.

1.2. Symmetrizable GKM algebras. A GGCM $A = (a_{ij})_{1 \leq i, j \leq n}$ is said to be *symmetrizable* if there exist an invertible diagonal matrix $D = \text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and a symmetric matrix $B = (b_{ij})_{1 \leq i, j \leq n}$ such that $A = DB$. In this case, we may assume that $\varepsilon_i > 0$ ($1 \leq i \leq n$) and that $b_{ij} \in \mathbf{R}$ ($1 \leq i, j \leq n$). In this subsection, we assume that the GGCM A is symmetrizable, and take (and fix) the above decomposition of A . Let $(\mathfrak{h}, \Pi = \{\alpha_i\}_{i=1}^n, \Pi^\vee = \{\alpha_i^\vee\}_{i=1}^n)$ be a realization of A , and fix a subspace \mathfrak{h}'' complementary to $\mathfrak{h}' := \sum_{i=1}^n \mathbf{C}\alpha_i^\vee$ in \mathfrak{h} . Define a symmetric bilinear form $(\cdot \mid \cdot)$ on \mathfrak{h} by

(B1) $(\alpha_i^\vee \mid h) = \langle \alpha_i, h \rangle \varepsilon_i \quad (h \in \mathfrak{h}, 1 \leq i \leq n),$
 (B2) $(h' \mid h'') = 0 \quad (h', h'' \in \mathfrak{h}'')$

Then, the bilinear form $(\cdot \mid \cdot)$ is non-degenerate on \mathfrak{h} , so we have a linear isomorphism $v: \mathfrak{h} \rightarrow \mathfrak{h}^*$ defined by $\langle v(h), h_1 \rangle = (h \mid h_1)$ ($h, h_1 \in \mathfrak{h}$), and the induced bilinear form on \mathfrak{h}^* , which is denoted by the same symbol $(\cdot \mid \cdot)$. Note that this induced bilinear form $(\cdot \mid \cdot)$ on \mathfrak{h}^* is W -invariant (cf. [4]).

As is well-known, we can extend this bilinear form $(\cdot \mid \cdot)$ on \mathfrak{h} to a non-degenerate symmetric invariant bilinear form on the GKM algebra $\mathfrak{g}(A)$. This bilinear form $(\cdot \mid \cdot)$ on $\mathfrak{g}(A)$ (or \mathfrak{h}^*) is called a *standard invariant bilinear form*.

1.3. Highest weight modules over generalized Kac-Moody algebras. Let $\mathfrak{g}(A)$ be the GKM algebra associated to a GGCM A , and \mathfrak{h} the Cartan subalgebra of $\mathfrak{g}(A)$. We say an \mathfrak{h} -module V to be \mathfrak{h} -diagonalizable if V admits a weight space decomposition $V = \sum_{\lambda \in \mathfrak{h}^*} V_\lambda$ where $V_\lambda := \{v \in V \mid h(v) = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}\}$ ($\lambda \in \mathfrak{h}^*$). In this case, let $\mathcal{P}(V) := \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq \{0\}\}$ denote the set of all weights of V .

For $\lambda \in \mathfrak{h}^*$, we set $D(\lambda) := \{\lambda - \beta \mid \beta \in Q_+ = \sum_{i=1}^n \mathbf{Z}_{\geq 0} \alpha_i\}$. Now, let \mathcal{O} be the category of all \mathfrak{h} -diagonalizable modules V with finite-dimensional weight spaces, such that there exist a finite number of elements $\lambda_1, \lambda_2, \dots, \lambda_s \in \mathfrak{h}^*$ satisfying $\mathcal{P}(V) \subset \bigcup_{i=1}^s D(\lambda_i)$. Note that any submodule and quotient module of a module from the category \mathcal{O} are also in \mathcal{O} , and that a direct sum and a tensor product of a finite number of modules from \mathcal{O} are again in \mathcal{O} .

For each $\lambda \in \mathfrak{h}^*$, there exists a unique irreducible highest weight module $L(\lambda)$ with highest weight λ , which is defined as a unique irreducible quotient of the Verma module $M(\lambda) := U(\mathfrak{g}(A))/J(\lambda)$. Here, $U(\mathfrak{g}(A))$ is the universal enveloping algebra of $\mathfrak{g}(A)$, and $J(\lambda)$ is the left ideal of $U(\mathfrak{g}(A))$ generated by $\mathfrak{n}_+ \cup \{h - \langle \lambda, h \rangle 1 \mid h \in \mathfrak{h}\}$.

An \mathfrak{h} -diagonalizable $\mathfrak{g}(A)$ -module V is said to be *integrable* if the Chevalley generators e_i and f_i are locally nilpotent on V when $a_{ii} = 2$. Note that, for any integrable

module V over the GKM algebra $\mathfrak{g}(A)$, $\dim_{\mathbb{C}} V_{w(\mu)} = \dim_{\mathbb{C}} V_{\mu}$ ($\mu \in \mathfrak{h}^*$, $w \in W$), and that $L(\lambda)$ is integrable if and only if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for every i with $a_{ii} = 2$, as in the case of Kac-Moody algebras (cf. [4]). Now, we set $P_+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ if } a_{ii} = 2, \text{ and } \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i\}$. Then, we have the following.

PROPOSITION 1.3 (cf. [4]). *Let V be an integrable $\mathfrak{g}(A)$ -module. Assume that there exists a $\Lambda \in P_+$ such that $\mathcal{P}(V) \subset D(\Lambda)$. Then, any weight $\lambda \in \mathcal{P}(V)$ is W -equivalent to a weight $\mu \in \mathcal{P}(V) \cap P_+$.*

Further, when the GGCM A is symmetrizable, we have the following.

PROPOSITION 1.4 ([4]). *Let $\mathfrak{g}(A)$ be the GKM algebra associated to a symmetrizable GGCM A , $(\cdot \mid \cdot)$ a standard invariant bilinear form on $\mathfrak{g}(A)$. Let $\Lambda \in P_+$, and $\lambda, \mu \in \mathcal{P}(L(\Lambda))$. Then, $(\Lambda \mid \Lambda) - (\lambda \mid \mu) \geq 0$ and the equality holds if and only if $\lambda = \mu \in W \cdot \Lambda$.*

1.4. Character formulas. Let $\mathfrak{g}(A)$ be a GKM algebra. In this subsection, we introduce the formal character of modules from the category \mathcal{O} . First, we define a certain algebra \mathcal{E} over \mathbb{C} . The elements of \mathcal{E} are series of the form $\sum_{\lambda \in \mathfrak{h}^*} c_{\lambda} e(\lambda)$, where $c_{\lambda} \in \mathbb{C}$ and $c_{\lambda} = 0$ for λ outside the union of a finite number of sets of the form $D(\mu)$ ($\mu \in \mathfrak{h}^*$). Then \mathcal{E} becomes a commutative associative algebra if we define its multiplication by $e(\lambda) \cdot e(\mu) = e(\lambda + \mu)$ ($\lambda, \mu \in \mathfrak{h}^*$). The elements $e(\lambda)$ are called *formal exponentials*. They are linearly independent and are in one-to-one correspondence with the elements $\lambda \in \mathfrak{h}^*$.

Second, we define the action of the Weyl group W on the elements of \mathcal{E} . W acts on the complex vector space $\tilde{\mathcal{E}}$ of all (possibly infinite) linear combinations of formal exponentials by: $w(e(\lambda)) = e(w(\lambda))$ ($\lambda \in \mathfrak{h}^*$, $w \in W$). $\tilde{\mathcal{E}}$ contains \mathcal{E} as a subspace. Note that \mathcal{E} itself is not stable under the action of W .

Now, let V be a module from \mathcal{O} and let $V = \sum_{\lambda \in \mathfrak{h}^*}^{\oplus} V_{\lambda}$ be its weight space decomposition. We define the formal character of V by $\text{ch } V := \sum_{\lambda \in \mathfrak{h}^*} (\dim_{\mathbb{C}} V_{\lambda}) e(\lambda) \in \tilde{\mathcal{E}}$.

From now on, we assume that the GGCM A is symmetrizable. Let \mathcal{S} be the set of all sums of distinct pairwise perpendicular elements from Π^{im} . Here, λ, μ ($\in \mathfrak{h}^*$) are perpendicular means that $(\lambda \mid \mu) = 0$, where $(\cdot \mid \cdot)$ is a standard invariant bilinear form. Note that $\{0\} \cup \Pi^{\text{im}}$ is contained in \mathcal{S} by definition. For each $\beta \in \mathcal{S}$, we put $\varepsilon(\beta) = (-1)^m$ if β is a sum of m elements from Π^{im} . Then, we have the following character formula.

THEOREM 1.1 ([2] and [4]). *Let $\Lambda \in P_+$, and let $\mathcal{S}(\Lambda)$ be the set of all elements of \mathcal{S} perpendicular to Λ . We put*

$$S_{\Lambda} := e(\Lambda + \rho) \cdot \sum_{\beta \in \mathcal{S}(\Lambda)} \varepsilon(\beta) e(-\beta), \quad R := \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha},$$

where $\text{mult } \alpha = \dim_{\mathbb{C}} \mathfrak{g}_{\alpha}$ ($\alpha \in \Delta_+$). Then,

$$e(\rho) \cdot R \cdot \text{ch } L(\Lambda) = \sum_{w \in W} (\det w) \cdot w(S_{\Lambda}).$$

COROLLARY 1.1 ([2] and [4]). *We put $S := e(\rho) \cdot \sum_{\beta \in \mathcal{S}} \varepsilon(\beta) e(-\beta)$. Then,*

$$e(\rho) \cdot \prod_{\alpha \in \mathcal{A}_+} (1 - e(-\alpha))^{\text{mult } \alpha} = \sum_{w \in W} (\det w) \cdot w(S).$$

REMARK 1.1. In the original statement of Theorem 1.1 (resp. Corollary 1.1), which is Theorem 11.13.3 (resp. Corollary 11.13.2) in [4], S_A (resp. S) is defined to be $e(A + \rho) \cdot \sum_{\beta \in \mathcal{S}(A)} \varepsilon(\beta) e(\beta)$ (resp. $e(\rho) \cdot \sum_{\beta \in \mathcal{S}} \varepsilon(\beta) e(\beta)$). However, these are obviously wrong, and the corrected version is given above.

2. Homology and cohomology of \mathfrak{n}_{\pm} with coefficients in $L(\lambda)$. Let $\mathfrak{g}(A)$ be the GKM algebra associated to a GGCM A , $L(\lambda)$ the irreducible highest weight $\mathfrak{g}(A)$ -module with highest weight $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}(A)$. In this section, we review the notion of homology of \mathfrak{n}_- and cohomology of \mathfrak{n}_+ with coefficients in $L(\lambda)$. We denote by $\bigwedge \mathfrak{n}_{\pm}$ the exterior algebra of \mathfrak{n}_{\pm} , and by $\bigwedge^j \mathfrak{n}_{\pm}$ the space of the j -th homogeneous elements in $\bigwedge \mathfrak{n}_{\pm}$ ($j \geq 0$). So we have $\bigwedge \mathfrak{n}_{\pm} = \sum_{j \geq 0}^{\oplus} \bigwedge^j \mathfrak{n}_{\pm}$.

The vector space $C_c^j(\mathfrak{n}_+, L(\lambda))$ of j -cochains is defined by $C_c^j(\mathfrak{n}_+, L(\lambda)) := \text{Hom}_{\mathbb{C}}^{\mathfrak{h}}(\bigwedge^j \mathfrak{n}_+, L(\lambda))$ ($j \geq 0$). Here, for \mathfrak{h} -diagonalizable modules $V = \sum_{\mu \in \mathfrak{h}^*}^{\oplus} V_{\mu}$ and $W = \sum_{\tau \in \mathfrak{h}^*}^{\oplus} W_{\tau}$ with finite-dimensional weight spaces, we put

$$\text{Hom}_{\mathbb{C}}^{\mathfrak{h}}(V, W) := \{f \in \text{Hom}_{\mathbb{C}}(V, W) \mid f(V_{\mu}) = 0 \text{ for all but finitely many weights } \mu \in \mathfrak{h}^* \text{ of } V\}.$$

Then, $\text{Hom}_{\mathbb{C}}^{\mathfrak{h}}(V, W)$ becomes an \mathfrak{h} -module in the standard way (see [8]). The coboundary operator $d^j: C_c^j(\mathfrak{n}_+, L(\lambda)) \rightarrow C_c^{j+1}(\mathfrak{n}_+, L(\lambda))$ is defined by

$$\begin{aligned} (d^j f)(x_1 \wedge \cdots \wedge x_j \wedge x_{j+1}) &:= \sum_{i=1}^{j+1} (-1)^{i+1} x_i (f(x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_{j+1})) \\ &+ \sum_{1 \leq r < t \leq j+1} (-1)^{r+t} f([x_r, x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_r \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_{j+1}), \end{aligned}$$

where $x_1, \dots, x_{j+1} \in \mathfrak{n}_+$, $f \in C_c^j(\mathfrak{n}_+, L(\lambda))$, and the symbol \hat{x}_i indicates a term to be omitted. It is easy to verify (cf. [5] for example) that $\{C_c^j(\mathfrak{n}_+, L(\lambda)), d^j\}_{j \geq -1}$ is a cochain complex, where $C_c^{-1}(\mathfrak{n}_+, L(\lambda)) := \{0\}$. The corresponding cohomology is called the j -th cohomology of \mathfrak{n}_+ with coefficients in $L(\lambda)$, and is denoted by $H_c^j(\mathfrak{n}_+, L(\lambda))$ ($j \geq 0$). Note that the coboundary operator $d^j: C_c^j(\mathfrak{n}_+, L(\lambda)) \rightarrow C_c^{j+1}(\mathfrak{n}_+, L(\lambda))$ commutes with the action of \mathfrak{h} , so that $H_c^j(\mathfrak{n}_+, L(\lambda))$ is also an \mathfrak{h} -module ($j \geq 0$).

For the homology, we define the vector space $C_j(\mathfrak{n}_-, L(\lambda))$ of j -chains to be $\bigwedge^j \mathfrak{n}_- \otimes_{\mathbb{C}} L(\lambda)$, which is an \mathfrak{h} -module in the usual sense ($j \geq 0$). The boundary operator $d_j: C_j(\mathfrak{n}_-, L(\lambda)) \rightarrow C_{j-1}(\mathfrak{n}_-, L(\lambda))$ is defined by

$$\begin{aligned} d_j(y_1 \wedge \cdots \wedge y_j \otimes v) &:= \sum_{i=1}^j (-1)^i (y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_j) \otimes y_i(v) \\ &+ \sum_{1 \leq r < t \leq j} (-1)^{r+t} ([y_r, y_t] \wedge y_1 \wedge \cdots \wedge \hat{y}_r \wedge \cdots \wedge \hat{y}_t \wedge \cdots \wedge y_j) \otimes v, \end{aligned}$$

where $y_1, \dots, y_j \in \mathfrak{n}_-, v \in L(\lambda)$. The homology of this chain complex is the usual Lie algebra homology with coefficients in $L(\lambda)$, which we denote by $H_j(\mathfrak{n}_-, L(\lambda))$ ($j \geq 0$) (see [3]). As in the case of cohomology, the homology $H_j(\mathfrak{n}_-, L(\lambda))$ is an \mathfrak{h} -module in the standard way ($j \geq 0$).

REMARK 2.1. The cohomology $H_c^j(\mathfrak{n}_+, L(\lambda))$ of \mathfrak{n}_+ defined in this section is slightly different from the usual Lie algebra cohomology $H^j(\mathfrak{n}_+, L(\lambda))$, since we have employed $\text{Hom}_{\mathbb{C}}^c(\bigwedge^j \mathfrak{n}_+, L(\lambda))$ instead of $\text{Hom}_{\mathbb{C}}(\bigwedge^j \mathfrak{n}_+, L(\lambda))$ as the space of j -cochains ($j \geq 0$) (see [3] and [8]).

3. The results of Liu. In this section, we explain briefly the results of Liu [8] about \mathfrak{h} -modules $H_j(\mathfrak{n}_-, L(\lambda))$ and $H_c^j(\mathfrak{n}_+, L(\lambda))$ for Kac-Moody algebras, and present the analogs for GKM algebras. His proof is applicable to GKM algebras with no modifications. For details, see [8] and also the appendix of [3].

3.1. The duality theorem between homology and cohomology. Let $\mathfrak{g}(A)$ be the GKM algebra associated to a GGCM A , $L(\lambda)$ the irreducible highest weight $\mathfrak{g}(A)$ -module with an arbitrary highest weight $\lambda \in \mathfrak{h}^*$. Note that $L(\lambda), \bigwedge^j \mathfrak{n}_-$ ($j \geq 0$) are in the category \mathcal{O} . Since \mathcal{O} is closed under the operation of taking tensor products and quotients, $(\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\lambda)$, and so $H_j(\mathfrak{n}_-, L(\lambda))$ ($j \geq 0$) are also in the category \mathcal{O} . Therefore, $H_j(\mathfrak{n}_-, L(\lambda))$ is a direct sum of its irreducible components $C(\mu)$'s ($\mu \in \mathfrak{h}^*$) as \mathfrak{h} -modules, and for each $\mu \in \mathfrak{h}^*$, $C(\mu)$ occurs only finitely many times as irreducible components. Here, $C(\mu)$ is the (one-dimensional) irreducible \mathfrak{h} -module with weight $\mu \in \mathfrak{h}^*$.

Now, we have the following, due to Liu.

PROPOSITION 3.1 (cf. [8]). $H_c^j(\mathfrak{n}_+, L(\lambda))$ is isomorphic to $H_j(\mathfrak{n}_-, L(\lambda))$ as \mathfrak{h} -modules for any j ($j \geq 0$).

Owing to this proposition, it is enough for us to consider $H_j(\mathfrak{n}_-, L(\lambda))$ ($j \geq 0$).

3.2. A necessity condition for weights of $H_j(\mathfrak{n}_-, L(\lambda))$. Here, we assume that the GGCM A is symmetrizable. Let $(\cdot | \cdot)$ be a standard invariant bilinear form. Then, we have the following.

PROPOSITION 3.2 (cf. [8]). Every irreducible component of $H_j(\mathfrak{n}_-, L(\lambda))$ is of the form $C(\mu)$ ($\mu \in \mathfrak{h}^*$), with $(\mu + \rho | \mu + \rho) = (\lambda + \rho | \lambda + \rho)$.

REMARK 3.1. In the above proposition, the highest weight λ of $L(\lambda)$ can be arbitrary, not necessarily belonging to P_+ .

4. Kostant's formula for GKM algebras. In this section, we prove our Theorem, which is "Kostant's formula" for GKM algebras.

4.1. Determination of weights of $H_j(\mathfrak{n}_-, L(\lambda))$. Let $\mathfrak{g}(A)$ be the GKM algebra associated to a symmetrizable GGCM $A = (a_{ij})_{1 \leq i, j \leq n}$ and $P_+ = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$

if $a_{ii}=2$, and $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all i . Recall that $\rho \in \mathfrak{h}^*$ is a fixed element such that $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$ ($1 \leq i \leq n$), \mathcal{S} is the set of all sums of distinct pairwise perpendicular elements from Π^{im} , and $\mathcal{S}(\lambda)$ is the set of all elements of \mathcal{S} perpendicular to $\lambda \in \mathfrak{h}^*$. Since \mathcal{S} is a finite subset of \mathfrak{h}^* , we put $\mathcal{S} = \{\beta_j\}_{j=1}^\ell$. Note that $\{0\} \cup \Pi^{\text{im}}$ is a subset of \mathcal{S} .

From now on, we assume that the GGCM $A = (a_{ij})_{1 \leq i, j \leq n}$ satisfies the following condition ($\hat{C}1$):

$$(\hat{C}1) \quad \text{either } a_{ii}=2 \text{ or } a_{ii}=0 \text{ (} 1 \leq i \leq n \text{)}.$$

Since $\beta_j \in \mathcal{S}$ is a sum of simple imaginary roots, $\langle \beta_j, \alpha_i^\vee \rangle \in \mathbf{Z}_{\leq 0}$ for i with $a_{ii}=2$ (by (C2)), and $\langle \beta_j, \alpha_i^\vee \rangle \leq 0$ for all i . So, by the condition ($\hat{C}1$), $\rho - \beta_j \in P_+$ ($1 \leq j \leq \ell$), for $\langle \rho, \alpha_i^\vee \rangle = 0$ if $a_{ii}=0$. Therefore, from Theorem 1.1, we have character formulas for $L(\rho - \beta_j)$ ($1 \leq j \leq \ell$) as follows:

$$e(\rho) \cdot R \cdot \text{ch } L(\rho - \beta_j) = \sum_{w \in W} (\det w) \cdot w(S_{\rho - \beta_j}),$$

where $R = \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha}$ and

$$S_{\rho - \beta_j} = e(2\rho - \beta_j) \cdot \sum_{1 \leq i \leq \ell, \langle \beta_i, \beta_j \rangle = 0} \varepsilon(\beta_i) e(-\beta_i) \quad (1 \leq j \leq \ell).$$

LEMMA 4.1. $e(\rho) \cdot \text{ch}(\bigwedge n_-) = \text{ch}(\sum_{1 \leq j \leq \ell}^\oplus L(\rho - \beta_j))$.

PROOF. First, by Corollary 1.1, we have

$$e(\rho) \cdot \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha} = \sum_{w \in W} (\det w) \cdot w(S),$$

where $S = e(\rho) \cdot \sum_{i=1}^\ell \varepsilon(\beta_i) e(-\beta_i)$. Therefore, we get

$$e(2\rho) \cdot \prod_{\alpha \in \Delta_+} (1 - e(-2\alpha))^{\text{mult } \alpha} = \sum_{w \in W} (\det w) \cdot w(\bar{S}),$$

where $\bar{S} := e(2\rho) \cdot \sum_{i=1}^\ell \varepsilon(\beta_i) e(-2\beta_i)$, since $\gamma_1 + \cdots + \gamma_r = \tau \in \mathfrak{h}^*$ implies $2\gamma_1 + \cdots + 2\gamma_r = 2\tau \in \mathfrak{h}^*$, for $\gamma_i \in \Delta_+$ ($1 \leq i \leq r$). On the other hand, we have

$$\begin{aligned} e(\rho) \cdot \text{ch}(\bigwedge n_-) &= e(\rho) \cdot \text{ch}\left(\sum_{j \geq 0}^\oplus \bigwedge^j n_-\right) = e(\rho) \cdot \prod_{\alpha \in \Delta_+} (1 + e(-\alpha))^{\text{mult } \alpha} \\ &= e(\rho) \cdot \frac{\prod_{\alpha \in \Delta_+} (1 - e(-2\alpha))^{\text{mult } \alpha}}{\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\text{mult } \alpha}} = \frac{e(2\rho) \cdot \prod_{\alpha \in \Delta_+} (1 - e(-2\alpha))^{\text{mult } \alpha}}{e(\rho) \cdot R}. \end{aligned}$$

Here, we note that $(1 - e(-\alpha))^{-1} = \sum_{k \geq 0} e(-k\alpha)$ for $\alpha \in \Delta_+$.

Therefore, by Theorem 1.1, we have only to show the following:

CLAIM. $\sum_{w \in W} (\det w) \cdot w(\bar{S}) = \sum_{j=1}^{\ell} \sum_{w \in W} (\det w) \cdot w(S_{\rho - \beta_j})$.

PROOF OF THE CLAIM. By Proposition 1.1 (cf. the proof of Proposition 4.2), we have only to show that $\bar{S} = \sum_{j=1}^{\ell} S_{\rho - \beta_j}$, or

$$\sum_{k=1}^{\ell} \varepsilon(\beta_k) e(-2\beta_k) = \sum_{1 \leq j \leq \ell} \sum_{1 \leq i \leq \ell, (\beta_i | \beta_j) = 0} \varepsilon(\beta_i) e(-\beta_i - \beta_j).$$

Since $(\beta_i | \beta_i) = 0$ for all i ($1 \leq i \leq \ell$) by $(\hat{C}1)$, the above equality is nothing but

$$0 = \sum_{1 \leq i \neq j \leq \ell, (\beta_i | \beta_j) = 0} \varepsilon(\beta_i) e(-\beta_i - \beta_j).$$

Moreover, the right hand side is equal to

$$(\$) \quad \sum_{\substack{1 \leq i \neq j \leq \ell, (\beta_i | \beta_j) = 0 \\ \varepsilon(\beta_i) \cdot \varepsilon(\beta_j) = 1}} \varepsilon(\beta_i) e(-\beta_i - \beta_j).$$

Now, we assume that $\{1 \leq i \leq n | a_{ii} \leq 0\} = \{1, 2, \dots, m\}$ ($1 \leq m \leq n$) for notational simplicity. Then, $(\$)$ is equal to $\sum_{\beta} c_{\beta} e(-\beta)$, where β runs through all elements of the form

$$2(\alpha_{i_1} + \dots + \alpha_{i_p}) + \alpha_{i_{p+1}} + \dots + \alpha_{i_{p+q}},$$

where $(\alpha_{i_r} | \alpha_{i_s}) = 0$ ($1 \leq r, s \leq p+q$), $1 \leq i_r \neq i_s \leq m$ ($1 \leq r \neq s \leq p+q$), $0 \leq p$, $1 \leq q$, and $p+q \leq m$. Note that $q \neq 0$ and q must be even, since the sum in $(\$)$ is for $1 \leq i \neq j \leq \ell$ such that $\varepsilon(\beta_j) \cdot \varepsilon(\beta_j) = 1$. We calculate $c_{\beta} \in \mathbb{Z}$ for such β 's and want to show that $c_{\beta} = 0$ for all β . Actually, c_{β} is equal to $(-1)^p \cdot \sum_{i=0}^{2t} (-1)^i \cdot \begin{bmatrix} 2t \\ i \end{bmatrix}$, where $q = 2t$ and $\begin{bmatrix} 2t \\ i \end{bmatrix}$ is the binomial coefficient, because $\beta_i \in \mathcal{S}$ is a sum of distinct elements from Π^{im} and $\varepsilon(\beta_i) = (-1)^m$ if β_i is a sum of m elements. Hence $c_{\beta} = (-1)^p \cdot (1-1)^{2t} = 0$ (note that $t \neq 0$ as seen above). Thus, the claim has been proved.

This completes the proof of Lemma 4.1. □

From Lemma 4.1, it follows that, for every $\lambda \in \mathfrak{h}^*$,

$$e(\rho) \cdot \text{ch}((\bigwedge n_{-}) \otimes_{\mathbb{C}} L(\lambda)) = \text{ch} \left(\left(\sum_{1 \leq j \leq \ell}^{\oplus} L(\rho - \beta_j) \right) \otimes_{\mathbb{C}} L(\lambda) \right).$$

Hence, from the above, we can deduce that

$$\mu \in \mathcal{P}((\bigwedge n_{-}) \otimes_{\mathbb{C}} L(\lambda)) \iff \mu + \rho \in \mathcal{P} \left(\left(\sum_{1 \leq j \leq \ell}^{\oplus} L(\rho - \beta_j) \right) \otimes_{\mathbb{C}} L(\lambda) \right),$$

and the multiplicities for μ and $\mu + \rho$ coincide with each other.

LEMMA 4.2. *Let $\lambda \in P_+$. If, for some i ($i \geq 0$), μ is a weight of $(\bigwedge^i n_{-}) \otimes_{\mathbb{C}} L(\lambda)$ and satisfies $(\mu + \rho | \mu + \rho) = (\lambda + \rho | \lambda + \rho)$, then*

(1) there exist a $\beta_j \in \mathcal{S}$ with $(\Lambda | \beta_j) = 0$ and a $w \in W$, such that $\ell(w) + \text{ht}(\beta_j) = i$ and $\mu = w(\Lambda + \rho - \beta_j) - \rho$;

(2) the multiplicity of μ in $(\bigwedge^i \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$ is equal to one.

Here, $\text{ht}(\beta_j) = m$ if β_j is a sum of m distinct elements from Π^{im} for $\beta_j \in \mathcal{S}$ ($1 \leq j \leq \ell$).

PROOF. From the above consideration, $\mu + \rho$ is a weight of $(\sum_{1 \leq j \leq \ell}^{\oplus} L(\rho - \beta_j)) \otimes_{\mathbb{C}} L(\Lambda)$ with the same multiplicity as μ in $(\bigwedge^i \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$. We remark that $\mathcal{P}((\sum_{1 \leq j \leq \ell}^{\oplus} L(\rho - \beta_j)) \otimes_{\mathbb{C}} L(\Lambda)) = \mathcal{P}(\sum_{1 \leq j \leq \ell}^{\oplus} (L(\rho - \beta_j) \otimes_{\mathbb{C}} L(\Lambda))) = \bigcup_{j=1}^{\ell} \mathcal{P}(L(\rho - \beta_j) \otimes_{\mathbb{C}} L(\Lambda))$.

Now, suppose that $\mu + \rho \in \mathcal{P}(L(\rho - \beta_j) \otimes_{\mathbb{C}} L(\Lambda))$ for some j ($1 \leq j \leq \ell$). Since $\rho - \beta_j$ and Λ are elements of P_+ , $L(\rho - \beta_j)$, $L(\Lambda)$, and so $L(\rho - \beta_j) \otimes_{\mathbb{C}} L(\Lambda)$ are integrable (see Section 1). So, there exists a $\tilde{w} \in W$ such that $\tilde{w}(\mu + \rho) \in P_+ \cap \mathcal{P}(L(\rho - \beta_j) \otimes_{\mathbb{C}} L(\Lambda))$ by Proposition 1.3. We put $\tilde{w}(\mu + \rho) = \Lambda + \rho - \beta_j - \varphi$ with $\varphi \in Q_+$. Then, since $(\cdot | \cdot)$ is W -invariant, we have

$$\begin{aligned} (\Lambda + \rho | \Lambda + \rho) &= (\mu + \rho | \mu + \rho) = (\tilde{w}(\mu + \rho) | \tilde{w}(\mu + \rho)) = (\Lambda + \rho - \beta_j - \varphi | \Lambda + \rho - \beta_j - \varphi) \\ &= (\Lambda + \rho | \Lambda + \rho) - 2(\Lambda + \rho | \beta_j) + (\beta_j | \beta_j) - (\Lambda + \rho - \beta_j | \varphi) - (\Lambda + \rho - \beta_j - \varphi | \varphi) \\ &= (\Lambda + \rho | \Lambda + \rho) - 2(\Lambda | \beta_j) - (\Lambda + \rho - \beta_j | \varphi) - (\Lambda + \rho - \beta_j - \varphi | \varphi) \end{aligned}$$

(note that $(\rho | \beta_j) = 0 = (\beta_j | \beta_j)$ by $(\hat{C}1)$). Therefore, $2(\Lambda | \beta_j) + (\Lambda + \rho - \beta_j | \varphi) + (\Lambda + \rho - \beta_j - \varphi | \varphi) = 0$. Now, since $\tilde{w}(\mu + \rho) = \Lambda + \rho - \beta_j - \varphi \in P_+$ and $\varphi \in Q_+$, we have $(\Lambda + \rho - \beta_j - \varphi | \varphi) \geq 0$. Since Λ , $\rho - \beta_j \in P_+$, and $\varphi \in Q_+$, we have $(\Lambda + \rho - \beta_j | \varphi) \geq 0$. So, $(\Lambda | \beta_j) = (\Lambda + \rho - \beta_j | \varphi) = (\Lambda + \rho - \beta_j - \varphi | \varphi) = 0$, since $\Lambda \in P_+$.

We would like to show that $\varphi = 0$. For this purpose, put $\varphi = \sum_{i=1}^n k_i \alpha_i$, $k_i \in \mathbb{Z}_{\geq 0}$. Note that $(\rho | \alpha_i) = (1/2) \cdot (\alpha_i | \alpha_i) > 0$ for all $\alpha_i \in \Pi^{\text{re}}$, and that $(\beta_j | \varphi) \leq 0$ since β_j is a sum of elements from Π^{im} . Hence, $(\Lambda + \rho - \beta_j | \varphi) = 0$ implies that $(\Lambda | \varphi) = (\rho | \varphi) = (\beta_j | \varphi) = 0$. Further, $k_i = 0$ if $\alpha_i \in \Pi^{\text{re}}$. On the other hand, since $\tilde{w}(\mu + \rho) = \Lambda + \rho - \beta_j - \varphi$ is a weight of $L(\rho - \beta_j) \otimes_{\mathbb{C}} L(\Lambda)$, we have $\Lambda + \rho - \beta_j - \varphi = (\rho - \beta_j - \varphi_1) + (\Lambda - \varphi_2)$, where $\rho - \beta_j - \varphi_1 \in \mathcal{P}(L(\rho - \beta_j))$ and $\Lambda - \varphi_2 \in \mathcal{P}(L(\Lambda))$ with $\varphi_1, \varphi_2 \in Q_+$. So, we have $\varphi = \varphi_1 + \varphi_2$. Since $(\beta_j | \varphi) = (\Lambda | \varphi) = 0$, we get $(\beta_j | \varphi_1) = (\Lambda | \varphi_2) = 0$. Then, $(\rho - \beta_j | \rho - \beta_j) - (\rho - \beta_j | \rho - \beta_j - \varphi_1) = 0$ since $(\rho | \alpha_i) = 0$ if $\alpha_i \in \Pi^{\text{im}}$. Moreover, $(\Lambda | \Lambda) - (\Lambda | \Lambda - \varphi_2) = 0$. Therefore, we deduce from Proposition 1.4 that $\varphi_1 = \varphi_2 = 0$, hence $\varphi = 0$.

Thus, we get $\tilde{w}(\mu + \rho) = \Lambda + \rho - \beta_j$. So, we have shown that $\mu = w(\Lambda + \rho - \beta_j) - \rho$ with $w := (\tilde{w})^{-1}$. It is clear that the multiplicity of $\mu + \rho$ in $L(\rho - \beta_j) \otimes_{\mathbb{C}} L(\Lambda)$ is 1, since this $\mathfrak{g}(\Lambda)$ -module is integrable (see Section 1). Moreover, from the above argument, we see that $\beta_j \in \mathcal{S}$ is uniquely determined (see the proof of Proposition 4.2 below). Therefore, the multiplicity of μ in $(\bigwedge^i \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\Lambda)$ is 1.

Now, by Proposition 1.2, $w(\rho) - \rho = -\sum_{\alpha \in \Phi_w} \alpha$, where $\Phi_w = \{\alpha \in \Delta_+ | w^{-1}(\alpha) \in \Delta_-\}$. Therefore, $\mu = w(\Lambda + \rho - \beta_j) - \rho = w(\Lambda) - w(\beta_j) - (\sum_{\alpha \in \Phi_w} \alpha)$. We express β_j as $\beta_j = \alpha_{i_1} + \dots + \alpha_{i_m}$, where $m = \text{ht}(\beta_j)$, $\alpha_{i_k} \in \Pi^{\text{im}}$ ($1 \leq k \leq m$), and $i_r \neq i_t$ ($1 \leq r \neq t \leq m$). Remark that $w(\alpha_{i_k}) \in \Delta_+ \setminus \Phi_w$ ($1 \leq k \leq m$) and that $w(\Lambda) \in \mathcal{P}(L(\Lambda))$, since Δ and $\mathcal{P}(L(\Lambda))$ are both W -stable (see Section 1 and [4]). We take non-zero root vectors $E_k \in \mathfrak{g}_{-w(\alpha_{i_k})}$ ($1 \leq k \leq m$),

$E_\alpha \in \mathfrak{g}_{-\alpha}$ ($\alpha \in \Phi_w$), and a non-zero weight vector $v \in L(\lambda)_{w(\lambda)}$. Then, $0 \neq (E_1 \wedge \cdots \wedge E_m) \wedge (\bigwedge_{\alpha \in \Phi_w} E_\alpha) \otimes v \in (\bigwedge \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\lambda)$ is a weight vector of weight μ . Because μ is a weight of $(\bigwedge \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\lambda) \cong \sum_{k \geq 0}^{\oplus} ((\bigwedge^k \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\lambda))$ with multiplicity 1, and $\mu \in \mathcal{P}((\bigwedge^i \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\lambda))$ by assumption, we deduce that $m + \#(\Phi_w) = i$, where $\#(\Phi_w)$ denotes the cardinality of Φ_w . So, by Proposition 1.2, we have $i = \text{ht}(\beta_j) + \ell(w)$. \square

By Proposition 3.2 and Lemma 4.2, we have the following.

PROPOSITION 4.1. *Let $\lambda \in P_+$. If $\mathcal{C}(\mu)$ ($\mu \in \mathfrak{h}^*$) is an irreducible component of $H_i(\mathfrak{n}_-, L(\lambda))$ ($i \geq 0$), then*

- (1) $\mu = w(\lambda + \rho - \beta_j) - \rho$, for some $\beta_j \in \mathcal{S}$ with $(\lambda | \beta_j) = 0$, and some $w \in W$, such that $\ell(w) + \text{ht}(\beta_j) = i$;
- (2) $H_i(\mathfrak{n}_-, L(\lambda))$ has only one copy of $\mathcal{C}(\mu)$ in itself.

4.2. A sufficiency condition for weights of $H_i(\mathfrak{n}_-, L(\lambda))$. We assume that the GGCM A is symmetrizable and satisfies the condition $(\hat{C}1)$. Then we have the following sufficiency condition for weights of $H_i(\mathfrak{n}_-, L(\lambda))$.

PROPOSITION 4.2. *Let $\lambda \in P_+$ and fix $i \in \mathbb{Z}_{\geq 0}$. Then, for each $\beta_k \in \mathcal{S}$ with $(\lambda | \beta_k) = 0$ and each $w \in W$ such that $\ell(w) + \text{ht}(\beta_k) = i$, $\mu := w(\lambda + \rho - \beta_k) - \rho$ is a weight of $H_i(\mathfrak{n}_-, L(\lambda))$.*

PROOF. By Theorem 1.1, we have

$$R \cdot e(\rho) \cdot \text{ch } L(\lambda) = \sum_{w \in W} (\det w) \cdot w(S_\lambda),$$

where $R = \prod_{\alpha \in A_+} (1 - e(-\alpha))^{\text{mult } \alpha}$ and

$$S_\lambda = e(\lambda + \rho) \cdot \sum_{1 \leq k \leq \ell, (\lambda | \beta_k) = 0} \varepsilon(\beta_k) e(-\beta_k).$$

Recall that the boundary operator $d_j : (\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\lambda) \rightarrow (\bigwedge^{j-1} \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\lambda)$ commutes with the action of \mathfrak{h} ($j \geq 0$). Then, by the Euler-Poincaré formula (cf. [3]), we have

$$\begin{aligned} \sum_{j \geq 0} (-1)^j \cdot \text{ch}(H_j(\mathfrak{n}_-, L(\lambda))) &= \sum_{j \geq 0} (-1)^j \cdot \text{ch}((\bigwedge^j \mathfrak{n}_-) \otimes_{\mathbb{C}} L(\lambda)) \\ &= \left(\sum_{j \geq 0} (-1)^j \cdot \text{ch } \bigwedge^j \mathfrak{n}_- \right) \cdot \text{ch } L(\lambda) = \left(\prod_{\alpha \in A_+} (1 - e(-\alpha))^{\text{mult } \alpha} \right) \cdot \text{ch } L(\lambda) \\ &= R \cdot \text{ch } L(\lambda) = e(-\rho) \cdot \sum_{w \in W} (\det w) \cdot w(S_\lambda) \\ &= \sum_{w \in W} \sum_{1 \leq k \leq \ell, (\lambda | \beta_k) = 0} (\det w) \cdot \varepsilon(\beta_k) e(w(\lambda + \rho - \beta_k) - \rho) \end{aligned}$$

$$= \sum_{w \in W} \sum_{1 \leq k \leq \ell, \langle A | \beta_k \rangle = 0} (-1)^{\ell(w) + \text{ht}(\beta_k)} \cdot e(w(\Lambda + \rho - \beta_k) - \rho).$$

Now, we show that $w(\Lambda + \rho - \beta_k) - \rho$ differs if w or β_k differs. Suppose that $w_1(\Lambda + \rho - \beta_r) - \rho = w_2(\Lambda + \rho - \beta_t) - \rho$ for $w_1, w_2 \in W$, and $\beta_r, \beta_t \in \mathcal{S}$. Then, $w_2^{-1}w_1(\Lambda + \rho - \beta_r) = \Lambda + \rho - \beta_t$. Since $\Lambda + \rho - \beta_r, \Lambda + \rho - \beta_t \in C^\vee$, we get $\Lambda + \rho - \beta_r = \Lambda + \rho - \beta_t$ by Proposition 1.1(b). So, $\beta_r = \beta_t$. Therefore, we have $w_2^{-1}w_1(\Lambda + \rho - \beta_r) = \Lambda + \rho - \beta_r$. Then, since $\langle \rho, \alpha_i^\vee \rangle = 1$ and $\langle \beta_k, \alpha_i^\vee \rangle \leq 0$ for all i with $a_{ii} = 2$, it follows that $w_2^{-1}w_1 = 1$ from Proposition 1.1(a). So, we have $w_1 = w_2$.

From the above argument and Proposition 4.1, the Proposition now follows. \square

From Propositions 3.1, 4.1, and 4.2, we get the following theorem, which is our final goal.

THEOREM 4.1 (Kostant’s formula). *Let $\mathfrak{g}(A)$ be the GKM algebra associated to a symmetrizable GGCM A satisfying $(\hat{C}1)$. Let $L(\Lambda)$ be the irreducible highest weight $\mathfrak{g}(A)$ -module with highest weight $\Lambda \in P_+$. Then,*

$$H_c^j(\mathfrak{n}_+, L(\Lambda)) \cong H_j(\mathfrak{n}_-, L(\Lambda)) \cong \sum_{\beta \in \mathcal{S}(A)}^\oplus \sum_{w \in W, \ell(w) = j - \text{ht}(\beta)}^\oplus C(w(\Lambda + \rho - \beta) - \rho)$$

as \mathfrak{h} -modules ($j \geq 0$). Here, the sum is a direct sum of inequivalent irreducible \mathfrak{h} -modules.

COROLLARY 4.1 (Bott’s formula). *Under the same assumption as in Theorem 4.1, we have*

$$\begin{aligned} \dim_{\mathbb{C}} H_c^j(\mathfrak{n}_+, L(\Lambda)) &= \dim_{\mathbb{C}} H_j(\mathfrak{n}_-, L(\Lambda)) \\ &= \#\{(\beta, w) \in \mathcal{S}(A) \times W \mid \ell(w) + \text{ht}(\beta) = j\} < +\infty. \end{aligned}$$

REMARK 4.1. When A is a GCM (i.e., $a_{ii} = 2$ for all i), $\mathcal{S}(A)$ consists of only one element $0 \in \mathfrak{h}^*$. Hence, in this case, Theorem 4.1 is nothing but the well-known formula of Kostant for Kac-Moody algebras (cf. [3] and [7]).

REMARK 4.2. In Theorem 4.1 and Corollary 4.1, the assumption that the GGCM A satisfies $(\hat{C}1)$ is essential, because the element $\rho \in \mathfrak{h}^*$ must belong to P_+ , while $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$ for all i .

5. A presentation of GKM algebras. In this section, as an application of Theorem 4.1, we get a presentation by generators and relations of the GKM algebra $\mathfrak{g}(A)$ associated to a symmetrizable GGCM A satisfying $(\hat{C}1)$. Though such a presentation is already known for an arbitrary symmetrizable GGCM A ([4]), its proof is rather complicated. Here, using Theorem 4.1, we give a simple cohomological proof by the method of Mathieu [9].

Let $\tilde{\mathfrak{g}}(A)$ be the Lie algebra defined in Section 1, and $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $e_i, 1 \leq i \leq n$ (resp. $f_i, 1 \leq i \leq n$). Then, we know the following.

PROPOSITION 5.1 ([4]). (a) $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$.

(b) $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) is freely generated by $e_i, 1 \leq i \leq n$ (resp. $f_i, 1 \leq i \leq n$).

(c) The map determined by $e_i \mapsto -f_i, f_i \mapsto -e_i (1 \leq i \leq n), h \mapsto -h (h \in \mathfrak{h})$, can be uniquely extended to an involution $\tilde{\omega}$ of the Lie algebra $\tilde{\mathfrak{g}}(A)$.

(d) With respect to \mathfrak{h} , we have the root space decomposition:

$$\tilde{\mathfrak{g}}(A) = \left(\sum_{\alpha \in Q_+ \setminus \{0\}}^{\oplus} \tilde{\mathfrak{g}}_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left(\sum_{\alpha \in Q_+ \setminus \{0\}}^{\oplus} \tilde{\mathfrak{g}}_{\alpha} \right),$$

where $\tilde{\mathfrak{g}}_{\alpha} := \{x \in \tilde{\mathfrak{g}}(A) \mid [h, x] = \langle \alpha, h \rangle x, \text{ for all } h \in \mathfrak{h}\}$. Furthermore, $\tilde{\mathfrak{g}}_{\alpha} \subset \tilde{\mathfrak{n}}_{\pm}$ for $\pm \alpha \in Q_+ \setminus \{0\}$.

(e) $\mathfrak{r} = \mathfrak{r}_- \oplus \mathfrak{r}_+$ (direct sum of ideals), with $\mathfrak{r}_{\pm} = \sum_{\alpha \in Q_+}^{\oplus} (\mathfrak{r} \cap \tilde{\mathfrak{g}}_{\pm \alpha})$. Moreover, $\mathfrak{r} \cap \tilde{\mathfrak{g}}_{\pm \alpha_i} = \{0\} (1 \leq i \leq n)$.

In order to determine $\mathfrak{r} = \mathfrak{r}_- \oplus \mathfrak{r}_+$, it is enough to consider \mathfrak{r}_- only, since the result for \mathfrak{r}_+ follows by the application of the involution $\tilde{\omega}$ of $\tilde{\mathfrak{g}}(A)$. Then, as a special case of Mathieu’s general result [10, Chap. XVI, §4, Lemme 116], we have the following.

PROPOSITION 5.2. As \mathfrak{h} -modules,

$$\mathfrak{r}_- / [\tilde{\mathfrak{n}}_-, \mathfrak{r}_-] \cong H_2(\mathfrak{n}_-, L(0)).$$

Now, we assume that the GGCM $A = (a_{ij})_{1 \leq i, j \leq n}$ is symmetrizable, and satisfies $(\hat{C}1)$. From Theorem 4.1, we have

$$(\&) \quad H_2(\mathfrak{n}_-, L(0)) \cong \sum_{\beta \in \mathcal{S}}^{\oplus} \sum_{w \in W, \ell(w) = 2 - \text{ht}(\beta)}^{\oplus} C(w(\rho - \beta) - \rho)$$

as \mathfrak{h} -modules.

We see that the sum on the right hand side actually runs through the disjoint union of:

$$S_1 := \{(\alpha_i + \alpha_j, 1) \in \mathcal{S} \times W \mid \alpha_i, \alpha_j \in \Pi^{\text{im}}, a_{ij} = 0 (1 \leq i \neq j \leq n)\},$$

$$S_2 := \{(\alpha_j, r_i) \in \mathcal{S} \times W \mid \alpha_j \in \Pi^{\text{im}}, a_{ii} = 2 (1 \leq i \neq j \leq n)\},$$

$$S_3 := \{(0, r_i r_j) \in \mathcal{S} \times W \mid a_{ii} = a_{jj} = 2 (1 \leq i \neq j \leq n)\}.$$

For $(\alpha_i + \alpha_j, 1) \in S_1$, the corresponding weight $w(\rho - \beta) - \rho$ is $-(\alpha_i + \alpha_j)$. For $(\alpha_j, r_i) \in S_2$ and $(0, r_i r_j) \in S_3$, it is $-(\alpha_j + (1 - a_{ij})\alpha_i)$. Therefore, we can easily deduce from Propositions 5.1 and 5.2 that the ideal \mathfrak{r}_- of $\tilde{\mathfrak{n}}_-$ is generated by the following elements:

$$(\text{ad } f_i)^{1 - a_{ij}} f_j \quad (1 \leq i \neq j \leq n, a_{ii} = 2),$$

$$[f_i, f_j] \quad (1 \leq i \neq j \leq n, a_{ii} = a_{jj} = a_{ij} = 0).$$

Hence, we have recovered the following theorem.

THEOREM 5.1 ([4]). Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a symmetrizable GGCM satisfying $(\hat{C}1)$.

Then, the GKM algebra $\mathfrak{g}(A)$ is isomorphic to the Lie algebra given by generators $\mathfrak{h} \cup \{e_i, f_i\}_{i=1}^n$ and the relations (F1)–(F3) and the following:

$$\begin{aligned} (\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 & \quad \text{if } a_{ii} = 2 \text{ and } i \neq j; \\ [e_i, e_j] = 0, \quad [f_i, f_j] = 0 & \quad \text{if } a_{ii} = a_{jj} = a_{ij} = 0. \end{aligned}$$

Here, the triple $(\mathfrak{h}, \Pi = \{\alpha_i\}_{i=1}^n, \Pi^\vee = \{\alpha_i^\vee\}_{i=1}^n)$ is a realization of the GGCM A .

REFERENCES

- [1] F. ARIBAUD, Une nouvelle démonstration d'un théorème de R. Bott et B. Kostant, *Bull. Soc. math. France* 95 (1967), 205–242.
- [2] R. BORCHERS, Generalized Kac-Moody algebras, *J. Algebra* 115 (1988), 501–512.
- [3] H. GARLAND AND J. LEPOWSKY, Lie algebra homology and the Macdonald-Kac formulas, *Invent. math.* 34 (1976), 37–76.
- [4] V. G. KAC, *Infinite dimensional Lie algebras* (3rd edition), Cambridge Univ. Press, 1990.
- [5] A. W. KNAPP, *Lie groups, Lie algebras, and cohomology*, Princeton Univ. Press, 1988.
- [6] B. KOSTANT, Lie algebra cohomology and generalized Borel-Weil theorem, *Ann. of Math.* 74 (1961), 329–387.
- [7] J. LEPOWSKY, Generalized Verma modules, loop space cohomology and Macdonald-type identities, *Ann. Sci. École Norm. Sup.* 12 (1979), 169–234.
- [8] L. LIU, Kostant's formula for Kac-Moody Lie algebras, dissertation, Yale University (1990).
- [9] O. MATHIEU, Une démonstration cohomologique du théorème de Gabber-Kac, 1984, not published.
- [10] O. MATHIEU, Formules de caractères pour les algèbres de Kac-Moody générales, *Astérisque* 159–160, 1988.
- [11] S. NAITO, On regular subalgebras of Kac-Moody algebras and their associated invariant forms—Symmetrizable case—, *J. Math. Soc. Japan* 44 (1992), 157–177.
- [12] S. NAITO, Embedding into Kac-Moody algebras and construction of folding subalgebras for generalized Kac-Moody algebras, *Japan. J. Math.* 18 (1992), 155–171.

DEPARTMENT OF MATHEMATICS
SHIZUOKA UNIVERSITY
SHIZUOKA, 422
JAPAN