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HOLOMORPHIC CONFORMAL STRUCTURES AND CHARACTERISTIC FORMS

Dedicated to Professor Tadashi Nagano on his sixtieth birthday

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Abstract. On a complex manifold of dimension more than two which admits a holomorphic conformal structure, we define *conformal Weyl forms*, a kind of characteristic forms, by means of the holomorphic conformal Weyl curvature tensor, and prove a formula which relates these forms with Chern forms.

Our result is a conformal analogue in the holomorphic case of our previous result [Kt] on projective connections, and gives a more precise description of a theorem of Kobayashi-Ochiai [KO, Theorem 3.20] in the case of dimension more than two. At present, we do not know whether a similar formula exists in the general case where the manifold admits only differentiable conformal structures.

In Section 1, we shall give the definition of holomorphic conformal structures and holomorphic conformal connections. In Section 2, we shall calculate the conformal Weyl curvature tensor explicitly. The process of the calculation will be used in Section 3. In Section 3, we shall prove our main result (Theorem 3.2). I would like to express my sincere gratitude to Professor Tadashi Nagano who suggested to me that there would be a conformal analogue of my previous result [Kt, Theorem 3.1].

1. Holomorphic conformal structures. In this section, we shall give the definitions of holomorphic conformal structures and holomorphic conformal connections together with some preparations for later sections. Let X be a complex manifold of dimension $n \geq 1$. Take a locally finite open covering $\mathscr{U} = \{U_\alpha\}$ of X so that on each U_α, there is a system of local coordinates $z_\alpha = (z_\alpha^1, z_\alpha^2, \dots, z_\alpha^n)$.

Put

$$\varphi_{\alpha\beta} = z_\alpha \circ z_\beta^{-1}$$

and denote by $\tau_{\alpha\beta}$ the Jacobian matrix of $\varphi_{\alpha\beta}$. On $U_\alpha \cap U_\beta$, we consider an $n \times n$-matrix-valued holomorphic 1-form

$$a_{\alpha\beta} = \tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta}$$

and a scalar-valued holomorphic 1-form

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$$\sigma_{\alpha\beta}=\frac{1}{n}\,\mathrm{Trace}\,a_{\alpha\beta}=\frac{1}{n}\,d\log\det\tau_{\alpha\beta}\,.$$

We write $\sigma_{\alpha\beta}$ as

$$\sigma_{\alpha\beta}=\sigma_{\alpha\beta j}dz_{\beta}^{j}$$

and define another $n\times n$-matrix-valued holomorphic 1-form $\rho_{\alpha\beta}=(\rho_{\alpha\beta k}^{j})$ by

$$\rho_{\alpha\beta k}^{j}=\sigma_{\alpha\beta k}dz_{\beta}^{j}\,.$$

It is well-known and easy to check that the sets $\{a_{\alpha\beta}\}$, $\{\sigma_{\alpha\beta}\}$, $\{\rho_{\alpha\beta}\}$ are 1-cocycles which define elements of

$$H^1(X,\Omega^1(\mathrm{End}(\Theta)))\,,\quad H^1(X,\Omega^1)\,,\quad \text{and}\quad H^1(X,\Omega^1(\mathrm{End}(\Theta)))\,,$$

respectively.

We say that a complex manifold X of dimension $n\geq 1$ admits a *holomorphic conformal structure* if the structure group of the tangent bundle reduces as a holomorphic bundle to the conformal group $CO(n, \boldsymbol{C})$. Let $S\subset GL(n, \boldsymbol{C})/\boldsymbol{C}^*$ be the set of non-singular symmetric matrices factored by the non-zero scalar matrices. We form a holomorphic fibre bundle

$$Z=\left(\bigcup_{\alpha} U_\alpha\times S\right)\Big/\sim$$

on X with the typical fibre S by identifying $(z_\alpha, s_\alpha)\in U_\alpha\times S$ with $(z_\beta, s_\beta)\in U_\beta\times S$ if and only if $z_\alpha=z_\beta$ and $s_\beta={}^t\tau_{\alpha\beta}s_\alpha\tau_{\alpha\beta}$. Let $\pi\colon Z\to X$ be the natural projection. That X admits a holomorphic conformal structure is equivalent to saying the π admits a holomorphic section. A holomorphic section g of π is also called a holomorphic conformal structure of X.

Now suppose that X admits a holomorphic conformal structure g. Then on each U_α, g is represented by a holomorphic symmetric (2, 0)-form

$$g_\alpha=g_{\alpha ij}dz_\alpha^i dz_\alpha^j$$

such that

(1) $$g_\beta=f_{\beta\alpha}g_\alpha \qquad \text{on} \quad U_\alpha\cap U_\beta\,,$$

where $f_{\beta\alpha}$ is a nowhere vanishing holomorphic function defined on $U_\alpha\cap U_\beta$ and

(2) $$\det(g_{\alpha ij}(x))\neq 0 \qquad \text{for all} \quad x\in U_\alpha\,.$$

Let F be the holomorphic line bundle on X formed by the 1-cocycle $\{f_{\alpha\beta}\}$. Then $\{g_\alpha\}$ can be regarded as an element of $\Gamma(X, \mathscr{S}^2(\Omega^1)\otimes F)$, where $\mathscr{S}^2(\Omega^1)$ indicates the second symmetric power of Ω^1. Note that two sections $\{g_\alpha\}$ and $\{h_\alpha\}$ in $\Gamma(X, \mathscr{S}^2(\Omega^1)\otimes F)$ represent the same conformal structure if and only if on each U_α

there is a nowhere vanishing holomorphic function f_α such that $g_\alpha = f_\alpha h_\alpha$. Put

$$G_\alpha = (g_{\alpha ij}),$$

where the (i, j)-component is given by $g_{\alpha ij}$. By (1), we have

$$g_{\beta rs} = f_{\beta\alpha} g_{\alpha ij} \tau^i_{\alpha\beta r} \tau^j_{\alpha\beta s} \quad \text{on} \quad U_\alpha \cap U_\beta . \tag{3}$$

It follows easily from (2) and (3) that the first Chern class of X has a following property.

PROPOSITION 1.1 (cf. [KO]). $nc_1[F] + 2c_1(X) = 0$.

We put

$$\rho^*_{\alpha\beta} = -G_\beta^{-1}\, {}^t\rho_{\alpha\beta} G_\beta ,$$

or

$$\rho^{*j}_{\alpha\beta k} = -g^{jr}_\beta \rho^s_{\alpha\beta r} g_{\beta sk} ,$$

where g^{jk}_β is the (j, k)-component of G_β^{-1}. Then on $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$\rho^*_{\alpha\gamma} = \tau^{-1}_{\beta\gamma} \rho^*_{\alpha\beta} \tau_{\beta\gamma} + \rho^*_{\beta\gamma} .$$

This implies that the set $\{\rho^*_{\alpha\beta}\}$ defines a 1-cocycle in $Z^1(X, \Omega^1 \otimes \mathrm{End}\, \Theta)$. Now we shall define a 1-cocyle $\{c_{\alpha\beta}\}$ by

$$c_{\alpha\beta} = a_{\alpha\beta} - \rho_{\alpha\beta} - \rho^*_{\alpha\beta} - \sigma_{\alpha\beta} I ,$$

which is also an element of $Z^1(X, \Omega^1 \otimes \mathrm{End}\, \Theta)$. As we see by the following argument, *the cohomology class represented by* $\{c_{\alpha\beta}\}$ *turns out to be zero.*

By means of the representative $\{g_\alpha\}$ of g, we can construct explicitly a 0-cochain $\{c_\alpha\} \in C^0(\mathscr{U}, \Omega^1 \otimes \mathrm{End}\, \Theta)$ whose coboundary coincides with the 1-cocycle $\{c_{\alpha\beta}\}$, i.e.,

$$c_\beta = c_{\alpha\beta} + \tau^{-1}_{\alpha\beta} c_\alpha \tau_{\alpha\beta} . \tag{4}$$

The 0-cochain $\{c_\alpha\}$ is called a *holomorphic conformal connection* of X. We define the Christoffel symbols $\left\{ \begin{matrix} l \\ ij \end{matrix} \right\}_\alpha$ associated with the symmetric tensor g_α on U_α by

$$\left\{ \begin{matrix} l \\ ij \end{matrix} \right\}_\alpha = \frac{1}{2} g^{lk}_\alpha \left(\frac{\partial g_{\alpha ik}}{\partial z^j_\alpha} + \frac{\partial g_{\alpha jk}}{\partial z^i_\alpha} - \frac{\partial g_{\alpha ij}}{\partial z^k_\alpha} \right).$$

The conformal connection $\{c_\alpha\}$ associated with the conformal structure $\{g_\alpha\}$ is defined by

$$c_\alpha = (c^l_{\alpha i}),$$

$$c^l_{\alpha i} = c^l_{\alpha ij} dz^j_\alpha,$$

$$c^l_{\alpha ij} = \left\{ \begin{matrix} l \\ ij \end{matrix} \right\}_\alpha - \frac{\delta^l_i}{n} \left\{ \begin{matrix} a \\ aj \end{matrix} \right\}_\alpha - \frac{\delta^l_j}{n} \left\{ \begin{matrix} a \\ ai \end{matrix} \right\}_\alpha + \frac{1}{n} g^{lb}_\alpha g_{\alpha ij} \left\{ \begin{matrix} a \\ ab \end{matrix} \right\}_\alpha .$$

Then by a direct calculation, we have

LEMMA 1.1.

$$c^j_{\alpha ij} = 0, \tag{5}$$

$$c^l_{\alpha ij} = c^l_{\alpha ji}, \tag{6}$$

$$c_\beta = c_{\alpha\beta} + \tau^{-1}_{\alpha\beta} c_\alpha \tau_{\alpha\beta}. \tag{7}$$

REMARK 1.1. In view of (7), we see easily that the cochain $\{c_\alpha\}$ is determined by g and is independent of the choice of $\{g_\alpha\}$.

2. Conformal Weyl curvature tensors. From this section, we assume that $n = \dim X \geq 3$. Using the notation in Section 1, we shall calculate the conformal Weyl curvature tensor W_α on each U_α associated with a holomorphic conformal structure g. In the proof of Proposition 3.1 in the next section, we shall make use of the following tensor calculation, which is due to Thomas [T1], [T2]. Put

$$e_{\alpha\beta} = c_\beta \rho_{\alpha\beta} + \rho_{\alpha\beta} c_\beta + d\rho_{\alpha\beta} + \rho_{\alpha\beta} \wedge \rho_{\alpha\beta},$$

$$e^*_{\alpha\beta} = c_\beta \rho^*_{\alpha\beta} + \rho^*_{\alpha\beta} c_\beta + d\rho^*_{\alpha\beta} + \rho^*_{\alpha\beta} \wedge \rho^*_{\alpha\beta},$$

and

$$\Delta_{\alpha\beta} = e_{\alpha\beta} + e^*_{\alpha\beta} + \rho_{\alpha\beta} \wedge \rho^*_{\alpha\beta} + \rho^*_{\alpha\beta} \wedge \rho_{\alpha\beta}. \tag{8}$$

Then from (4) and (7), if follows that

$$dc_\beta + c_\beta \wedge c_\beta = \tau^{-1}_{\alpha\beta} (dc_\alpha + c_\alpha \wedge c_\alpha) \tau_{\alpha\beta} - \Delta_{\alpha\beta}. \tag{9}$$

Put

$$F_\alpha = dc_\alpha + c_\alpha \wedge c_\alpha .$$

Then the equation (9) is equivalent to

$$F_\beta = \tau^{-1}_{\alpha\beta} F_\alpha \tau_{\alpha\beta} - \Delta_{\alpha\beta}. \tag{10}$$

We denote the (j, k)-components of c_α, F_α, $e_{\alpha\beta}$ and $\Delta_{\alpha\beta}$ by $c^j_{\alpha k}$, $F^j_{\alpha k}$, $e^j_{\alpha\beta k}$ and $\Delta^j_{\alpha\beta k}$, respectively. Put

$$c^j_{\alpha k} = c^j_{\alpha kr} dz^r_\alpha ,$$
$$F^j_{\alpha k} = F^j_{\alpha krs} dz^r_\alpha \wedge dz^s_\alpha ,$$
$$e^j_{\alpha\beta k} = e^j_{\alpha\beta krs} dz^r_\alpha \wedge dz^s_\alpha ,$$
$$e^{*j}_{\alpha\beta k} = e^{*j}_{\alpha\beta krs} dz^r_\beta \wedge dz^s_\beta ,$$
$$\Delta^j_{\alpha\beta k} = \Delta^j_{\alpha\beta krs} dz^r_\beta \wedge dz^s_\beta .$$

Then we have

(11)
$$2F^j_{\alpha klm} = \frac{\partial c^j_{\alpha km}}{\partial z^l_\alpha} - \frac{\partial c^j_{\alpha kl}}{\partial z^m_\alpha} + c^j_{\alpha rl} c^r_{\alpha km} - c^j_{\alpha rm} c^r_{\alpha kl} ,$$

$$2e^j_{\alpha\beta klm} = \delta^j_l \left(\sigma_{\alpha\beta r} c^r_{\beta km} - \frac{\partial \sigma_{\alpha\beta k}}{\partial z^m_\beta} + \sigma_{\alpha\beta m}\sigma_{\alpha\beta k} \right) - \delta^j_m \left(\sigma_{\alpha\beta r} c^r_{\beta kl} - \frac{\partial \sigma_{\alpha\beta k}}{\partial z^l_\beta} + \sigma_{\alpha\beta l}\sigma_{\alpha\beta k} \right),$$

(12)
$$2e^{*j}_{\alpha\beta klm} = c^j_{\beta rm} g^{rs}_\beta \sigma_{\alpha\beta s} g_{\beta lk} - c^j_{\beta rl} g^{rs}_\beta \sigma_{\alpha\beta s} g_{\beta mk} + g^{jr}_\beta \sigma_{\alpha\beta r} g_{\beta ms} c^s_{\beta kl} - g^{jr}_\beta \sigma_{\alpha\beta r} g_{\beta ls} c^s_{\beta km}$$
$$+ \frac{\partial}{\partial z^m_\beta} (g^{jr}_\beta \sigma_{\alpha\beta r} g_{\beta lk}) - \frac{\partial}{\partial z^l_\beta} (g^{jr}_\beta \sigma_{\alpha\beta r} g_{\beta mk}) - g^{jr}_\beta \sigma_{\alpha\beta r} \sigma_{\alpha\beta m} g_{\beta lk} + g^{jr}_\beta \sigma_{\alpha\beta r} \sigma_{\alpha\beta l} g_{\beta mk} .$$

By a direct calculation, we have

LEMMA 2.1.

$$\frac{\partial}{\partial z^m_\beta} (g^{jr}_\beta g_{\beta lk}) - \frac{\partial}{\partial z^l_\beta} (g^{jr}_\beta g_{\beta mk}) = g^{jr}_\beta g_{\beta sl} c^s_{\beta km} - g^{jr}_\beta g_{\beta sm} c^s_{\beta kl} + g^{sj}_\beta g_{\beta km} c^r_{\beta sl} - g^{sj}_\beta g_{\beta kl} c^r_{\beta sm}$$
$$+ g^{sr}_\beta g_{\beta km} c^j_{\beta sl} - g^{sr}_\beta g_{\beta kl} c^j_{\beta sm} .$$

By Lemma 2.1, (12) can be written as

$$2e^{*j}_{\alpha\beta klm} = g^{jr}_\beta g_{\beta mk} \left(\sigma_{\alpha\beta s} c^s_{\beta rl} - \frac{\partial \sigma_{\alpha\beta r}}{\partial z^l_\beta} + \sigma_{\alpha\beta r}\sigma_{\alpha\beta l} \right) - g^{jr}_\beta g_{\beta lk} \left(\sigma_{\alpha\beta s} c^s_{\beta rm} - \frac{\partial \sigma_{\alpha\beta r}}{\partial z^m_\beta} + \sigma_{\alpha\beta r}\sigma_{\alpha\beta m} \right).$$

Put

(13)
$$2e_{\alpha\beta jk} = \sigma_{\alpha\beta s} c^s_{\beta jk} - \frac{\partial \sigma_{\alpha\beta j}}{\partial z^k_\beta} + \sigma_{\alpha\beta j}\sigma_{\alpha\beta k} .$$

Then we have

$$e^j_{\alpha\beta klm} = \delta^j_l e_{\alpha\beta km} - \delta^j_m e_{\alpha\beta kl} ,$$
$$e^{*j}_{\alpha\beta klm} = g^{jr}_\beta g_{\beta mk} e_{\alpha\beta rl} - g^{jr}_\beta g_{\beta lk} e_{\alpha\beta rm} .$$

Note that $\rho^*_{ab} \wedge \rho_{ab} = 0$ (cf. (28)). By (8), it follows that

$$(14)\qquad \Delta^{j}_{\alpha\beta klm}=\delta^{j}_{l}e_{\alpha\beta km}-\delta^{j}_{m}e_{\alpha\beta kl}+g^{jr}_{\beta}g_{\beta km}e_{\alpha\beta rl}-g^{jr}_{\beta}g_{\beta kl}e_{\alpha\beta rm}-\delta^{j}_{l}\sigma_{\alpha\beta r}g^{rs}_{\beta}\sigma_{\alpha\beta s}g_{\beta km}+\delta^{j}_{m}\sigma_{\alpha\beta r}g^{rs}_{\beta}\sigma_{\alpha\beta s}g_{\beta kl}\,.$$

We set

$$(15)\qquad \Delta_{\alpha\beta kl}=\Delta^{m}_{\alpha\beta klm}\,,$$

$$(16)\qquad F_{\beta kl}=F^{m}_{\beta klm}\,,$$

$$(17)\qquad \Phi_{\beta}=g^{kl}_{\beta}F_{\beta kl}\,,$$

$$(18)\qquad \phi_{\alpha\beta}=g^{rs}_{\beta}\sigma_{\alpha\beta r}\sigma_{\alpha\beta s}\,.$$

From (14),

$$(19)\qquad \Delta_{\alpha\beta kl}=-(n-2)e_{\alpha\beta kl}-g_{\beta kl}g^{rs}_{\beta}e_{\alpha\beta rs}+(n-1)g_{\beta kl}\phi_{\alpha\beta}\,.$$

On the other hand, contraction of (10) gives

$$(20)\qquad F_{\beta kl}=F_{\alpha rs}\tau^{r}_{\alpha\beta k}\tau^{s}_{\alpha\beta l}-\Delta_{\alpha\beta kl}\,.$$

Hence we have

$$(n-2)e_{\alpha\beta kl}+g_{\beta kl}g^{rs}_{\beta}e_{\alpha\beta rs}=F_{\beta kl}-F_{\alpha rs}\tau^{r}_{\alpha\beta k}\tau^{s}_{\alpha\beta l}+(n-1)g_{\beta kl}\phi_{\alpha\beta}\,.$$

Multiplying the above equation by g^{kl}_{β}, we obtain

$$(21)\qquad 2(n-1)g^{kl}_{\beta}e_{\alpha\beta kl}=\Phi_{\beta}-\Phi_{\alpha}f_{\alpha\beta}+n(n-1)\phi_{\alpha\beta}\,.$$

Substitution from (21) in (19) gives

$$\Delta_{\alpha\beta kl}=-(n-2)e_{\alpha\beta kl}-\frac{1}{2(n-1)}g_{\beta kl}(\Phi_{\beta}-\Phi_{\alpha}f_{\alpha\beta})+\frac{n-2}{2}g_{\beta kl}\phi_{\alpha\beta}\,.$$

From this equality and (20), it follows that

$$F_{\beta kl}=F_{\alpha rs}\tau^{r}_{\alpha\beta k}\tau^{s}_{\alpha\beta l}+(n-2)e_{\alpha\beta kl}+\frac{1}{2(n-1)}g_{\beta kl}(\Phi_{\beta}-\Phi_{\alpha}f_{\alpha\beta})-\frac{n-2}{2}g_{\beta kl}\phi_{\alpha\beta}\,.$$

Thus, since $n\geq 3$, we have

$$(22)\qquad e_{\alpha\beta kl}=\frac{1}{n-2}(F_{\beta kl}-F_{\alpha rs}\tau^{r}_{\alpha\beta k}\tau^{s}_{\alpha\beta l})-\frac{1}{2(n-1)(n-2)}g_{\beta kl}(\Phi_{\beta}-\Phi_{\alpha}f_{\alpha\beta})+\frac{1}{2}g_{\beta kl}\phi_{\alpha\beta}\,.$$

When $e_{\alpha\beta jk}$ are eliminated from (14) by means of (22), $\phi_{\alpha\beta}$ turns out to cancel out. Therefore from (10), we see that the quantities

$$W^j_{\beta klm}=F^j_{\beta klm}+\frac{1}{n-2}(\delta^j_l F_{\beta km}-\delta^j_m F_{\beta kl})+\frac{1}{n-2}g^{jr}_\beta(g_{\beta km}F_{\beta rl}-g_{\beta kl}F_{\beta rm})$$

$$+\frac{1}{(n-1)(n-2)}(\delta^j_m g_{\beta kl}-\delta^j_l g_{\beta km})\Phi_\beta$$

satisfy

(23) $$W^j_{\beta klm}=(\tau^{-1}_{\alpha\beta})^j_i W^i_{\alpha rst}\tau^r_{\alpha\beta k}\tau^s_{\alpha\beta l}\tau^t_{\alpha\beta m}\,.$$

Define an $n\times n$-matrix-valued holomorphic 2-form W_β by

(24) $$W_\beta=(W^j_{\beta k})\,,\qquad W^j_{\beta k}=W^j_{\beta klm}dz^l_\beta\wedge dz^m_\beta\,.$$

Then (23) is written as

(25) $$W_\beta=\tau^{-1}_{\alpha\beta}W_\alpha\tau_{\alpha\beta}\,,$$

i.e., $\{W_\alpha\}$ is an element of $\Gamma(X, \Omega^2\otimes \mathrm{End}\,\Theta)$, which is called *the conformal Weyl curvature tensor* associated with the holomorphic conformal structure g. For the modern discription of the conformal Weyl curvature tensor in terms of Cartan connections, see Kobayashi [Kb, page 137].

REMARK 2.1. By Remark 1.1, $\{W_\alpha\}$ is defined independently of the choice of $\{g_\alpha\}$ which represents g.

3. Conformal Weyl forms and their relations with Chern forms. Let X be a complex manifold of dimension $n\geq 3$ which admits a holomorphic conformal structure g. Let $\mathscr{U}=\{U_\alpha\}$ be an open covering of X and $(z^1_\alpha, z^2_\alpha,\dots,z^n_\alpha)$ a system of local coordinates on U_α. The canonical line bundle K_X of X is represented by the 1-cocycle $\{K_{\alpha\beta}\}$, $K_{\alpha\beta}=(\det\tau_{\alpha\beta})^{-1}\in\Gamma(U_\alpha\cap U_\beta, \mathcal{O}^*_X)$. On each U_α, there is a nowhere vanishing C^∞ positive-valued function h_α such that

$$h_\beta=|K_{\alpha\beta}|^2h_\alpha \qquad \text{on} \quad U_\alpha\cap U_\beta\,.$$

Suppose that the conformal structure g is represented by a holomorphic symmetric (2, 0)-form

$$g_\alpha=g_{\alpha ij}dz^i_\alpha dz^j_\alpha$$

on each U_α with the relations

$$g_\beta=f_{\beta\alpha}g_\alpha \qquad \text{on} \quad U_\alpha\cap U_\beta\,.$$

Using the metric h_α, we put $\sigma_{\alpha j}=(-1/n)(\partial\log h_\alpha/\partial z^j_\alpha)$ and define a C^∞-(1, 0)-form σ_α by $\sigma_\alpha=\sigma_{\alpha j}dz^j_\alpha$. Put $\rho^j_{\alpha k}=\sigma_{\alpha k}dz^j_\alpha$ and define a $n\times n$-matrix-valued C^∞-(1, 0)-form ρ_α whose (j, k)-component is $\rho^j_{\alpha k}$. Put $\rho^*_\alpha=-G^{-1}_\alpha{}^t\rho_\alpha G_\alpha$. Then we have

$$\sigma_{\alpha\beta}=\sigma_\beta-\sigma_\alpha\,,\quad \rho_{\alpha\beta}=\rho_\beta-\tau_{\alpha\beta}^{-1}\rho_\alpha\tau_{\alpha\beta}\,,\quad \rho^*_{\alpha\beta}=\rho^*_\beta-\tau_{\alpha\beta}^{-1}\rho^*_\alpha\tau_{\alpha\beta}\,.$$

Put

$$a_\alpha=c_\alpha+\rho_\alpha+\rho^*_\alpha+\sigma_\alpha I\,.$$

Then we have easily

$$a_{\alpha\beta}=a_\beta-\tau_{\alpha\beta}^{-1}a_\alpha\tau_{\alpha\beta}\,.$$

Thus $\theta=\{a_\alpha\}$ gives an affine connection of the tangent bundle Θ. Let t be an indeterminate and A an $n\times n$ matrix. Define polynomials $\varphi_0, \varphi_1, \dots, \varphi_n$ by

$$\det\left(I-\frac{t}{2\pi i}A\right)=\sum_{k=0}^{n}\varphi_k(A)t^k\,.$$

First, the Chern forms $c_k(\theta)$, $k=0, 1, \dots, n$, associated with the affine connection θ is defined by

$$c_k(\theta)=\varphi_k(R_\alpha)\,,\qquad k=0, 1\,, \dots, n\,,$$

where $R=\{R_\alpha\}$ is the curvature tensor

$$R_\alpha=da_\alpha+a_\alpha\wedge a_\alpha \tag{26}$$

of the affine connection θ. Next, we shall define holomorphic $2k$-forms $\mathscr{C}_k(g)$, $k=0, 1, \dots, n$, associated with the holomorphic conformal structure g by

$$\mathscr{C}_k(g)=\varphi_k(W_\alpha)\,,$$

where W_α is the conformal Weyl curvature tensor defined by (24). In view of (25), the $\mathscr{C}_k(g)$ are indeed defined on the whole X.

THEOREM 3.1. *The holomorphic $2k$-forms $\mathscr{C}_k(g)$ are d-closed. The de Rham cohomology classes $[\mathscr{C}_k(g)]$, $k=0, \dots, n$, are real and are independent of the choice of the holomorphic conformal structure g.*

This theorem follows as a corollary from the following main result.

THEOREM 3.2. *Let X be a complex manifold of dimension $n\geq 3$ which admits a holomorphic conformal structure g on X. Then there exists a C^∞-affine connection θ on X which satisfies the equality*

$$\sum_{k=0}^{n}\mathscr{C}_k(g)t^k=(1-a^2t^2)\sum_{k=0}^{n}(1-at)^{n-k}t^kc_k(\theta)\,,$$

or equivalently,

$$\sum_{k=0}^{n}c_k(\theta)t^k=\frac{(1+at)^{n+2}}{1+2at}\sum_{k=0}^{n}\left(\frac{t}{1+at}\right)^k\mathscr{C}_k(g)\,,$$

where $a=(1/n)c_1(\theta)$ *and* $c_k(\theta)$ *(resp.* $\mathscr{C}_k(g)$*) is the k-th Chern forms associated with* θ *(resp. conformal Weyl form associated with g).*

DEFINITION 3.1. The d-closed holomorphic $2k$-form $\mathscr{C}_k(g)$ is called the k-th conformal Weyl form associated with the holomorphic conformal structure g.

Note that we obtain the second equality of Theorem 3.2 by replacing t of the first equality by $t/(1+at)$. Now we shall prove Theorem 3.2.

LEMMA 3.1.

$$c_\alpha \wedge \rho_\alpha = dc_\alpha \wedge \rho_\alpha = c_\alpha \wedge d\rho_\alpha = dc_\alpha \wedge d\rho_\alpha = 0 , \tag{27}$$

$$\rho^*_\alpha \wedge \rho_\alpha = d\rho^*_\alpha \wedge \rho_\alpha = \rho^*_\alpha \wedge d\rho_\alpha = d\rho^*_\alpha \wedge d\rho_\alpha = 0 . \tag{28}$$

PROOF. Proof of (27):

$$c^i_{\alpha j} \wedge \rho^j_{\alpha k} = c^i_{\alpha ja} dz^a_\alpha \wedge \sigma_{\alpha k} dz^j_\alpha = \sigma_{\alpha k} c^i_{\alpha ja} dz^a_\alpha \wedge dz^j_\alpha = 0 .$$

The rest can be proved similarly.

Proof of (28):

$$\rho^{*i}_{\alpha j} \wedge \rho^j_{\alpha k} = -g^{ia}_\alpha \sigma_{\alpha a} dz^b_\alpha g_{\alpha bj} \wedge \sigma_{\alpha k} dz^j_\alpha = -(g^{ia}_\alpha \sigma_{\alpha a} \sigma_{\alpha k}) g_{\alpha bj} dz^b_\alpha \wedge dz^j_\alpha = 0 .$$

The rest can be proved similarly. ■

The following proposition is useful to simplify our calculation.

PROPOSITION 3.1. *Let* $\{g_\gamma\}$ *be any representative of g. Let o be any point on X. Choosing a suitable system of local coordinates* $(z^1_\alpha, \dots, z^n_\alpha)$ *on a neighborhood* U_α *of o with* $o=(0, \dots, 0)$*, we have*

$$g_{\alpha ij}(o) = \delta_{ij} , \qquad 1 \le i, j \le n , \tag{29}$$

$$\frac{\partial g_{\alpha ij}}{\partial z^k_\alpha}(o) = 0 , \qquad 1 \le i, j, k \le n , \tag{30}$$

$$F_{\alpha ij}(o) = \phi \delta_{ij} , \qquad 1 \le i, j \le n , \tag{31}$$

where ϕ *is a certain constant.*

PROOF. Choose any coordinate system $z_\alpha = (z^1_\alpha, \dots, z^n_\alpha)$ on U_α with $o=(0, \dots, 0)$. Let g_α be a representative of g on U_α. Write g_α in terms of z_α as

$$g_\alpha = g_{\alpha ij} dz^i_\alpha dz^j_\alpha .$$

Since the $n \times n$ matrix $(g_{\alpha ij})$ is non-singular and symmetric, there is a non-singular constant matrix $A=(A^i_j)$ such that

$$g_{\alpha ij}(o) A^i_k A^j_l = \delta_{kl} .$$

Define a new system of local coordinates $z_\beta = (z^1_\beta, \dots, z^n_\beta)$ by

$$z_\alpha^i = A_j^i z_\beta^j .$$

Then we have

$$g_{\alpha ij} dz_\alpha^i dz_\alpha^j = \delta_{kl} dz_\beta^k dz_\beta^l .$$

Thus, if $(z_\alpha^1, \ldots, z_\alpha^n)$ is replaced by $(z_\beta^1, \ldots, z_\beta^n)$, then the $g_{\alpha ij}$ satisfy (29). Suppose that the $g_{\alpha ij}$ satisfy (29) with respect to a system of local coordinates $(z_\alpha^1, \ldots, z_\alpha^n)$ with $o = (0, \ldots, 0)$. Define a new system of local coordinates $(z_\beta^1, \ldots, z_\beta^n)$ by

$$z_\alpha^j = z_\beta^j - \frac{1}{2} A_{kl}^j z_\beta^k z_\beta^l ,$$

where

$$A_{kl}^j = \left\{ \begin{matrix} j \\ kl \end{matrix} \right\}_\alpha (o) .$$

Then g_α is written in terms of $(z_\beta^1, \ldots, z_\beta^n)$ as

$$g_{\alpha ij} dz_\alpha^i dz_\alpha^j = g_{\alpha ij} (\delta_r^i - A_{rk}^i z_\beta^k)(\delta_s^j - A_{sl}^j z_\beta^l) dz_\beta^r dz_\beta^s .$$

Put

$$g_{\beta rs} = g_{\alpha ij} (\delta_r^i - A_{rk}^i z_\beta^k)(\delta_s^j - A_{sl}^j z_\beta^l) .$$

Then using (29), we obtain

$$\frac{\partial g_{\beta rs}}{\partial z_\beta^t}(o) = \frac{\partial g_{\alpha rs}}{\partial z_\alpha^t}(o) - A_{st}^r - A_{rt}^s . \tag{32}$$

By

$$A_{st}^r = \frac{1}{2}\left(\frac{\partial g_{\alpha sr}}{\partial z_\alpha^t}(o) + \frac{\partial g_{\alpha tr}}{\partial z_\alpha^s}(o) - \frac{\partial g_{\alpha st}}{\partial z_\alpha^r}(o) \right)$$

and (32), we have easily

$$\frac{\partial g_{\beta rs}}{\partial z_\beta^t}(o) = 0 .$$

Thus, if $(z_\alpha^1, \ldots, z_\alpha^n)$ is replaced by $(z_\beta^1, \ldots, z_\beta^n)$, then the $g_{\alpha ij}$ satisfy (29) and (30). Suppose that the g_α satisfy (29) and (30) with respect to a system of local coordinates $(z_\alpha^1, \ldots, z_\alpha^n)$ with $o = (0, \ldots, 0)$. Define a new system of local coordinates $(z_\beta^1, \ldots, z_\beta^n)$ by

$$z_\alpha^j = z_\beta^j - \frac{1}{6} A_{klm}^j z_\beta^k z_\beta^l z_\beta^m ,$$

where

$$A^j_{klm}=\frac{n}{n-2}\left(\frac{\partial c^j_{\alpha lm}}{\partial z^k_\alpha}(o)+\frac{\partial c^j_{\alpha mk}}{\partial z^l_\alpha}(o)+\frac{\partial c^j_{\alpha kl}}{\partial z^m_\alpha}(o)\right).$$

By (13), (15), (18) and (19), we have

$$\sigma_{\alpha\beta j}(o)=0\,,\quad \phi_{\alpha\beta}(o)=0\,,\quad \frac{\partial\sigma_{\alpha\beta j}}{\partial z^k_\beta}(o)=-\frac{1}{n}A^l_{ljk}\,,$$

$$e_{\alpha\beta jk}(o)=\frac{1}{2n}A^l_{ljk}\,,\quad \Delta_{\alpha\beta jk}(o)=-\frac{n-2}{2n}A^l_{ljk}-\frac{\delta_{jk}}{2n}A^l_{lmm}\,,$$

where l and m are summed. Since

$$A^l_{ljk}=\frac{n}{n-2}\left(\frac{\partial c^l_{\alpha jk}}{\partial z^l_\alpha}(o)+\frac{\partial c^l_{\alpha kl}}{\partial z^j_\alpha}(o)+\frac{\partial c^l_{\alpha lj}}{\partial z^m_\alpha}(o)\right)=\frac{n}{n-2}\frac{\partial c^l_{\alpha jk}}{\partial z^l_\alpha}(o)\,,$$

we have

$$\Delta_{\alpha\beta jk}(o)=-\frac{1}{2}\frac{\partial c^l_{\alpha jk}}{\partial z^l_\alpha}(o)-\frac{\delta_{jk}}{2(n-2)}\frac{\partial c^l_{\alpha mm}}{\partial z^l_\alpha}(o)\,,$$

where l and m are summed. On the other hand, by (11) and (16), we have

$$F_{\alpha jk}(o)=-\frac{1}{2}\frac{\partial c^l_{\alpha jk}}{\partial z^l_\alpha}(o)\,.$$

Therefore it follows from (20) that

$$F_{\beta jk}(o)=F_{\alpha jk}(o)-\Delta_{\alpha\beta jk}(o)=\delta_{jk}\frac{1}{2(n-2)}\frac{\partial c^l_{\alpha mm}}{\partial z^l_\alpha}(o)\,,$$

where l and m are summed. Thus (31) is satisfied for the system of local coordinates $(z^1_\beta, \dots, z^n_\beta)$. Obviously (29) and (30) are also satisfied for this system of coordinates, it is enough to replace $(z^1_\alpha, \dots, z^n_\alpha)$ by $(z^1_\beta, \dots, z^n_\beta)$. ∎

Let o be any fixed point on X. Suppose that $o\in U_\alpha$. We fix a system of local coordinates $(z^1_\alpha, \dots, z^n_\alpha)$ of Proposition 3.1, and omit the subscript α for simplicity. By the definition of Christoffel symbols, (29) and (30), we have

$$c(o)=0\,. \tag{33}$$

Hence the curvature tensor (26) is given at o by

$$R=\bar{\partial}\sigma I+d\rho^*+\rho^*\wedge\rho^*+dc+\rho\wedge\rho^*+d\rho+\rho\wedge\rho\,.$$

Using Proposition 3.1, at o we obtain

$$R=\bar{\partial}\sigma I-{}^t d\rho+{}^t\rho\wedge{}^t\rho+dc-\rho\wedge{}^t\rho+d\rho+\rho\wedge\rho\,. \tag{34}$$

On the other hand, by (33), the conformal Weyl curvature tensor can be written as

$$W=dc+H+H^*+J \qquad \text{at} \quad o\,,$$

where

$$H=\frac{2}{n-2}(H_j^i)\,, \qquad H_j^i=F_{jk}dz^i\wedge dz^k\,,$$

$$H^*=\frac{2}{n-2}(H_j^{*i})\,, \qquad H_j^{*i}=g^{il}F_{lm}dz^m\wedge g_{jk}dz^k\,,$$

$$J=\frac{2\Phi}{(n-1)(n-2)}(J_j^i)\,, \qquad J_j^i=g_{jk}dz^k\wedge dz^i\,.$$

Hence, using Proposition 3.1, we obtain

$$W=dc+K \qquad \text{at} \quad o\,, \tag{35}$$

where

$$K=(K_j^i)\,, \qquad K_j^i=\frac{2\phi}{n-1}dz^i\wedge dz^j.$$

LEMMA 3.2. $dc_j^i=-dc_i^j$ at o.

PROOF. By (29) and (30), we have

$$\begin{aligned}\frac{\partial}{\partial z^l}\left\{\begin{matrix} i \\ jk\end{matrix}\right\}&=\frac{1}{2}g^{im}\left(\frac{\partial^2 g_{jm}}{\partial z^l\partial z^k}+\frac{\partial^2 g_{km}}{\partial z^l\partial z^j}-\frac{\partial^2 g_{jk}}{\partial z^l\partial z^m}\right)\\ &=\frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial z^l\partial z^k}+\frac{\partial^2 g_{ki}}{\partial z^l\partial z^j}-\frac{\partial^2 g_{jk}}{\partial z^l\partial z^i}\right).\end{aligned}$$

Hence we get

$$\frac{\partial}{\partial z^l}\left\{\begin{matrix} a \\ aj\end{matrix}\right\}=\frac{1}{2}\sum_a\frac{\partial^2 g_{aa}}{\partial z^l\partial z^j}\,, \qquad \frac{\partial}{\partial z^l}\left(g^{im}g_{jk}\left\{\begin{matrix} a \\ am\end{matrix}\right\}\right)=\frac{1}{2n}\delta_{jk}\sum_a\frac{\partial^2 g_{aa}}{\partial z^l\partial z^i}\,.$$

Therefore

$$\begin{aligned}\frac{\partial c_{jk}^i}{\partial z^l}&=\frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial z^l\partial z^k}+\frac{\partial^2 g_{ki}}{\partial z^l\partial z^j}-\frac{\partial^2 g_{jk}}{\partial z^l\partial z^i}\right)-\frac{\delta_j^i}{2n}\sum_a\frac{\partial^2 g_{aa}}{\partial z^l\partial z^k}-\frac{\delta_k^i}{2n}\sum_a\frac{\partial^2 g_{aa}}{\partial z^l\partial z^j}+\frac{\delta_{jk}}{2n}\sum_a\frac{\partial^2 g_{aa}}{\partial z^l\partial z^i}\\ &=\frac{1}{2}\frac{\partial^2 g_{ij}}{\partial z^l\partial z^k}-\frac{\delta_{ij}}{2n}\sum_a\frac{\partial^2 g_{aa}}{\partial z^l\partial z^k}+\frac{1}{2}\left(\frac{\partial^2 g_{ki}}{\partial z^l\partial z^j}-\frac{\partial^2 g_{jk}}{\partial z^l\partial z^i}\right)-\frac{1}{2n}\sum_a\left(\delta_{ik}\frac{\partial^2 g_{aa}}{\partial z^l\partial z^j}-\delta_{jk}\frac{\partial^2 g_{aa}}{\partial z^l\partial z^i}\right).\end{aligned}$$

Hence we obtain

$$
\begin{aligned}
dc^i_j &= \frac{\partial c^i_{jk}}{\partial z^l} dz^l \wedge dz^k \\
&= \frac{1}{2}\left(\left(\frac{\partial^2 g_{ki}}{\partial z^l \partial z^j} - \frac{\partial^2 g_{jk}}{\partial z^l \partial z^i}\right) - \frac{1}{n}\sum_a\left(\delta_{ik}\frac{\partial^2 g_{aa}}{\partial z^l \partial z^j} - \delta_{jk}\frac{\partial^2 g_{aa}}{\partial z^l \partial z^i}\right)\right) dz^l \wedge dz^k \\
&= -dc^j_i\,.
\end{aligned}
$$

■

LEMMA 3.3. ${}^t d\rho \wedge dc = dc \wedge K = 0$ *at* o.

PROOF. By Lemmas 3.1 and 3.2, ${}^t({}^t d\rho \wedge dc) = dc \wedge d\rho = 0$. The equality $dc \wedge K = 0$ follows from $dc^i_{kl} = dc^i_{lk}$ and the equality

$$
(dc \wedge K)^i_j = \frac{2\phi}{n-1} dz^j \wedge dc^i_{kl} \wedge dz^k \wedge dz^l\,.
$$

■

In view of the equations (34) and (35), we have by virtue of Lemmas 3.1 and 3.3 that

$$
\begin{aligned}
(36)\qquad I - \frac{t}{2\pi i} R = \lambda\left(I + \frac{t}{2\pi i\lambda}({}^t d\rho - {}^t\rho \wedge {}^t\rho)\right)\left(I - \frac{t}{2\pi i\lambda} dc\right) \\
\times\left(I + \frac{t}{2\pi i\lambda}\rho \wedge {}^t\rho\right)\left(I - \frac{t}{2\pi i\lambda}(d\rho + \rho \wedge \rho)\right),
\end{aligned}
$$

where

$$
\lambda = 1 - \frac{t}{2\pi i}\bar{\partial}\sigma\,,
$$

and

$$
(37)\qquad I - \frac{t}{2\pi i} W = \left(I - \frac{t}{2\pi i} dc\right)\left(I - \frac{t}{2\pi i} K\right).
$$

We set

$$
a = \frac{1}{n} c_1(\theta) = -\frac{\bar{\partial}\sigma}{2\pi i}\,.
$$

LEMMA 3.4.

$$
\det\left(I - \frac{t}{2\pi i}(d\rho + \rho \wedge \rho)\right) = \frac{1}{1 - at}\,.
$$

PROOF. The left hand side is equal to

$$\det\left(I-\frac{t}{2\pi i}(d\rho+\rho\wedge\rho)\right)=\sum_{q=0}^{n}\left(-\frac{t}{2\pi i}\right)^q{\sum_J}'\det Q_J^J\,,$$

where J runs through all q-tuples $\{j_1, j_2, \ldots, j_q\}$ with $j_1<j_2<\cdots<j_q$, $1\leq j_\lambda\leq n$, and Q_J^J is the $q\times q$-principal minor corresponding to J. Then

$${\sum_J}'\det Q_J^J=\left((d\sigma_k-\sigma_k\sigma)\wedge dz^k\right)^q=(d\sigma-\sigma\wedge\sigma)^q=(\bar{\partial}\sigma)^q\,,$$

since $\partial\sigma=0$. Hence the left hand side of 3.4 is equal to

$$\sum_{q=0}^{n}\left(-\frac{t}{2\pi i}\bar{\partial}\sigma\right)^q=\sum_{q=0}^{n}(at)^q=\frac{1}{1-at}\,.$$

■

LEMMA 3.5.

$$\det\left(I+\frac{t}{2\pi i}({}^t d\rho-{}^t\rho\wedge{}^t\rho)\right)=\frac{1}{1+at}\,.$$

PROOF. Taking the transpose of the matrix on the left hand side, we have

$$\det\left(I+\frac{t}{2\pi i}({}^t d\rho-{}^t\rho\wedge{}^t\rho)\right)=\det\left(I-\frac{-t}{2\pi i}(d\rho+\rho\wedge\rho)\right).$$

Hence the lemma follows from Lemma 3.4. ■

LEMMA 3.6.

$$\det\left(I+\frac{t}{2\pi i}\rho\wedge{}^t\rho\right)=1\,.$$

LEMMA 3.7.

$$\det\left(I-\frac{t}{2\pi i}K\right)=1\,.$$

The proofs of the two lemmas above are similar to that of Lemma 3.4.

By the four lemmas above, we have from (36) and (37) that

$$(1-a^2s^2)\sum_{k=0}^{n}c_k(\theta)s^k(1-as)^{n-k}=\det\left(I-\frac{s}{2\pi i}dc\right),$$

where $s=t/\lambda$, and

$$\sum_{k=0}^{n}\mathscr{C}_k(g)t^k=\det\left(I-\frac{t}{2\pi i}dc\right). \tag{38}$$

Combining these two equalities, we have Theorem 3.2. ■

THEOREM 3.3. *Let X be a complex manifold of dimension $n \geq 3$ with a holomorphic conformal structure g. Then*

$$\mathscr{C}_{2k+1}(g)=0\,, \qquad k=0,1,\ldots,\left[\frac{n}{2}\right].$$

PROOF. This follows immediately from Lemma 3.2 and (38). ■

By Theorems 3.2 and 3.3, the conformal Weyl forms are, for example,

$$\mathscr{C}_0(g)=1\,,$$

$$\mathscr{C}_1(g)=0\,,$$

$$\mathscr{C}_2(g)=\frac{-n^2+n-2}{2n^2}\,c_1^2(\theta)+c_2(\theta)\,,$$

$$\mathscr{C}_3(g)=\frac{(n-1)(n-2)}{3n^2}\,c_1^3(\theta)-\frac{n-2}{n}\,c_1(\theta)c_2(\theta)+c_3(\theta)=0\,,$$

$$\mathscr{C}_4(g)=-\frac{(n-1)(n^2-5n+2)}{8n^3}\,c_1^4(\theta)+\frac{(n-1)(n-4)}{2n^2}\,c_1^2(\theta)c_2(\theta)-\frac{n-3}{n}\,c_1(\theta)c_3(\theta)+c_4(\theta)\,.$$

The following is a consequence of Theorem 3.2.

COROLLARY 3.1. *The conformal Weyl forms are d-closed. The de Rham cohomology classes of the conformal Weyl forms are real cohomology classes and are independent of the choice of holomorphic conformal structures.*

For $n \geq 3$, the following corollary gives a refinement of [KO, Theorem (3.20)].

COROLLARY 3.2. *If a compact complex manifold with dimension $n \geq 3$ admits a holomorphic conformal structure, then all k-th conformal Weyl forms with $2k \geq n$ vanish. If, further, the manifold is of Kähler then all k-th, $k \geq 1$, conformal Weyl classes are zero.*

PROOF. All k-th conformal Weyl forms are holomorphic $2k$-forms. Therefore if $2k > n$ then the k-th conformal Weyl form vanishes. Since d-closed holomorphic n-form represents a real de Rham cohomology class only if it represents a zero class, we see that the n-th conformal Weyl class also vanishes. If the manifold is of Kähler then we can apply Hodge theory. Since the conformal Weyl forms are holomorphic, they are harmonic. On the other hand, the conformal Weyl classes are real by Corollary 3.1. Therefore they vanish by Hodge theory. ■

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