Tôhoku Math. J.
44 (1992), 523–534

PARAMETER SHIFT IN NORMAL GENERALIZED HYPERGEOMETRIC SYSTEMS

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(Received October 7, 1991, revised April 28, 1992)

Abstract. We treat the problem of shifting parameters of the generalized hypergeometric systems defined by Gelfand when their associated toric varieties are normal. In this context we define and determine the Bernstein-Sato polynomials for the natural morphisms of shifting parameters. We also give some examples.

Let $A=\{\chi_1,\dots,\chi_N\}\subset \boldsymbol{Z}^n$ be a finite subset with certain properties. In [G], [GGZ], [GZK1], [GZK2], [GKZ] and so on, Gelfand and his collaborators defined and studied generalized hypergeometric systems M_α associated to A with parameter α. Aomoto defined and studied a broader class of systems (cf. [A1]–[A4]). Generalized hypergeometric systems of this kind were also defined in [KKM] and [H], where they were named canonical systems. For $1\le j\le N$, there exists a natural morphism $f_{\chi_j}: M_{\alpha-\chi_j}\to M_\alpha$, which corresponds to the differentiation of solutions. In this paper, we treat the problem of determining when f_{χ_j} becomes isomorphic under the condition that a certain associated affine toric variety is normal.

In §1 and §2, we define the system M_α and the natural morphism f_{χ_j}, and give a necessary condition (Theorem 2.3) for the morphism f_{χ_j} to be an isomorphism. In §3, we introduce an assumption, which we call the normality and keep throughout this paper. In §4, §5, and §6, we define an ideal $B(\chi_j)$ of the b-functions for the morphism f_{χ_j}, and obtain a sufficient condition in terms of the b-functions (Corollary 5.4) for the morphism f_{χ_j} to be isomorphic. The ideal $B(\chi_j)$ turns out to be singly generated by a certain polynomial (Theorem 6.4). In §7, some example are given.

The author would like to thank Professors Ryoshi Hotta and Masa-Nori Ishida for helpful conversation.

1. Generalized hypergeometric systems.

First of all, we recall the definition of generalized hypergeometric systems following Gelfand et al. (cf. [GGZ]). Suppose we are given N integral vectors $\chi_j=(\chi_{1j},\dots,\chi_{nj})\in \boldsymbol{Z}^n$ $(j=1,\dots,N)$ satisfying two conditions.

(1) The vectors $\chi_1,\dots,\chi_N$ generate the lattice $\boldsymbol{Z}^n$.

* Partly supported by the Grants-in-Aid for Encouragement of Young Scientists, The Ministry of Education, Science and Culture, Japan.

1991 *Mathematics Subject Classification*. Primary 33C70; Secondary 14M25, 16W50, 39B32.

(2) All the vectors χ_j lie on some affine hyperplane $\sum_{i=1}^n c_i x_i = 1$ in $\boldsymbol{R}^n$, where $c_i \in \boldsymbol{Z}$.

We denote by L the subgroup in $\boldsymbol{Z}^n$ consisting of those $a=(a_j)_{j=1}^N$ satisfying $\sum_{j=1}^N a_j \chi_j = 0$. Let $(v_1, \dots, v_N)$ be a coordinate system on $V = \boldsymbol{C}^N$. Let $W = W_V$ denote the Weyl algebra on V, i.e.,

$$W = W_V = \boldsymbol{C}[v_1, \dots, v_N, D_1, \dots, D_N]$$

where $D_j = \partial/\partial v_j$ for $j=1, \dots, N$. We put for $a \in L$

$$\Box_a = \prod_{a_j > 0} D_j^{a_j} - \prod_{a_j < 0} D_j^{-a_j}.$$

For a parameter $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{C}^n$ we define a generalized hypergeometric system M_α on V as a W-module to be W modulo the left W-module generated by $\sum_{j=1}^N \chi_{ij}\theta_j - \alpha_i$ $(1 \le i \le n)$ and $\Box_a$ $(a \in L)$, i.e.,

$$M_\alpha := W \Big/ \left(\sum_{i=1}^n W\left(\sum_{j=1}^n \chi_{ij}\theta_j - \alpha_i \right) + \sum_{a \in L} W\Box_a \right).$$

Here $\theta_j = v_j D_j$ for $j = 1, \dots, N$, and $\sum_{\alpha \in L} W\Box_a$ denotes the left W-submodule of W consisting of all sums $\sum_{a \in L} w_a \Box_a$ with $w_a \in W$ such that only finitely many w_a are not zero. We denote by Q the Newton polyhedron, i.e., Q is the convex hull in $\boldsymbol{R}^n$ of the points $\chi_1, \dots, \chi_N$, by Λ the semigroup $\boldsymbol{Z}_{\ge 0}\chi_1 + \cdots + \boldsymbol{Z}_{\ge 0}\chi_N$, and by R the semigroup ring $\boldsymbol{C}[\Lambda]$ regarded as a $\boldsymbol{Z}^n$-graded ring in an obvious way.

2. Saturated subsets. We now define saturated subsets of $\{1, \dots, N\}$, which later turn out to correspond to faces of the polyhedron Q. Here the empty set $\varnothing$ is regarded as a face of the polyhedron Q. One might refer to [D] or [O] for the theory of toric varieties.

DEFINITION. Let I be a subset of $\{1, \dots, N\}$. We call I a saturated subset when for any $a \in L$ either $I \cap \{i \mid a_i \ne 0\} = \varnothing$ or there exist $i, j \in I$ such that $a_i > 0$ and $a_j < 0$.

We can regard R as the quotient of $\boldsymbol{C}[D_1, \dots, D_N]$ by the $\boldsymbol{C}[D_1, \dots, D_n]$-submodule generated by $\Box_a$ $(a \in L)$. Let R_λ $(\lambda \in \Lambda)$ denote the subspace of R generated by the image of $D_1^{b_1} \cdots D_N^{b_N}$ with $b_j \in \boldsymbol{Z}_{\ge 0}$ $(1 \le j \le N)$ satisfying $\lambda = \sum_{j=1}^N b_j \chi_j$. Then we have

$$R = \boldsymbol{C}[D_1, \dots, D_N] \Big/ \sum_{a \in L} \boldsymbol{C}[D_1, \dots, D_N]\Box_a = \bigoplus_{\lambda \in \Lambda} R_\lambda.$$

Here $\sum_{a \in L} \boldsymbol{C}[D_1, \dots, D_N]\Box_a$ denotes the ideal of $\boldsymbol{C}[D_1, \dots, D_N]$ consisting of all sums $\sum_{a \in L} p_a \Box_a$ with $p_a \in \boldsymbol{C}[D_1, \dots, D_N]$ such that only finitely many p_a are not zero. Clearly the images of $D_1^{b_1} \cdots D_N^{b_N}$ and $D_1^{b'_1} \cdots D_N^{b'_N}$ in R coincide if $\sum_{j=1}^N b_j \chi_j = \sum_{j=1}^N b'_j \chi_j$. Hence the subspace R_λ of R is one-dimensional. Elements in R_λ are said to be

Λ-homogeneous, and the ideals generated by Λ-homogeneous elements are also said to be Λ-homogeneous. For a saturated subset I, we denote by $P(I)$ the Λ-homogeneous ideal of R generated by all D_i for $i \in I$, where we use the same letter D_i for its image in R.

LEMMA 2.1. *$\{P(I) | I$ is saturated$\}$ is the set of Λ-homogeneous prime ideals of R.*

PROOF. We first prove that $P(I)$ is prime. Since $\dim R_\lambda = 1$ for all $\lambda \in \Lambda$, it is enough to show that $m_2 \in P(I)$ if $m_1 \notin P(I)$ and $m = m_1 m_2 \in P(I)$ for two monomials m_1, m_2. Set $m_1 = \prod_{j=1}^N D_j^{c_{1j}}$, $m_2 = \prod_{j=1}^N D_j^{c_{2j}}$ and $m = \sum_{j=1}^N D_j^{b_j}$. Then we have $\prod_{j=1}^N D_j^{b_j} = \prod_{j=1}^N D_j^{(c_{1j}+c_{2j})}$, and there exists $i \in I$ such that $b_i > 0$. Since I is saturated and $b_i > 0$, there exists $i' \in I$ such that $c_{1i'} + c_{2i'} > 0$. Since $m_1 \notin P(I)$, we have $c_{1i'} = 0$. Thus we obtain $c_{2i'} > 0$ and $m_2 \in P(I)$.

We next assume P to be a Λ-homogeneous prime ideal. Denote $I(P) := \{1 \le i \le N | D_i \in P\}$. Since $\dim R_\lambda = 1$ for all $\lambda \in \Lambda$, the Λ-homogeneous ideal P is generated by some monomials. Moreover, since P is prime, we see that P is generated by $\{D_i | i \in I(P)\}$. For $i \in I(P)$ and $\alpha \in L$ such that $a_i > 0$, we see that $\prod_{a_j > 0} D_j^{a_j} \in P$. Since $\prod_{a_j > 0} D_j^{a_j} = \prod_{a_j < 0} D_j^{-a_j}$ and P is prime, there exists k such that $a_k < 0$ and $D_k \in P$. We have thus proved $I(P)$ to be saturated. ■

Let Γ be a face of Q. We denote by $P(\Gamma)$ the ideal of R generated by all D_j for $\chi_j \notin \Gamma$.

LEMMA 2.2 (cf. [I]). *$\{P(\Gamma) | \Gamma$ is a face of $Q\}$ is the set of Λ-homogeneous prime ideals of R.*

As a result, for a saturated subset I, the χ_j $(j \notin I)$ span a face of Q. Conversely, for a face Γ, $I(\Gamma) = \{1 \le j \le N | \chi_j \notin \Gamma\}$ is a saturated subset. In particular, the set of nonempty minimal saturated subsets bijectively corresponds to the set of faces of codimension one. For a face Γ of Q of codimension one we denote by F_Γ the linear form for the hyperplane spanned by Γ such that the coefficients of F_Γ are integers, that their greatest common divisor is one, and that $F_\Gamma(\chi) \ge 0$ for any $\chi \in \Lambda$.

DEFINITION. We call a point $l = (l_1, \dots, l_N) \in (\mathbf{Z}_{\ge 0})^N$ a quotient point associated to a saturated subset I when $I = \{j | l_j \ne 0\}$ and for any $a \in L$ either $I \cap \{i | a_i \ne 0\} = \varnothing$ or there exist $i, j \in I$ such that $0 < l_i \le a_i$ and $0 > -l_j \ge a_j$.

For $\chi = \sum_{j=1}^N b_j \chi_j$ such that each b_j is a nonnegative integer, we denote by D^χ the operator $\prod_{j=1}^N D_j^{b_j}$. Since $(\sum_{j=1}^N \chi_{ij}\theta_j - \alpha_i) D^\chi = D^\chi(\sum_{j=1}^N \chi_{ij}\theta_j - \alpha_i - \sum_{j=1}^N b_j \chi_{ij})$, we have a natural morphism $f_\chi \colon M_{\alpha-\chi} \to M_\alpha$ by multiplying D^χ from the right.

THEOREM 2.3. *For $j_0 \in \{1, \dots, N\}$, the morphism $f_{\chi_{j_0}}$ is not isomorphic if there exist a face Γ of codimension d and a quotient point l associated to $I(\Gamma)$ such that Γ does not contain χ_{j_0}, and $F_{\Gamma_k}(\alpha) = \sum_{j \in I(\Gamma) - \{j_0\}} (l_j - 1) F_{\Gamma_k}(\chi_j)$ for $k = 1, \dots, d$, where $\Gamma = \Gamma_1 \cap \dots \cap \Gamma_d$ and the codimension of each Γ_k is one.*

PROOF. Suppose that there exist a face $\Gamma=\Gamma_1\cap\cdots\cap\Gamma_d$ and a quotient point l associated to $I(\Gamma)\ni j_0$ such that $F_{\Gamma_k}(\alpha)=\sum_{j\in I(\Gamma)-\{j_0\}}(l_j-1)F_{\Gamma_k}(\chi_j)$ for $k=1,\dots,d$. Let J be the complement of $I(\Gamma)$. Let $\boldsymbol{C}^{I(\Gamma)}=\{(v_i)|i\in I(\Gamma)\}$, $\boldsymbol{C}^J=\{(v_j)|j\in J\}$ and $L_J:=\{a\in L|a_i=0 \text{ for all } i\in I(\Gamma)\}$. Consider the quotient

$$M'=\mathrm{Coker}(f_{\chi_{j_0}})\Big/\Big(\sum_{j\in I(\Gamma)-\{j_0\}}W_VD_j^{l_j}+\sum_{j\in I(\Gamma)-\{j_0\}}W_V(\theta_j-(l_j-1))\Big)$$

$$=W_V\Big/\Big(W_VD_{j_0}+\sum_{i=1}^{n}W_V\Big(\sum_{j=1}^{N}\chi_{ij}\theta_j-\alpha_i\Big)+\sum_{j\in I(\Gamma)-\{j_0\}}W_VD_j^{l_j}$$

$$+\sum_{j\in I(\Gamma)-\{j_0\}}W_V(\theta_j-(l_j-1))+\sum_{a\in L_J}W_V\Box_a\Big)$$

$$=W_V\Big/\Big(W_VD_{j_0}+\sum_{i=1}^{n}W_V\Big(\sum_{j=1}^{N}\chi_{ij}\theta_j-\beta_i\Big)+\sum_{j\in I(\Gamma)-\{j_0\}}W_VD_j^{l_j}$$

$$+\sum_{j\in I(\Gamma)-\{j_0\}}W_V(\theta_j-(l_j-1))+\sum_{a\in L_J}W_V\Box_a\Big)$$

$$=W_{\boldsymbol{C}^J}\Big/\Big(\sum_{i=1}^{n}W_{\boldsymbol{C}^J}\sum_{j\in J}(\chi_{ij}\theta_j-\beta_i)+\sum_{a\in L_J}W_{\boldsymbol{C}^J}\Box_a\Big)\otimes_{\boldsymbol{C}}W_{\boldsymbol{C}^{I(\Gamma)}}\Big/$$

$$\Big(W_{\boldsymbol{C}^{I(\Gamma)}}D_{j_0}+\sum_{j\in I(\Gamma)-\{j_0\}}W_{\boldsymbol{C}^{I(\Gamma)}}D_j^{l_j}+\sum_{j\in I(\Gamma)-\{j_0\}}W_{\boldsymbol{C}^{I(\Gamma)}}(\theta_j-(l_j-1))\Big),$$

where $\beta_i=\alpha_i-\sum_{j\in I(\Gamma)-\{j_0\}}(l_j-1)\chi_{ij}$. We have $F_{\Gamma_k}(\beta)=0$ for any k and the module

$$W_{\boldsymbol{C}^J}\Big/\Big(\sum_{i=1}^{n}W_{\boldsymbol{C}^J}\sum_{j\in J}(\chi_{ij}\theta_j-\beta_i)+\sum_{a\in L_J}W_{\boldsymbol{C}^J}\Box_a\Big)$$

is a generalized hypergeometric system on $\boldsymbol{C}^J$ with respect to χ_j $(j\in J)$.

Furthermore, the module

$$W_{\boldsymbol{C}^{I(\Gamma)}}\Big/\Big(W_{\boldsymbol{C}^{I(\Gamma)}}D_{j_0}+\sum_{j\in I(\Gamma)-\{j_0\}}W_{\boldsymbol{C}^{I(\Gamma)}}D_j^{l_j}+\sum_{j\in I(\Gamma)-\{j_0\}}W_{\boldsymbol{C}^{I(\Gamma)}}(\theta_j-(l_j-1))\Big)$$

$$=W_{\boldsymbol{C}^{I(\Gamma)}}\prod_{j\in I(\Gamma)-\{j_0\}}v_j^{l_j-1}=\boldsymbol{C}[v_i|i\in I(\Gamma)]$$

is not zero. We thus deduce that M', hence accordingly $\mathrm{Coker}(f_{\chi_{j_0}})$ is not zero. ■

3. Normality assumption. For a $\boldsymbol{Z}^n$-graded R-module M we define a subset $\Lambda(M)\subset\boldsymbol{Z}^n$ by $\Lambda(M):=\{\lambda\in\boldsymbol{Z}^n|M_\lambda\neq0\}$, when $M=\bigoplus_{\lambda\in\boldsymbol{Z}^n}M_\lambda$. Since we have

$$\boldsymbol{R}_{\geq0}\chi_1+\cdots+\boldsymbol{R}_{\geq0}\chi_N=\bigcap_{\Gamma}\{\chi\in\boldsymbol{R}^n|F_\Gamma(\chi)\geq0\},$$

where Γ runs through the faces of codimension one, the following is the normality condition, i.e., the condition for the ring R to be normal (see, e.g., [S1]).

NORMALITY CONDITION.

$$\bigcap_{\Gamma}\{\chi\in \boldsymbol{R}^n \mid F_\Gamma(\chi)\geq 0\}\cap \boldsymbol{Z}^n=\Lambda\ ,$$

where Γ runs through the faces of codimension one.

From now on, we always assume the normality.

LEMMA 3.1. *Let $\chi_0\in\Lambda$, and let (D^{χ_0}) be the ideal of R generated by D^{χ_0}. Then we have*

$$\Lambda((D^{\chi_0}))=\boldsymbol{Z}^n\cap\bigcap_{\Gamma}\{\chi\in\boldsymbol{R}^n \mid F_\Gamma(\chi)\geq F_\Gamma(\chi_0)\}\ .$$

PROOF. Suppose that $\chi\in\boldsymbol{Z}^n$ and $F_\Gamma(\chi)\geq F_\Gamma(\chi_0)$ for any Γ of codimension one. Let $\chi':=\chi-\chi_0\in\boldsymbol{Z}^n$. Then we have $F_\Gamma(\chi')\geq 0$ for any Γ. By the normality we see that $\chi'\in\Lambda$. Therefore $\chi\in\chi_0+\Lambda=\Lambda((D_0^\chi))$. The other inclusion is clear. ■

4. Decomposition of ideals. Let (Γ, χ_0) be a pair of a face Γ of codimension one and $\chi_0\in\Lambda$. To such a pair (Γ,χ_0) we associate an ideal $D(\Gamma,\chi_0)$ of R defined as the one generated by all $\prod_{b_j\geq 0}D_j^{b_j}$ such that $F_\Gamma(\chi_0)\leq\sum_{b_j\geq 0}b_jF_\Gamma(\chi_j)$.

PROPOSITION 4.1. *We have the following decomposition of the ideal (D^{χ_0}):*

$$(D^{\chi_0})=\bigcap_{\Gamma}D(\Gamma,\chi_0)\ .$$

PROOF. Since D^{χ_0} belongs to $D(\Gamma,\chi_0)$ for any pair (Γ,χ_0), it is clear that (D^{χ_0}) is contained in the intersection $\bigcap_\Gamma D(\Gamma,\chi_0)$. In order to show the other inclusion, it is enough to verify that the intersection $\bigcap_\Gamma\Lambda(D(\Gamma,\chi_0))$ is a subset of $\Lambda((D^{\chi_0}))$. Suppose that $\chi\in\boldsymbol{Z}^n$ does not belong to $\Lambda((D^{\chi_0}))$. By Lemma 3.1 there exists a face Γ of codimension one such that $F_\Gamma(\chi)<F_\Gamma(\chi_0)$. By the definition of the ideal $D(\Gamma,\chi_0)$ we see that χ does not belong to $\Lambda(D(\Gamma,\chi_0))$. ■

Let I' denote the left ideal of W generated by all $\square_a$ $(a\in L)$, $I'(\chi_0)$ the one generated by I' and D^{χ_0}, and $I'(\Gamma,\chi_0)$ the one generated by I' and all $\prod_{b_j\geq 0}D_j^{b_j}$ such that $\sum_{b_j\geq 0}F_\Gamma(\chi_j)\geq F_\Gamma(\chi_0)$. For a left ideal J of W we denote by $\bar{J}$ the graded ideal with respect to the order filtration in W.

LEMMA 4.2. (1) *Let J be a left ideal of W generated by homogeneous operators $P_1,\dots,P_s$ in $\boldsymbol{C}[D_1,\dots,D_N]$. Then the graded ideal $\bar{J}$ is generated by $\bar{P}_1,\dots,\bar{P}_s$ in the graded ring $\bar{W}$, where $\bar{P}_j$ is the image of P_j in $\bar{W}$ for any j.*

(2) *Let J and J' be two left ideals of the algebra W. Suppose that $J\subset J'$ and*

$\bar{J}=\bar{J}'$. Then J coincides with J'.

The proof is straightforward.

PROPOSITION 4.3. *We have the following decomposition of the left ideal $I'(\chi_0)$:*

$$I'(\chi_0)=\bigcap_{\Gamma} I'(\Gamma, \chi_0).$$

PROOF. Clearly $I'(\chi_0)$ is contained in $\bigcap_\Gamma I'(\Gamma, \chi_0)$. We thus have $(I'(\chi_0))^{-}\subset(\bigcap_\Gamma I'(\Gamma, \chi_0))^{-}\subset\bigcap_\Gamma(I'(\Gamma, \chi_0)^{-})$. By Proposition 4.1 and Lemma 4.2 (1), we see that $(I'(\chi_0))^{-}=\bigcap_\Gamma(I'(\Gamma, \chi_0)^{-})$ in $\bar{W}$. We thus conclude that $I'(\chi_0)=\bigcap_\Gamma I'(\Gamma, \chi_0)$ from Lemma 4.2 (2). ■

We denote by $W[s]$ the noncommutative ring $\boldsymbol{C}[s_1, \dots, s_n]\otimes_{\boldsymbol{C}} W$, where each s_i is an indeterminate central element. Let I be the left ideal of $W[s]$ generated by $\sum_{j=1}^N \chi_{ij}\theta_j - s_i$ $(i=1, \dots, n)$ and $\square_a$ $(a\in L)$. We denote by $M[s]$ the quotient $W[s]/I$. Let $I(\chi_0)$ be the left ideal of $W[s]$ generated by I and D^{χ_0}, and $I(\Gamma, \chi_0)$ the one generated by I and all $\prod_{b_j\geq 0} D_j^{b_j}$ such that $\sum_{b_j\geq 0} b_j F_\Gamma(\chi_j)\geq F_\Gamma(\chi_0)$. To $P=\sum_c P_c s^c\in W[s]$, where $P_c\in W$ and $c=(c_1, \dots, c_n)\in(\boldsymbol{Z}_{\geq 0})^n$ is a multi-index, we associate the element $P':=\sum_c P_c(\sum_{j=1}^N \chi_{1j}\theta_j)^{c_1}\cdots(\sum_{j=1}^N \chi_{nj}\theta_j)^{c_n}\in W$.

PROPOSITION 4.4. *We have the following decomposition of the left ideal $I(\chi_0)$:*

$$I(\chi_0)=\bigcap_{\Gamma} I(\Gamma, \chi_0).$$

PROOF. Clearly $I(\chi_0)$ is contained in $\bigcap_\Gamma I(\Gamma, \chi_0)$. Suppose that P belongs to $\bigcap_\Gamma I(\Gamma, \chi_0)$. Since we have $[\sum_{j=1}^N \chi_{ij}\theta_j, \prod_{b_j\geq 0} D_j^{b_j}]=(-\sum_{b_j\geq 0} b_j\chi_{ij})\prod_{b_j\geq 0} D_j^{b_j}$ and $[\sum_{j=1}^N \chi_{ij}\theta_j, \square_a]=(-\sum_{a_j>0} a_j\chi_{ij})\square_a$, $P\in I(\Gamma, \chi_0)$ implies that $P'\in I'(\Gamma, \chi_0)$ for any Γ. We thus see that P' belongs to $I'(\chi_0)$ and accordingly P to $I(\chi_0)$. ■

5. *b*-functions. Let $B(\chi_0)$ be the kernel of the natural morphism $\boldsymbol{C}[s]\to W[s]/I(\chi_0)$. We call a nonzero element of $B(\chi_0)$ a b-function of $M[s]$ with respect to χ_0.

PROPOSITION 5.1. *For a polynomial $b(s)\in B(\chi_0)$ there exists an operator $Q\in W$ such that $b(s)=QD^{\chi_0}$ in $M[s]$.*

The proof is clear. In the situation of Proposition 5.1, we have $b(\alpha)=QD^{\chi_0}$ in M_α for any $\alpha\in\boldsymbol{C}^n$.

LEMMA 5.2. *For $d, e\in\boldsymbol{Z}_{\geq 0}$ and any $1\leq j\leq N$, we have in W*

$$D_j^d v_j^e=\sum_{k=0}^{\min\{d,e\}}\binom{d}{k}\left(\prod_{r=0}^{k-1}(e-r)\right)v_j^{e-k}D_j^{d-k},$$

and

$$\sum_{k=0}^{\min\{d,e\}} \binom{d}{k} \left(\prod_{r=0}^{k-1} (e-r) \right) \left(\prod_{q=0}^{e-k-1} (\theta_j - q) \right) = \prod_{r=0}^{e-1} (\theta_j + d - r) .$$

The proof is omitted.

PROPOSITION 5.3. *Let $d_1, \dots, d_N \in \boldsymbol{Z}_{\geq 0}$, $Q \in W$, and $P \in \boldsymbol{C}[\theta_1, \dots, \theta_N]$. Suppose that we have in $M[s]$*

$$QD_1^{d_1} \cdots D_N^{d_N} = P(\theta_1, \dots, \theta_N) .$$

Then we have in $M[s]$

$$D_1^{d_1} \cdots D_N^{d_N} Q = P(\theta_1 + d_1, \dots, \theta_N + d_N) .$$

PROOF. Let $e_1, \dots, e_{2N} \in \boldsymbol{Z}_{\geq 0}$ satisfy $\sum_{j=1}^N e_j \chi_j = \sum_{j=1}^N (e_{N+j} + d_j) \chi_j$. Then we have in $M[s]$

$$v_1^{e_1} \cdots v_N^{e_N} D_1^{e_{N+1}} \cdots D_N^{e_{2N}} D_1^{d_1} \cdots D_N^{d_N} = v_1^{e_1} D_1^{e_1} \cdots v_N^{e_N} D_N^{e_N} = \prod_{j=1}^N \prod_{r_j=0}^{e_j-1} (\theta_j - r_j) .$$

By Lemma 5.2, we see in $M[s]$

$$D_1^{d_1} \cdots D_N^{d_N} v_1^{e_1} \cdots v_N^{e_N} D_1^{e_{N+1}} \cdots D_N^{e_{2N}} = \prod_{j=1}^N \prod_{r_j=0}^{e_j-1} (\theta_j + d_j - r_j) .$$

Since Q is a linear sum of terms of the form of $v_1^{e_1} \cdots v_N^{e_N} D_1^{e_{N+1}} \cdots D_N^{e_{2N}}$ with the relation $\sum_{j=1}^N e_j \chi_j = \sum_{j=1}^N (e_{N+j} + d_j) \chi_j$, we reach the assertion. ■

COROLLARY 5.4. *Suppose that there exists a polynomial $b(s) \in B(\chi_0)$ such that $b(\alpha) \neq 0$. Then the morphism f_{χ_0}: $M_{\alpha - \chi_0} \to M_\alpha$ is isomorphic.*

PROOF. Let $\chi_0 = \sum_{j=1}^N d_j \chi_j$ with $d_j \in \boldsymbol{Z}_{\geq 0}$ $(j = 1, \dots, N)$. In this case, there exists an operator $Q \in W$ such that

$$QD^{\chi_0} = QD_1^{d_1} \cdots D_N^{d_N} = b(s) = b(s_1, \dots, s_n) = b\left(\sum_{j=1}^N \chi_{1j} \theta_j, \dots, \sum_{j=1}^N \chi_{nj} \theta_j \right)$$

is $M[s]$. By Proposition 5.3, we see that

$$D_1^{d_1} \cdots D_N^{d_N} Q = b\left(\sum_{j=1}^N \chi_{1j} (\theta_j + d_j), \dots, \sum_{j=1}^N \chi_{nj} (\theta_j + d_j) \right) = b(s + \chi_0)$$

in $M[s]$. Hence we obtain $QD^{\chi_0} = b(\alpha) \neq 0$ in M_α, and $D^{\chi_0} Q = b(\alpha - \chi_0 + \chi_0) = b(\alpha) \neq 0$ in $M_{\alpha - \chi_0}$. Therefore the morphism f_{χ_0} is bijective. ■

Let $B(\Gamma, \chi_0)$ be the kernel of the natural morphism $\boldsymbol{C}[s] \to W[s]/I(\Gamma, \chi_0)$. Since we have $I(\chi_0) = \bigcap_\Gamma I(\Gamma, \chi_0)$, we obtain:

LEMMA 5.5.

$$B(\chi_0)=\bigcap_{\Gamma} B(\Gamma, \chi_0).$$

We remark that $B(\Gamma, \chi_0)=\boldsymbol{C}[s]$ for $\chi_0 \in \boldsymbol{Z}_{\geq 0}\Gamma$. Suppose that χ_0 does not belong to $\boldsymbol{Z}_{\geq 0}\Gamma$. For $m \in \boldsymbol{Z}_{\geq 0}$ we denote by $\Theta(\Gamma, m)$ the ideal of $\boldsymbol{C}[\theta_j | \chi_j \notin \Gamma]$ generated by all $\prod_{b_j>0} \theta_j(\theta_j-1)\cdots(\theta_j-b_j+1)$ for $\sum_{b_j \geq 0} b_j F_\Gamma(\chi_j) \geq m$. Clearly $\Theta(\Gamma, F_\Gamma(\chi_0))$ is contained in $I(\Gamma, \chi_0)$. For $\chi_j \notin \Gamma$ there exists an integer $c_j>0$ such that $c_j F_\Gamma(\chi_j) \geq m$, and thus $\theta_j(\theta_j-1)\cdots(\theta_j-c_j+1)$ belongs to $\Theta(\Gamma, m)$. Consequently, we see that the zero set $V(\Theta(\Gamma, m))$ is a finite set contained in $(\boldsymbol{Z}_{\geq 0})^{|I(\Gamma)|}$, and the multiplicity of $\boldsymbol{C}[\theta_j | \chi_j \notin \Gamma]/\Theta(\Gamma, m)$ at each point of $V(\Theta(\Gamma, m))$ is one. Therefore $\Theta(\Gamma, m)$ is a radical ideal. We define a finite subset $Z(\Gamma, m)$ of $\boldsymbol{Z}_{\geq 0}$ by

$$Z(\Gamma, m):=\left\{\sum_{\chi_j \notin \Gamma} v_j F_\Gamma(\chi_j) \in \boldsymbol{Z}_{\geq 0} \,\middle|\, v \in V(\Theta(\Gamma, m))\right\}.$$

PROPOSITION 5.6. *The polynomial* $b(\Gamma, \chi_0) \in \boldsymbol{C}[s]$ *defined by*

$$b(\Gamma, \chi_0):=\prod_{z \in Z(\Gamma, F_\Gamma(\chi_0))} (F_\Gamma(s)-z)$$

belongs to $B(\Gamma, \chi_0)$.

PROOF. We denote by $b(\theta)$ the polynomial $\prod_{z \in Z(\Gamma, F_\Gamma(\chi_0))}(\sum_{\chi_j \notin \Gamma} F_\Gamma(\chi_j)\theta_j - z)$ in $\boldsymbol{C}[\theta_j | \chi_j \notin \Gamma]$. Then we see that $b(v)=0$ for all $v \in V(\Theta(\Gamma, F_\Gamma(\chi_0)))$. Since $\Theta(\Gamma, F_\Gamma(\chi_0))$ is a radical ideal, the polynomial $b(\theta)$ belongs to $\Theta(\Gamma, F_\Gamma(\chi_0))$, in particular, to $I(\Gamma, \chi_0)$. Since $b(\Gamma, \chi_0)=b(\theta)$ in $M[s]$, we conclude that $b(\Gamma, \chi_0) \in B(\Gamma, \chi_0)$. ■

COROLLARY 5.7. *We define a polynomial* $b_{\chi_0} \in \boldsymbol{C}[s]$ *by* $b_{\chi_0}:=\prod_\Gamma b(\Gamma, \chi_0)$. *Then the polynomial* b_{χ_0} *belongs to* $B(\chi_0)$.

The proof is clear.

COROLLARY 5.8. *Let* $j_0 \in \{1, \ldots, N\}$. *Assume that for any* $a \in L$ *and any face* Γ *of codimension one not containing* χ_{j_0} *we have either* $\sum_{a_j>0} a_j F_\Gamma(\chi_j)=0$ *or* $\sum_{a_j>0} a_j F_\Gamma(\chi_j) \geq F_\Gamma(\chi_{j_0})$. *Then the morphism* $f_{\chi_{j_0}}\colon M_{\alpha-\chi_{j_0}} \to M_\alpha$ *is isomorphic if and only if* $b_{\chi_{j_0}}(\alpha) \neq 0$.

PROOF. Suppose that $b_{\chi_{j_0}}(\alpha)=0$. Then there exists a face Γ of Q of codimension one not containing j_0 with $b(\Gamma, \chi_{j_0})(\alpha)=0$. Hence there exists $z \in Z(\Gamma, F_\Gamma(\chi_{j_0}))$ such that $F_\Gamma(\alpha)=z$. In other words, there exists $v=(v_j)_{j \in I(\Gamma)} \in V(\Theta(\Gamma, F_\Gamma(\chi_{j_0})))$ such that $F_\Gamma(\alpha)=\sum_{j \in I(\Gamma)} v_j F_\Gamma(\chi_j)$. Define $v'=(v'_j)_{j=1}^N \in \boldsymbol{Z}^N$ by $v'_j=v_j+1$ for $j \in I(\Gamma)$ and $v'_j=0$ for $j \notin I(\Gamma)$. Under the assumption, the condition $v \in V(\Theta(\Gamma, F_\Gamma(\chi_{j_0})))$ implies that v' is a quotient point associated to $I(\Gamma)$. By Theorem 2.3, the morphism $f_{\chi_{j_0}}$ is not isomorphic.

When $b_{\chi_{j_0}}(\alpha) \neq 0$, the morphism $f_{\chi_{j_0}}$ is isomorphic by Corollary 5.4 and Corollary 5.7. ∎

6. The set $Z(\Gamma, m)$.

LEMMA 6.1. *The set $Z(\Gamma, m)$ is contained in $\{0, 1, \dots, m-1\}$.*

PROOF. We use induction on m. When $m=1$, it is clear that $\Theta(\Gamma, 1)$ contains θ_i for any $i \in I(\Gamma)$. We thus see that $V(\Theta(\Gamma, 1)) = \{(0, \dots, 0)\}$ and $Z(\Gamma, 1) = \{0\}$.

Let $v=(v_i;\, i \in I(\Gamma))$ belong to $V(\Theta(\Gamma, m))$. Suppose that $v_{i_0} \neq 0$ for some $i_0 \in I(\Gamma)$. We define $v' \in V(\Theta(\Gamma, m))$ by $v'_{i_0}=0$ and $v'_i = v_i$ for all $i \in I(\Gamma)-\{i_0\}$. If $F_\Gamma(\sum_{i \in I(\Gamma)-\{i_0\}} b_i \chi_i) \geq m - v_{i_0} F_\Gamma(\chi_{i_0})$, then $F_\Gamma(\sum_{i \in I(\Gamma)-\{i_0\}} b_i \chi_i + v_{i_0} \chi_{i_0}) \geq m$, and thus $\theta_{i_0}(\theta_{i_0}-1)\cdots(\theta_{i_0}-v_{i_0}+1) \times \prod_{i \in I(\Gamma)-\{i_0\}} \theta_i(\theta_i-1)\cdots(\theta_i-b_i+1)$ belongs to $\Theta(\Gamma, m)$. Hence we obtain $\prod_{i \in I(\Gamma)-\{i_0\}} v_i(v_i-1)\cdots(v_i-b_i+1)=0$. We thus see that $v' \in V(\Theta(\Gamma, m-v_{i_0}F_\Gamma(\chi_{i_0})))$. By the induction hypothesis, $\sum_{i \neq i_0} v_i F_\Gamma(\chi_i)$ belongs to $\{0, 1, \dots, m-v_{i_0}F_\Gamma(\chi_{i_0})-1\}$. Therefore the sum $\sum_{i \in I(\Gamma)} v_i F_\Gamma(\chi_i)$ belongs to $\{v_{i_0}F_\Gamma(\chi_{i_0}), v_{i_0}F_\Gamma(\chi_{i_0})+1, \dots, m-1\}$. ∎

LEMMA 6.2. *Fix a face Γ of codimension one. Then there exists $k \in \{1, \dots, N\}$ such that $F_\Gamma(\chi_k)=1$.*

PROOF. Since the greatest common divisor of the coefficients of F_Γ is one, there exists $\chi \in \mathbf{Z}^n$ such that $F_\Gamma(\chi)=1$. If necessary, translate χ by an element of $\mathbf{Z}^n \cap (F_\Gamma=0) \cap \bigcap_{\Gamma' \neq \Gamma}(F_{\Gamma'} \geq 0)$, and we see that there exists $\chi \in \Lambda$ such that $F_\Gamma(\chi)=1$. By the normality assumption, we conclude that there exists $k \in \{1, \dots, N\}$ such that $F_\Gamma(\chi_k)=1$. ∎

LEMMA 6.3.

$$Z(\Gamma, m) = \{0, 1, \dots, m-1\}\,.$$

PROOF. Suppose that $F_\Gamma(\chi_k)=1$ and $j \in \{0, 1, \dots, m-1\}$. Define $v \in (\mathbf{Z}_{\geq 0})^{|I(\Gamma)|}$ by $v_k = j$ and $v_i = 0$ for all $i \in I(\Gamma)-\{k\}$. Then $v \in V(\Theta(\Gamma, m))$. Hence j belongs to the set $Z(\Gamma, m)$. ∎

THEOREM 6.4. *The ideal $B(\chi_0)$ is singly generated by the polynomial b_{χ_0}.*

PROOF. Let $\alpha \in \mathbf{C}^n$ satisfy $F_{\Gamma'}(\alpha) \notin \mathbf{Z}_{\geq 0}$ for any face Γ' of codimension one different from Γ. Suppose that $F_\Gamma(\chi_k)=1$. Since $F_\Gamma(\chi_0 - F_\Gamma(\chi_0)\chi_k)=0$, we see that $\chi_0 - F_\Gamma(\chi_0)\chi_k$ belongs to $\mathbf{Z}\Gamma$. Hence the morphism $f_{\chi_0}\colon M_{\alpha-\chi_0} \to M_\alpha$ is isomorphic if and only if so is $f_k^{F_\Gamma(\chi_0)}$. Consequently, f_{χ_0} is isomorphic if and only if $F_\Gamma(\alpha) \neq 0, 1, \dots, F_\Gamma(\chi_0)-1$. ∎

REMARK (cf. [S2]). When we are given an example explicitly, we can calculate not only the b-functions but also operators Q in the notation of Proposition 5.1. This calculation gives us the contiguity relations which generalize the relations of the following type:

$$(c-a)F(a-1, b; c; x)=\left\{x(1-x)\frac{d}{dx}-bx+c-a\right\}F(a, b; c; x),$$

where F is the classical hypergeometric function.

7. Examples. All of the following examples satisfy the normality assumption (see [S1]). We denote f_j (resp. b_j) instead of f_{χ_j} (resp. b_{χ_j}).

EXAMPLE 1. Let $V=\boldsymbol{C}^{2p}$, and

$$M_{\alpha\beta}=W\Big/\!\!\Big/\left(\sum_{i=1}^{p}W(\theta_i+\theta_{2p}-\alpha_i)+\sum_{i=1}^{p-1}W(\theta_{p+i}-\theta_{2p}-\beta_i)\right.$$
$$\left.+W(D_1\cdots D_p-D_{p+1}\cdots D_{2p})\right).$$

(1) Let $1\le i\le p$. Then $b_i(\alpha, \beta)=\alpha_i(\alpha_i+\beta_1)(\alpha_i+\beta_2)\cdots(\alpha_i+\beta_{p-1})$, and f_i is isomorphic if and only if $\alpha_i\ne 0$, $\alpha_i+\beta_1\ne 0, \dots, \alpha_i+\beta_{p-1}\ne 0$.

(2) Let $1\le i\le p-1$. Then $b_{p+i}(\alpha, \beta)=(\alpha_1+\beta_i)(\alpha_2+\beta_i)\cdots(\alpha_p+\beta_i)$, and f_{p+i} is isomorphic if and only if $\alpha_1+\beta_i\ne 0, \dots, \alpha_p+\beta_i\ne 0$.

(3) $b_{2p}(\alpha, \beta)=\alpha_1\alpha_2\cdots\alpha_p$, and f_{2p} is isomorphic if and only if $\alpha_1\ne 0, \dots, \alpha_p\ne 0$.

EXAMPLE 2. Let $V=\boldsymbol{C}^{(k+1)l}=\{(v_{ij})\,|\,1\le i\le l, 0\le j\le k\}$ and

$$M_{\alpha\beta}=W\Big/\!\!\Big/\left(\sum_{j=1}^{k}W\left(\sum_{i=1}^{l}\theta_{ij}-\alpha_j\right)+\sum_{i=1}^{l}W\left(\sum_{j=0}^{k}\theta_{ij}-\beta_i\right)+\sum_{i\ne i', j\ne j'}W(D_{ij}D_{i'j'}-D_{ij'}D_{i'j})\right).$$

We put $\alpha_0=\sum_{i=1}^{l}\beta_i-\sum_{j=1}^{k}\alpha_j$. Then $b_{ij}(\alpha, \beta)=\alpha_j\beta_i$, and f_{ij} is isomorphic if and only if $\alpha_j\ne 0$ and $\beta_i\ne 0$.

EXAMPLE 3. Let $V=\boldsymbol{C}^{n(n-1)/2}=\{(v_{ij})\,|\,1\le i<j\le n\}$ $(n\ge 4)$, and

$$M_{\alpha}=W\Big/\!\!\Big/\left(\sum_{k=1}^{n}W\left(\sum_{i=1}^{k-1}\theta_{ik}+\sum_{j=k+1}^{n}\theta_{kj}-\alpha_k\right)+\sum_{1\le i<j<k<l\le n}W(D_{ij}D_{kl}-D_{ik}D_{jl})\right.$$
$$\left.+\sum_{1\le i<j<k<l\le n}W(D_{ik}D_{jl}-D_{il}D_{jk})+\sum_{1\le i<j<k<l\le n}W(D_{ij}D_{kl}-D_{il}D_{jk})\right).$$

Then $2^{n-2}\cdot b_{st}(\alpha)=\alpha_s\alpha_t\prod_{k\ne s,t}(\sum_{i\ne k}\alpha_i-\alpha_k)$. f_{st} is isomorphic if and only if $\alpha_s\ne 0$, $\alpha_t\ne 0$ and $\sum_{i\ne k}\alpha_i-\alpha_k\ne 0$ for any $k\ne s, t$.

EXAMPLE 4. Let $V=\boldsymbol{C}^{n(n+1)/2}=\{(v_{ij})\,|\,1\le i\le j\le n\}$ $(n\ge 2)$, and

$$M_{\alpha}=W\Big/\!\!\Big/\left(\sum_{k=1}^{n}W\left(\sum_{i=1}^{k}\theta_{ik}+\sum_{j=k}^{n}\theta_{kj}-\alpha_k\right)+\sum_{1\le i\le j<k\le n}W(D_{ij}D_{kk}-D_{ik}D_{jk})\right.$$

$$+\sum_{1\le i<j\le k\le n} W(D_{ii}D_{jk}-D_{ij}D_{ik})+\sum_{1\le i<j\le k<l\le n} W(D_{ik}D_{jl}-D_{jk}D_{il})\Bigg).$$

(1) $b_{ss}(\alpha)=\alpha_s(\alpha_s-1)$, and f_{ss} is isomorphic if $\alpha_s\neq 0$, 1, and not isomorphic if $\alpha_s=0$.
(2) $b_{st}(\alpha)=\alpha_s\alpha_t$ for $s<t$, and f_{st} $(s<t)$ is isomorphic if and only if α_s, $\alpha_t\neq 0$.

EXAMPLE 5. Let $V=C^{2n-2}=\{(v_i)\,|\,i=\pm 1, \pm 2, \dots, \pm(n-1)\}$ $(n\geq 4)$ and

$$M_\alpha=W\Bigg/\left(\sum_{i=1}^{n-1} W(\theta_i-\theta_{-i}-\alpha_i)+W\left(\sum_{i=1}^{n-1}(\theta_i+\theta_{-i})-\alpha_n\right)+\sum_{i\neq\pm j} W(D_iD_{-i}-D_jD_{-j})\right).$$

For a subset I of $\{1, 2, \dots, n-1\}$, we denote by I' the complement of I.

(1) $2^{2^{n-2}}\cdot b_s(\alpha)=\prod_{I\ni s}(\alpha_n+\sum_{i\in I}\alpha_i-\sum_{i\in I'}\alpha_i)$ for $s>0$. f_s $(s>0)$ is isomorphic if and only if $\alpha_n+\sum_{i\in I}\alpha_i-\sum_{i\in I'}\alpha_i\neq 0$ for any $I\ni s$.

(2) $2^{2^{n-2}}\cdot b_{-s}(\alpha)=\prod_{I\ni s}(\alpha_n+\sum_{i\in I'}\alpha_i-\sum_{i\in I}\alpha_i)$ for $s>0$. f_{-s} $(s>0)$ is isomorphic if and only if $\alpha_n+\sum_{i\in I'}\alpha_i-\sum_{i\in I}\alpha_i\neq 0$ for any $I\ni s$.

EXAMPLE 6. Let $V=C^{2n-1}=\{(v_i)\,|\,-(n-1)\leq i\leq(n-1)\}$ $(n\geq 2)$ and

$$M_\alpha=W\Bigg/\left(\sum_{i=1}^{n-1} W(\theta_i-\theta_{-i}-\alpha_i)+W\left(\left(\sum_{-(n-1)\le i\le n-1}\theta_i\right)-\alpha_n\right)+\sum_{i=1}^{n-1} W(D_0^2-D_iD_{-i})\right).$$

As in Example 5, I' denotes the complement of I in $\{1, 2, \dots, n-1\}$.

(1) $b_0(\alpha)=\prod_I(\alpha_n+\sum_{i\in I}\alpha_i-\sum_{i\in I'}\alpha_i)$, and f_0 is isomorphic if and only if $\alpha_n+\sum_{i\in I}\alpha_i-\sum_{i\in I'}\alpha_i\neq 0$ for any subset I of $\{1, \dots, n-1\}$.

(2) $b_s(\alpha)=\prod_{I\ni s}(\alpha_n+\sum_{i\in I}\alpha_i-\sum_{i\in I'}\alpha_i)(\alpha_n+\sum_{i\in I}\alpha_i-\sum_{i\in I'}\alpha_i-1)$ for $s>0$. f_s $(s>0)$ is isomorphic if and only if $\alpha_n+\sum_{i\in I}\alpha_i-\sum_{i\in I'}\alpha_i\neq 0$, 1 for any $I\ni s$.

(3) $b_{-s}(\alpha)=\prod_{I\ni s}(\alpha_n+\sum_{i\in I'}\alpha_i-\sum_{i\in I}\alpha_i)(\alpha_n+\sum_{i\in I'}\alpha_i-\sum_{i\in I}\alpha_i-1)$ for $s>0$. f_{-s} $(s>0)$ is isomorphic if and only if $\alpha_n+\sum_{i\in I'}\alpha_i-\sum_{i\in I}\alpha_i\neq 0$, 1 for any $I\ni s$.

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