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ON PRODUCTS IN THE COHOMOLOGY OF THE DIHEDRAL GROUPS

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Abstract. Explicit diagonal approximations for efficient resolutions of the integers over the dihedral groups are constructed. As an application, the multiplicative structure of the cohomology of the dihedral groups arising from certain coefficient pairings is determined.

1. Introduction. The dihedral group D_{2n} of order $2n$ is generated by two elements x and y which satisfy the relations $x^n=y^2=1$, $xy=yx^{-1}$. Efficient free resolutions of the trivial D_{2n}-module $\boldsymbol{Z}$ over the integral group ring $\boldsymbol{Z}D_{2n}$ have been given by Wall [7] and Hamada [3]. Our main contribution is the construction of an explicit diagonal approximation for the Wall-Hamada resolution. This yields explicit cochain cup products with respect to any given coefficient pairings. It is well-known (e.g. [1, Ch. V], [2, Ch. 3]) that these cochain cup products induce the standard cohomology cup products.

In §2 we reformulate the Wall-Hamada resolution, and give our diagonal approximation. The construction of the latter proceeds via a well-known inductive technique which uses a contracting homotopy, and is presented in §3. In particular, we construct an explicit contracting homotopy for the Wall-Hamada resolution, providing an alternate proof that the latter is indeed a resolution of $\boldsymbol{Z}$ over $\boldsymbol{Z}D_{2n}$. In §4 we explicitly determine the cochain complexes arising from the Wall-Hamada resolution for general D_{2n}-modules and determine the cochain cup products arising from our diagonal approximation. This is applied in §5 to calculate the cohomology rings $H^*(D_{2n}; \boldsymbol{Z})$, $H^*(D_{2n}; \boldsymbol{Z}/2\boldsymbol{Z})$, as well as $H^*(D_{2n}; M)$ as a module over $H^*(D_{2n}; \boldsymbol{Z})$ for certain non-trivial D_{2n}-modules M. Some of these results for trivial coefficients have been previously obtained by other methods (e.g. [4, Prop. 3.5], [6, Ch. 1]).

I thank the referee for insisting that I include proofs which, in the original version, were left to the reader. The result is a more easily verifiable paper.

2. The resolution and a diagonal approximation. Let $n\geq 2$. We first reformulate the Wall-Hamada resolution of $\boldsymbol{Z}$ over $\boldsymbol{Z}D_{2n}$. For each $q\geq 0$, let C_q be the free $\boldsymbol{Z}D_{2n}$-module on generators $c_q^1, c_q^2, \dots, c_q^{q+1}$. For notational convenience, interpret c_q^i

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as 0 if either $i<1$ or $i>q+1$ or $q<0$. Using the notation of §1, let $N=\sum_{i=0}^{n-1} x^i \in ZD_{2n}$. Let

$$\varepsilon_i=\begin{cases}-1 & \text{if } i\equiv 0 \text{ or } 3 \pmod 4,\\ 1 & \text{if } i\equiv 1 \text{ or } 2 \pmod 4.\end{cases}$$

Then the augmentation $\varepsilon\colon C_0\to Z$ and the boundary operators $\partial_q\colon C_q\to C_{q-1}$ for $q>0$ are the right ZD_{2n}-homomorphisms determined by

$$\varepsilon(c_0^1)=1,$$

$$\partial_q(c_q^i)=\begin{cases} c_{q-1}^{i-1}(yx+\varepsilon_i\varepsilon_q)+c_{q-1}^{i}(x-1) & \text{if } q \text{ even}, \quad i \text{ even},\\ c_{q-1}^{i-1}(y-\varepsilon_i\varepsilon_q)+c_{q-1}^{i}N & \text{if } q \text{ even}, \quad i \text{ odd},\\ c_{q-1}^{i-1}(y-\varepsilon_i\varepsilon_q)-c_{q-1}^{i}N & \text{if } q \text{ odd}, \quad i \text{ even},\\ c_{q-1}^{i-1}(yx-\varepsilon_i\varepsilon_q)+c_{q-1}^{i}(x-1) & \text{if } q \text{ odd}, \quad i \text{ odd}.\end{cases}$$

THEOREM 2.1 (Wall-Hamada). *The above* $(C, \varepsilon, \partial)$ *is a free resolution of* Z *over* ZD_{2n}.

An explicit contracting homotopy for C will be given in §3, providing an alternative proof of Theorem 2.1. We turn now to the description of a diagonal approximation $\Delta\colon C\to C\otimes C$. $\otimes$ denotes $\otimes_Z$ and $C\otimes C$ is a bigraded ZD_{2n}-module via the diagonal action $(a\otimes b)g=(ag)\otimes(bg)$ for $a, b\in C$, $g\in D_{2n}$. To avoid excessive parentheses, $(ar)\otimes(bs)$ will be written as $ar\otimes bs$ whenever $a, b\in C$, $r, s\in ZD_{2n}$. $C\otimes C$ is another free resolution of Z over ZD_{2n} with augmentation $\varepsilon\otimes\varepsilon\colon C_0\otimes C_0\to Z\otimes Z=Z$ and the standard tensor product boundary. Δ is to be an augmentation-preserving ZD_{2n} chain map. For $1\le j\le n-1$ write $N_j=\sum_{i=0}^{j-1} x^i\in ZD_{2n}$, and $N_0=0$.

THEOREM 2.2. *For* $q\ge 0$, *let* $\Delta_q\colon C_q\to(C\otimes C)_q$ *denote the right* ZD_{2n}*-homomorphism determined as follows*:

For k *even and* $0\le k\le q-1$,

$$\begin{aligned}\Delta_q(c_q^{q-k})=&\sum_{\substack{i\text{ even}\\ r\ge 0}}(-1)^{rq}\Big((-1)^r c_i^{i+1-2r}\otimes c_{q-i}^{q-i+2r-k}+c_i^{i-2r}\otimes c_{q-i}^{q-i+2r-k}\\ &+c_i^{i-2r}\otimes c_{q-i}^{q-i+2r-k+1}y+rc_i^{i-2r}\otimes c_{q-i}^{q-i+2r-k}N\Big)\\ &+\sum_{\substack{i\text{ odd}\\ r\ge 0}}(-1)^{r(q+1)}\Big(c_i^{i-2r}\otimes c_{q-i}^{q-i+2r-k+1}x+c_i^{i-2r}\otimes c_{q-i}^{q-i+2r-k}yx\\ &+(-1)^r c_i^{i+1-2r}x\otimes c_{q-i}^{q-i+2r-k}yx+rc_i^{i-2r}\otimes c_{q-i}^{q-i+2r-k}yN\Big).\end{aligned}$$

For k *odd and* $-1\le k\le q-1$,

$$\Delta_q(c_q^{q-k})=\sum_{\substack{i \text{ even}\\ r\geq 0}}(-1)^{rq}\Big(c_i^{i+1-2r}\otimes c_{q-i}^{q-i+2r-k}+rc_i^{i+1-2r}\otimes c_{q-i}^{q-i+2r-k-1}yN$$
$$+(-1)^{q+r+1}\sum_{j=1}^{n-1}c_i^{i-2r}N_jx^{-j}\otimes c_{q-i}^{q-i+2r-k+1}yx^{-j}\Big)$$
$$+\sum_{\substack{i \text{ odd}\\ r\geq 0}}(-1)^{r(q+1)}\Big(c_i^{i+1-2r}\otimes c_{q-i}^{q-i+2r-k}y+rc_i^{i+1-2r}\otimes c_{q-i}^{q-i+2r-k-1}N$$
$$+(-1)^{q+r}\sum_{j=1}^{n-1}c_i^{i-2r}N_j\otimes c_{q-i}^{q-i+2r-k+1}x^j\Big).$$

Then the Δ_q constitute a diagonal approximation for C.

The proof of Theorem 2.2 is given in §§3 and 6. An alternative approach to the proof is to check directly that Δ_0 preserves augmentation (trivial) and that Δ commutes with the boundary maps. The latter task appears to be at least as tedious as the approach we have taken.

3. Constructing diagonal approximations. Let G be any group and $(X, \varepsilon, \partial)$ a free resolution of the trivial G-module $\boldsymbol{Z}$ over $\boldsymbol{Z}G$. Write $X_{-1}=\boldsymbol{Z}$ and $\partial_0=\varepsilon\colon X_0\to X_{-1}$. Recall that a contracting homotopy T for X consists of a sequence of $\boldsymbol{Z}$-homomorphisms $T_q\colon X_q\to X_{q+1}$, $q\geq -1$, such that $\partial_{q+1}T_q+T_{q-1}\partial_q=1_{X_q}$ for each $q\geq 0$.

PROPOSITION 3.1. *Let G be a group, $(X, \varepsilon, \partial)$ a free resolution of $\boldsymbol{Z}$ over $\boldsymbol{Z}G$, and U a contracting homotopy for $X\otimes X$. Suppose that for each $q\geq 0$, B_q is a $\boldsymbol{Z}G$-basis for X_q such that $\varepsilon(b)=1$ for each $b\in B_0$. Let $\psi_0\colon X_0\to X_0\otimes X_0$ be the right $\boldsymbol{Z}G$-module homomorphism determined by $\psi_0(b)=b\otimes b$ for $b\in B_0$. For $q>0$ let $\psi_q\colon X_q\to(X\otimes X)_q$ be the right $\boldsymbol{Z}G$-module homomorphism determined inductively by $\psi_q(b)=U_{q-1}\psi_{q-1}\partial_q(b)$ for $b\in B_q$. Then ψ is a diagonal approximation for X.*

PROOF. Trivially, $(\varepsilon\otimes\varepsilon)\psi_0(b)=1=\varepsilon(b)$ for all $b\in B_0$. Write $\partial^{\otimes}$ for the boundary operator on $X\otimes X$. Let $q\geq 0$ and assume, inductively, $\partial_q^{\otimes}\psi_q=\psi_{q-1}\partial_q$ (where $\psi_{-1}=1_{\boldsymbol{Z}}$). Let $b\in B_{q+1}$. Then $\partial_{q+1}^{\otimes}\psi_{q+1}(b)=\partial_{q+1}^{\otimes}U_q\psi_q\partial_{q+1}(b)=(1_{(X\otimes X)_q}-U_{q-1}\partial_q^{\otimes})\psi_q\partial_{q+1}(b)=\psi_q\partial_{q+1}(b)-U_{q-1}(\partial_q^{\otimes}\psi_q)\partial_{q+1}(b)=\psi_q\partial_{q+1}(b)-U_{q-1}(\psi_{q-1}\partial_q)\partial_{q+1}(b)=\psi_q\partial_{q+1}(b)$. Since B_{q+1} is a $\boldsymbol{Z}G$-basis for X_{q+1}, it follows that $\partial_{q+1}^{\otimes}\psi_{q+1}=\psi_q\partial_{q+1}$. □

Note that since the U_q are not necessarily $\boldsymbol{Z}G$-homomorphisms, the formula $\psi_q(u)=U_{q-1}\psi_{q-1}\partial_q(u)$ is not necessarily valid for all $u\in X_q$, but only for $u\in B_q$.

PROPOSITION 3.2. *Let G be a group, $(X, \varepsilon, \partial)$ a free resolution of $\boldsymbol{Z}$ over $\boldsymbol{Z}G$, and T a contracting homotopy for X. Extend $T_{-1}\varepsilon\colon X_0\to X_0$ to a chain map $T_{-1}\varepsilon\colon X\to X$ over $\boldsymbol{Z}$ by defining $(T_{-1}\varepsilon)_i=0$ if $i\neq 0$. Let $U_q\colon (X\otimes X)_q\to(X\otimes X)_{q+1}$ for $q\geq -1$ be the $\boldsymbol{Z}$-homomorphisms given by $U_{-1}=T_{-1}\otimes T_{-1}\colon \boldsymbol{Z}=\boldsymbol{Z}\otimes\boldsymbol{Z}\to X_0\otimes X_0$, and $U_q(u\otimes v)=$*

$T_i(u)\otimes v+(T_{-1}\varepsilon)_i(u)\otimes T_{q-i}(v)$ for $u\in X_i$, $v\in X_{q-i}$, $0\le i\le q$. Then the U_q constitute a contracting homotopy for $X\otimes X$.

PROOF. T may be regarded as a chain homotopy from 1_X to $T_{-1}\varepsilon$. It is standard (e.g. [5, Prop. 9.1]) that whenever s is a chain homotopy from f_1 to g_1 and t is a chain homotopy from f_2 to g_2, then u given by $u(a\otimes b)=s(a)\otimes g_1(b)+(-1)^{|a|}f_2(a)\otimes t(b)$ is a chain homotopy from $f_1\otimes g_1$ to $f_2\otimes g_2$. Applying this with $f_1=g_1=1_X$, $f_2=g_2=T_{-1}\varepsilon$, $s=t=T$, the u that results is U as defined above. □

THEOREM 3.3. *The following defines a contracting homotopy T for the Wall-Hamada resolution C:*

$$T_{-1}(1)=c_0^1.$$

If $q\ge 0$ is even, then

$$T_q(c_q^r y^j x^i)=\begin{cases} c_{q+1}^1 N_i & \text{if } j=0,\ r=1, \text{ and } 0\le i\le n-1,\\ -\varepsilon_q c_{q+1}^1 N_i + c_{q+1}^2 x^i & \text{if } j=1,\ r=1, \text{ and } 0\le i\le n-1,\\ 0 & \text{if } j=0,\ 2\le r\le q+1, \text{ and all } i,\\ c_{q+1}^{r+1}x^{i-1} & \text{if } j=1,\ 2\le r\le q,\ r \text{ even, and all } i,\\ c_{q+1}^{r+1}x^{i} & \text{if } j=1,\ 3\le r\le q+1,\ r \text{ odd, and all } i.\end{cases}$$

If $q\ge 1$ is odd, then

$$T_q(c_q^r y^j x^i)=\begin{cases} 0 & \text{if } j=0,\ r=1, \text{ and } 0\le i\le n-2,\\ c_{q+1}^1 & \text{if } j=0,\ r=1, \text{ and } i=n-1,\\ -\varepsilon_q c_{q+1}^1 + c_{q+1}^2 x^{-1} & \text{if } j=1,\ r=1, \text{ and } i=0,\\ c_{q+1}^2 x^{i-1} & \text{if } j=1,\ r=1, \text{ and } 1\le i\le n-1,\\ 0 & \text{if } j=0,\ 2\le r\le q+1, \text{ and all } i,\\ c_{q+1}^{r+1}x^{i} & \text{if } j=1,\ 2\le r\le q+1,\ r \text{ even, and all } i,\\ c_{q+1}^{r+1}x^{i-1} & \text{if } j=1,\ 3\le r\le q,\ r \text{ odd, and all } i.\end{cases}$$

PROOF. We must check

$$(*)\qquad (\partial_{q+1}T_q+T_{q-1}\partial_q)(c_q^r y^j x^i)=c_q^r y^j x^i$$

whenever $1\le r\le q+1$, $0\le i\le n-1$, and $j=0$ or 1.

From the definition of T and the boundary formula we obtain $\partial_1 T_0(c_0^1 y^j x^i)=c_0^1(y^j x^i-1)$, $T_{-1}\varepsilon(c_0^1 y^j x^i)=c_0^1$ which establishes $(*)$ for $q=0$.

For the case $q>0$, $r=1$, and $j=0$, we obtain

$$\partial_{q+1}T_q(c_q^1 x^i)=\begin{cases} c_q^1(x^i-1) & \text{if } q \text{ even, } 0\le i\le n-1,\\ 0 & \text{if } q \text{ odd, } 0\le i\le n-2,\\ c_q^1 N & \text{if } q \text{ odd, } i=n-1,\end{cases}$$

and

$$T_{q-1}\partial_q(c_q^1x^i)=\begin{cases} c_q^1 & \text{if } q \text{ even, } 0\le i\le n-1,\\ c_q^1x^i & \text{if } q \text{ odd, } 0\le i\le n-2,\\ -c_q^1N_{n-1} & \text{if } q \text{ odd, } i=n-1,\end{cases}$$

which combine to yield (∗) in this case.

For the case $q>0$, $r=1$, and $j=1$ we obtain, from the definitions and the fact that $\varepsilon_{q+1}=(-1)^{q+1}\varepsilon_q$ for all q,

$$\partial_{q+1}T_q(c_q^1yx^i)=\begin{cases} c_q^1(yx^i+\varepsilon_q)-c_q^2N & \text{if } q \text{ even, } 0\le i\le n-1,\\ c_q^1(-\varepsilon_qN_{n-1}+y)+c_q^2(1-x^{-1}) & \text{if } q \text{ odd, } i=0,\\ c_q^1(yx^i+\varepsilon_qx^{i-1})+c_q^2(x^i-x^{i-1}) & \text{if } q \text{ odd, } 1\le i\le n-1,\end{cases}$$

and

$$T_{q-1}\partial_q(c_q^1yx^i)=\begin{cases} -\varepsilon_qc_q^1+c_q^2N & \text{if } q \text{ even, } 0\le i\le n-1,\\ \varepsilon_qc_q^1N_{n-1}+c_q^2(x^{-1}-1) & \text{if } q \text{ odd, } i=0,\\ -\varepsilon_qc_q^1x^{i-1}+c_q^2(x^{i-1}-x^i) & \text{if } q \text{ odd, } 1\le i\le n-1,\end{cases}$$

which combine to yield (∗) in this case.

For the case $2\le r\le q+1$, r even, and $j=0$, we obtain $\partial_{q+1}T_q(c_q^rx^i)=0$ for all i. The computational details of $T_{q-1}\partial_q(c_q^rx^i)$ are different for the subcases $r=2$ and $r\ge 4$ due to the presence of a c_q^1 term in $\partial_q(c_q^2)$ and the anomaly in the definition of the $T_{q-1}(c_{q-1}^1x^i)$. In both subcases, one obtains $T_{q-1}\partial_q(c_q^rx^i)=c_q^rx^i$ for all i, thus establishing (∗) in this case.

For the case $2\le r\le q+1$, r even, $j=1$, and i arbitrary, we obtain

$$\partial_{q+1}T_q(c_q^ryx^i)=\begin{cases} c_q^r(yx^i-\varepsilon_{r+1}\varepsilon_{q+1}x^{i-1})+c_q^{r+1}(x^i-x^{i-1}) & \text{if } q \text{ even},\\ c_q^r(yx^i-\varepsilon_{r+1}\varepsilon_{q+1}x^i)+c_q^{r+1}N & \text{if } q \text{ odd},\end{cases}$$

and

$$T_{q-1}\partial_q(c_q^ryx^i)=\begin{cases} \varepsilon_r\varepsilon_qc_q^rx^{i-1}+c_q^{r+1}(x^{i-1}-x^i) & \text{if } q \text{ even},\\ -\varepsilon_r\varepsilon_qc_q^rx^i-c_q^{r+1}N & \text{if } q \text{ odd}.\end{cases}$$

Again, the subcases $r=2$ and $r\ge 4$ require separate treatment. Using $\varepsilon_{k+1}=(-1)^{k+1}\varepsilon_k$, (∗) now follows for this case.

For the case $3\le r\le q+1$, r odd, and $j=0$, we obtain $\partial_{q+1}T_q(c_q^rx^i)=0$ and $T_{q-1}\partial_q(c_q^rx^i)=c_q^rx^i$ for all i, thus establishing (∗) for this case.

For the case $3\le r\le q+1$, r odd, and $j=1$, we obtain

$$\partial_{q+1}T_q(c_q^ryx^i)=\begin{cases} c_q^r(yx^i-\varepsilon_{r+1}\varepsilon_{q+1}x^i)-c_q^{r+1}N & \text{if } q \text{ even},\\ c_q^r(yx^i+\varepsilon_{r+1}\varepsilon_{q+1}x^{i-1})+c_q^{r+1}(x^i-x^{i-1}) & \text{if } q \text{ odd},\end{cases}$$

and

$$T_{q-1}\partial_q(c_q^r y x^i) = \begin{cases} -\varepsilon_r\varepsilon_q c_q^r x^i + c_q^{r+1} N & \text{if } q \text{ even},\\ -\varepsilon_r\varepsilon_q c_q^r x^{i-1} + c_q^{r+1}(x^{i-1} - x^i) & \text{if } q \text{ odd} \end{cases}$$

for all i, which combine to yield $(*)$ in this case. □

THEOREM 3.4. *The diagonal approximation Δ for the Wall-Hamada resolution which results from Theorem* 3.3, *Proposition* 3.2, *and Proposition* 3.1 *with* $B_q = \{c_q^1, \ldots, c_q^{q+1}\}$ *is given by Theorem* 2.2.

PROOF. Let ψ denote the diagonal approximation which results from Theorem 3.3, Proposition 3.2, and Proposition 3.1 with $B_q = \{c_q^1, \ldots, c_q^{q+1}\}$. We must prove $\psi_q(c_q^{q-k}) = \Delta_q(c_q^{q-k})$ for all q and k. The contracting homotopy U of Proposition 3.2 which results from the T of Proposition 3.3 is given by $U(u \otimes v) = T(u) \otimes v + \varepsilon(u) c_0^1 \otimes T(v)$. ψ is determined inductively by

$$\psi_0(c_0^1) = c_0^1 \otimes c_0^1,\qquad\qquad (1)$$
$$\psi_q(c_q^i) = U_{q-1}\psi_{q-1}\partial_q(c_q^i) \quad \text{if } q \geq 1.$$

To make the notation less cumbersome, write

$$A_1(q,k) = \sum_{\substack{i \text{ even} \\ r \geq 0}} (-1)^{r(q+1)} c_i^{i+1-2r} \otimes c_{q-i}^{q-i+2r-k},$$

$$A_2(q,k) = \sum_{\substack{i \text{ even} \\ r \geq 0}} (-1)^{rq} c_i^{i-2r} \otimes c_{q-i}^{q-i+2r-k},$$

$$A_3(q,k) = \sum_{\substack{i \text{ even} \\ r \geq 0}} (-1)^{rq} c_i^{i-2r} \otimes c_{q-i}^{q-i+2r-k+1} y,$$

$$A_4(q,k) = \sum_{\substack{i \text{ even} \\ r \geq 0}} (-1)^{rq} r c_i^{i-2r} \otimes c_{q-i}^{q-i+2r-k} N,$$

$$A_5(q,k) = \sum_{\substack{i \text{ odd} \\ r \geq 0}} (-1)^{r(q+1)} c_i^{i-2r} \otimes c_{q-i}^{q-i+2r-k+1} x,$$

$$A_6(q,k) = \sum_{\substack{i \text{ odd} \\ r \geq 0}} (-1)^{r(q+1)} c_i^{i-2r} \otimes c_{q-i}^{q-i+2r-k} yx,$$

$$A_7(q,k) = \sum_{\substack{i \text{ odd} \\ r \geq 0}} (-1)^{rq} c_i^{i+1-2r} x \otimes c_{q-i}^{q-i+2r-k} yx,$$

$$A_8(q,k) = \sum_{\substack{i \text{ odd} \\ r \geq 0}} (-1)^{r(q+1)} r c_i^{i-2r} \otimes c_{q-i}^{q-i+2r-k} yN,$$

$$B_1(q,k)=\sum_{\substack{i\ \mathrm{even}\\ r\ge 0}}(-1)^{rq}c_i^{i+1-2r}\otimes c_{q-i}^{q-i+2r-k},$$

$$B_2(q,k)=\sum_{\substack{i\ \mathrm{even}\\ r\ge 0}}(-1)^{rq}rc_i^{i+1-2r}\otimes c_{q-i}^{q-i+2r-k-1}yN,$$

$$B_3(q,k)=\sum_{\substack{i\ \mathrm{even}\\ r\ge 0}}\sum_{j=1}^{n-1}(-1)^{(r+1)(q+1)}c_i^{i-2r}N_jx^{-j}\otimes c_{q-i}^{q-i+2r-k+1}yx^{-j},$$

$$B_4(q,k)=\sum_{\substack{i\ \mathrm{odd}\\ r\ge 0}}(-1)^{r(q+1)}c_i^{i+1-2r}\otimes c_{q-i}^{q-i+2r-k}y,$$

$$B_5(q,k)=\sum_{\substack{i\ \mathrm{odd}\\ r\ge 0}}(-1)^{r(q+1)}rc_i^{i+1-2r}\otimes c_{q-i}^{q-i+2r-k-1}N,$$

$$B_6(q,k)=\sum_{\substack{i\ \mathrm{odd}\\ r\ge 0}}\sum_{j=1}^{n-1}(-1)^{q(r+1)}c_i^{i-2r}N_j\otimes c_{q-i}^{q-i+2r-k+1}x^j.$$

Thus we must prove that for all $k\ge -1$ and all $q\ge k+1$,

$$\psi_q(c_q^{q-k})=\begin{cases}\sum_{t=1}^{8}A_t(q,k) & \text{if } k \text{ even},\\ \sum_{t=1}^{6}B_t(q,k) & \text{if } k \text{ odd}.\end{cases}\tag{2}$$

Let $P(q,k)$ denote the statement that $\psi_q(c_q^{q-k})$ is given by (2). The plan of the proof is the following induction scheme:

Step 1: Establish $P(q,q-1)$ for all $q\ge 0$ by induction on q.

Step 2: Establish $P(q,-1)$ for all $q\ge 0$ by induction on q. The case $q=0$ in Step 1 starts the induction here.

Step 3: Let $k>-1$. Assuming $P(p,k-1)$ holds for all $p\ge k$, deduce that $P(q,k)$ holds for all $q\ge k+1$ by induction on q. The case $q=k+1$ in Step 1 starts the induction here.

We proceed with Step 1. The statement $P(q,q-1)$ reduces to

$$\psi_q(c_q^1)=\begin{cases}\sum_{i\ \mathrm{even}}c_i^1\otimes c_{q-i}^1+\sum_{i\ \mathrm{odd}}c_i^1\otimes c_{q-i}^1x & \text{if } q \text{ odd},\\ \sum_{i\ \mathrm{even}}c_i^1\otimes c_{q-i}^1+\sum_{i\ \mathrm{odd}}\sum_{j=1}^{n-1}c_i^1N_j\otimes c_{q-i}^1x^j & \text{if } q \text{ even}.\end{cases}\tag{3}$$

In the case of q odd, the summations which appear in (3) are $A_1(q,q-1)$ and $A_5(q,q-1)$, respectively; the other $A_t(q,q-1)$ are all 0. In the case of q even, the summations which appear in (3) are $B_1(q,q-1)$ and $B_6(q,q-1)$, respectively; the other $B_t(q,q-1)$ are all 0.

The statement $P(0, -1)$ is immediate from (1). Let $q>0$ and suppose, inductively, $P(q-1, q-2)$ holds. The cases q odd and q even must be treated separately.

Suppose q is odd. Then $\partial_q(c_q^1)=c_{q-1}^1(x-1)$ and so by (1),

$$\psi_q(c_q^1)=U_{q-1}(\psi_{q-1}(c_{q-1}^1)x)-U_{q-1}\psi_{q-1}(c_{q-1}^1).$$

By the inductive hypothesis,

$$\begin{aligned}\psi_q(c_q^1)=&\sum_{i\text{ even}} U_{q-1}(c_i^1x\otimes c_{q-1-i}^1x)+\sum_{i\text{ odd}}\sum_{j=1}^{n-1} U_{q-1}(c_i^1xN_j\otimes c_{q-1-i}^1x^{j+1})\\ &-\sum_{i\text{ even}} U_{q-1}(c_i^1\otimes c_{q-1-i}^1)-\sum_{i\text{ odd}}\sum_{j=1}^{n-1} U_{q-1}(c_i^1N_j\otimes c_{q-1-i}^1x^j).\end{aligned}\tag{4}$$

From Theorem 3.3 and the definition of U we obtain

$$\begin{aligned}&\sum_{i\text{ even}} U_{q-1}(c_i^1\otimes c_{q-1-i}^1)=0=\sum_{i\text{ odd}}\sum_{j=1}^{n-1} U_{q-1}(c_i^1N_j\otimes c_{q-1-i}^1x^j),\\ &\sum_{i\text{ even}} U_{q-1}(c_i^1x\otimes c_{q-1-i}^1x)=\sum_{i\text{ even}} c_{i+1}^1\otimes c_{q-1-i}^1x+c_0^1\otimes c_q^1\\ &\qquad=\sum_{i\text{ odd}} c_i^1\otimes c_{q-i}^1x+c_0^1\otimes c_q^1,\\ &\sum_{i\text{ odd}}\sum_{j=1}^{n-1} U_{q-1}(c_i^1xN_j\otimes c_{q-1-i}^1x^{j+1})=\sum_{\substack{i\text{ odd}\\ i>0}} c_{i+1}^1\otimes c_{q-1-i}^1=\sum_{\substack{i\text{ even}\\ i>0}} c_i^1\otimes c_{q-i}^1.\end{aligned}\tag{5}$$

In this last summation, the only non-zero contributions come from the $j=n-1$ terms. (4) and (5) imply (3) if q is odd, and so $P(q-1, q-2)$ implies $P(q, q-1)$ in this case.

Suppose q is even. Then $\partial_q(c_q^1)=c_{q-1}^1N$ and so by (1),

$$\psi_q(c_q^1)=\sum_{j=0}^{n-1} U_{q-1}(\psi_{q-1}(c_{q-1}^1)x^j).$$

By the inductive hypothesis we have

$$\psi_q(c_q^1)=\sum_{j=0}^{n-1}\sum_{i\text{ even}} U_{q-1}(c_i^1x^j\otimes c_{q-1-i}^1x^j)+\sum_{j=0}^{n-1}\sum_{i\text{ odd}} U_{q-1}(c_i^1x^j\otimes c_{q-1-i}^1x^{j+1}).\tag{6}$$

From Theorem 3.3 and the definition of U we obtain

$$\sum_{i\text{ even}} U_{q-1}(c_i^1x^j\otimes c_{q-1-i}^1x^j)=\begin{cases}\displaystyle\sum_{i\text{ even}} c_{i+1}^1N_j\otimes c_{q-1-i}^1x^j & \text{if } 0\le j\le n-2,\\ \displaystyle\sum_{i\text{ even}} c_{i+1}^1N_j\otimes c_{q-1-i}^1x^j+c_0^1\otimes c_q^1 & \text{if } j=n-1\end{cases}$$

and so, since $N_0=0$, we obtain

(7) $$\sum_{j=0}^{n-1}\sum_{i\text{ even}} U_{q-1}(c_i^1x^j\otimes c_{q-1-i}^1x^j)=\sum_{j=1}^{n-1}\sum_{i\text{ odd}} c_i^1N_j\otimes c_{q-i}^1x^j+c_0^1\otimes c_q^1.$$

From Theorem 3.3 and the definition of U we obtain

$$\sum_{i\text{ odd}} U_{q-1}(c_i^1x^j\otimes c_{q-1-i}^1x^{j+1})=\begin{cases}0 & \text{if } 0\leq j\leq n-2,\\ \sum_{\substack{i\text{ odd}\\ i>0}} c_{i+1}^1\otimes c_{q-1-i}^1 & \text{if } j=n-1\end{cases}$$

and so

(8) $$\sum_{j=0}^{n-1}\sum_{i\text{ odd}} U_{q-1}(c_i^1x^j\otimes c_{q-1-i}^1x^{j+1})=\sum_{\substack{i\text{ even}\\ i>0}} c_i^1\otimes c_{q-i}^1.$$

(6), (7) and (8) yield (3) for q even. This completes Step 1 of the proof.

We proceed with Step 2. The statement $P(q,-1)$ reduces to

(9) $$\psi_q(c_q^{q+1})=\sum_{i\text{ even}} c_i^{i+1}\otimes c_{q-i}^{q-i+1}+\sum_{i\text{ odd}} c_i^{i+1}\otimes c_{q-i}^{q-i+1}y \quad \text{for } q\geq 0.$$

The summations which appear in (6) are $B_1(q,-1)$ and $B_4(q,-1)$, respectively; the other $B_t(q,-1)$ are all 0. We already know $P(0,-1)$ is true.

Let $q>0$ and suppose, inductively, $P(q-1,-1)$ holds. We have $\partial_q(c_q^{q+1})=c_{q-1}^q(y\pm 1)$ and so by (1),

$$\psi_q(c_q^{q+1})=U_{q-1}(\psi_{q-1}(c_{q-1}^q)y)\pm U_{q-1}\psi_{q-1}(c_{q-1}^q).$$

By the inductive hypothesis,

(10) $$\begin{aligned}\psi_q(c_q^{q+1})=&\sum_{i\text{ even}} U_{q-1}(c_i^{i+1}y\otimes c_{q-1-i}^{q-i}y)+\sum_{i\text{ odd}} U_{q-1}(c_i^{i+1}y\otimes c_{q-1-i}^{q-i})\\ &\pm\Bigg(\sum_{i\text{ even}} U_{q-1}(c_i^{i+1}\otimes c_{q-1-i}^{q-i})+\sum_{i\text{ odd}} U_{q-1}(c_i^{i+1}\otimes c_{q-1-i}^{q-i}y)\Bigg).\end{aligned}$$

From Theorem 3.3 and the definiton of U we obtain

(11) $$\begin{aligned}\sum_{i\text{ even}} U_{q-1}(c_i^{i+1}y\otimes c_{q-1-i}^{q-i}y)&=\sum_{i\text{ even}} c_{i+1}^{i+2}\otimes c_{q-1-i}^{q-i}y+c_0^1\otimes c_q^{q+1}\\ &=\sum_{i\text{ odd}} c_i^{i+1}\otimes c_{q-i}^{q-i+1}y+c_0^1\otimes c_q^{q+1},\\ \sum_{i\text{ odd}} U_{q-1}(c_i^{i+1}y\otimes c_{q-1-i}^{q-i})&=\sum_{\substack{i\text{ odd}\\ i>0}} c_{i+1}^{i+2}\otimes c_{q-1-i}^{q-i}=\sum_{\substack{i\text{ even}\\ i>0}} c_i^{i+1}\otimes c_{q-i}^{q-i+1},\\ \sum_{i\text{ even}} U_{q-1}(c_i^{i+1}\otimes c_{q-1-i}^{q-i})&=0=\sum_{i\text{ odd}} U_{q-1}(c_i^{i+1}\otimes c_{q-1-i}^{q-i}y).\end{aligned}$$

(10) and (11) imply (9), completing Step 2.

To facilitate Step 3 we interpose six lemmas whose proofs are deferred until §6.

LEMMA 3.5. *Suppose $k\geq 0$ is even and $q>k$. Then*

(a) $U_q(A_1(q,k)yx)=A_7(q+1,k)+\sum_{r\geq 0}(-1)^{rq}c^1_{2r+1}\otimes c^{q-k}_{q-2r}yx+c^1_0\otimes c^{q+1-k}_{q+1}$;

(b) $U_q(A_2(q,k)yx)=A_6(q+1,k)-\sum_{r\geq 0}(-1)^{rq}c^1_{2r+1}\otimes c^{q-k}_{q-2r}yx$;

(c) $U_q(A_3(q,k)yx)=A_5(q+1,k)-\sum_{r\geq 0}(-1)^{rq}c^1_{2r+1}\otimes c^{q+1-k}_{q-2r}x$;

(d) $U_q(A_4(q,k)yx)=A_8(q+1,k)-\sum_{r\geq 0}(-1)^{rq}rc^1_{2r+1}\otimes c^{q-k}_{q-2r}yN$;

(e) $U_q(A_5(q,k)yx)=A_3(q+1,k)$;

(f) $U_q(A_6(q,k)yx)=A_2(q+1,k)$;

(g) $U_q(A_7(q,k)yx)=A_1(q+1,k)-\sum_{r\geq 0}(-1)^{rq}c^1_{2r}\otimes c^{q+1-k}_{q+1-2r}$;

(h) $U_q(A_8(q,k)yx)=A_4(q+1,k)$.

LEMMA 3.6. *Suppose $k\geq 0$ is even and $q>k$. Then $U_q(A_t(q,k))=0$ for $1\leq t\leq 8$.*

LEMMA 3.7. *Suppose $k\geq 0$ is even and $q>k+1$. Then*

(a) $U_q(A_1(q,k)N)=\sum_{r\geq 0}\sum_{j=1}^{n-1}(-1)^{r(q+1)}c^1_{2r+1}N_j\otimes c^{q-k}_{q-2r}x^j$;

(b) $U_q(A_5(q,k)N)=\sum_{r>0}(-1)^{(r+1)(q+1)}c^1_{2r}\otimes c^{q-k}_{q+1-2r}$;

(c) $U_q(A_6(q,k)N)=\sum_{r>0}(-1)^{(r+1)(q+1)}c^1_{2r}\otimes c^{q-1-k}_{q+1-2r}y$;

(d) $U_q(A_8(q,k)N)=\sum_{r>0}(-1)^{(r+1)(q+1)}(r-1)c^1_{2r}\otimes c^{q-1-k}_{q+1-2r}yN$;

(e) $U_q(A_t(q,k)N)=0$ *for* $t=2,3,4,$ *and* 7.

LEMMA 3.8. *Suppose $k\geq -1$ is odd and $q>k$. Then $U_q(B_t(q,k))=0$ for $1\leq t\leq 6$.*

LEMMA 3.9. *Suppose $k\geq 1$ is odd and $q>k$. Then*

(a) $U_q(B_1(q,k)y)=B_4(q+1,k)+c^1_0\otimes c^{q+1-k}_{q+1}$;

(b) $U_q(B_2(q,k)y)=B_5(q+1,k)$;

(c) $U_q(B_3(q,k)y)=B_6(q+1,k)-\sum_{r\geq 0}\sum_{j=1}^{n-1}(-1)^{(q+1)(r+1)}c^1_{2r+1}N_j\otimes c^{q+1-k}_{q-2r}x^j$;

(d) $U_q(B_4(q, k)y) = B_1(q+1, k) - \sum_{r \geq 0} (-1)^{r(q+1)} c^1_{2r} \otimes c^{q+1-k}_{q+1-2r}$;

(e) $U_q(B_5(q, k)y) = B_2(q+1, k) - \sum_{r \geq 0} (-1)^{r(q+1)} r c^1_{2r} \otimes c^{q-k}_{q+1-2r} yN$;

(f) $U_q(B_6(q, k)y) = B_3(q+1, k) + \sum_{r > 0} (-1)^{r(q+1)} c^1_{2r} \otimes c^{q-k}_{q+1-2r} yN$

$$- \sum_{r > 0} (-1)^{r(q+1)} c^1_{2r} \otimes c^{q-k}_{q+1-2r} y.$$

LEMMA 3.10. *Suppose $k \geq -1$ is odd and $q > k+1$. Then*

(a) $U_q(B_1(q, k)x) = \sum_{r \geq 0} (-1)^{rq} c^1_{2r+1} \otimes c^{q-k}_{q-2r} x$;

(b) $U_q(B_2(q, k)x) = \sum_{r \geq 0} (-1)^{rq} r c^1_{2r+1} \otimes c^{q-1-k}_{q-2r} yN$;

(c) $U_q(B_6(q, k)x) = \sum_{r > 0} (-1)^{rq} c^1_{2r} \otimes c^{q-k}_{q+1-2r}$;

(d) $U_q(B_t(q, k)x) = 0$ *for* $t = 3, 4,$ *and* 5.

We proceed with Step 3. Suppose $k > -1$ and that $P(p, k-1)$ holds for all $p \geq k$. By Step 1, $P(k+1, k)$ holds. Let $q > k+1$ and suppose, inductively, $P(q-1, k)$ holds. The cases k even and k odd require separate treatment.

Suppose k is even. Then $\partial_q(c^{q-k}_q) = c^{q-k-1}_{q-1}(yx \pm 1) + c^{q-k}_{q-1}(x-1)$ and so by (1),

$$\psi_q(c^{q-k}_q) = U_{q-1}(\psi_{q-1}(c^{q-k-1}_{q-1})yx) \pm U_{q-1}\psi_{q-1}(c^{q-k-1}_{q-1}) + U_{q-1}(\psi_{q-1}(c^{q-k}_{q-1})x) - U_{q-1}\psi_{q-1}(c^{q-k}_{q-1}).$$

By the induction hypothesis,

$$\begin{aligned} \psi_q(c^{q-k}_q) = {} & \sum_{t=1}^{8} U_{q-1}(A_t(q-1, k)yx) \pm \sum_{t=1}^{8} U_{q-1}(A_t(q-1, k)) \\ & + \sum_{t=1}^{6} U_{q-1}(B_t(q-1, k-1)x) - \sum_{t=1}^{6} U_{q-1}(B_t(q-1, k-1)). \end{aligned} \tag{12}$$

Using Lemmas 3.5, 3.6, 3.8, and 3.10 to express the right-hand side of (12), one easily deduces that $\psi_q(c^{q-k}_q)$ is given by (2).

Suppose k is odd. Then $\partial_q(c^{q-k}_q) = c^{q-1-k}_{q-1}(y \pm 1) + (-1)^q c^{q-k}_{q-1} N$ and so by (1),

$$\psi_q(c^{q-k}_q) = U_{q-1}(\psi_{q-1}(c^{q-1-k}_{q-1})y) \pm U_{q-1}\psi_{q-1}(c^{q-1-k}_{q-1}) + (-1)^q U_{q-1}(\psi_{q-1}(c^{q-k}_{q-1})N).$$

By the induction hypothesis,

$$\psi_q(c_q^{q-k}) = \sum_{t=1}^{6} U_{q-1}(B_t(q-1,k)y) \pm \sum_{t=1}^{6} U_{q-1}(B_t(q-1,k)) + (-1)^q \sum_{t=1}^{8} U_{q-1}(A_t(q-1,k-1)N). \tag{13}$$

Using Lemmas 3.7, 3.8, and 3.9 to express the right-hand side of (13), one easily deduces that $\psi_q(c_q^{q-k})$ is given by (2).

This completes Step 3, modulo the proofs of Lemmas 3.5–3.10 (see §6). □

4. Cochain complexes and products. This section is concerned with cochain-level computations arising from the Wall-Hamada resolution and our diagonal approximation, in preparation for the cohomology determinations in §5.

Let A be a right ZD_{2n}-module. Write $C_A^q = \mathrm{Hom}_{ZD_{2n}}(C_q, A)$ where C is the Wall-Hamada resolution. We first describe the coboundary maps $\delta^q\colon C_A^q \to C_A^{q+1}$. If $a \in A$ and $1 \le i \le q+1$, let $a_q^i \in C_A^q$ denote the cochain characterized by $a_q^i(c_q^j) = \delta_i^j a$ where δ_i^j is the Kronecker delta. (We will sometimes write $(a)_q^i$ if a represents an expression consisting of more than one symbol.) Thus for any $z \in C_A^q$,

$$z = \sum_{i=1}^{q+1} (z(c_q^i))_q^i .$$

Following standard sign conventions, the coboundary maps δ^q are characterized by $\delta^q(\alpha)(u) = (-1)^{q+1}\alpha(\partial_{q+1}(u))$. The following is a routine consequence of the boundary formulas in §2:

Proposition 4.1. *Let $n \ge 2$ and suppose A is a right ZD_{2n}-module. Then for $a \in A$ and $1 \le i \le q+1$,*

$$\delta^q(a_q^i) = \begin{cases} -(a(x-1))_{q+1}^i - (a(y-\varepsilon_{i+1}\varepsilon_{q+1}))_{q+1}^{i+1} & \text{if } q \text{ even}, \ i \text{ odd}, \\ (aN)_{q+1}^i - (a(yx-\varepsilon_{i+1}\varepsilon_{q+1}))_{q+1}^{i+1} & \text{if } q \text{ even}, \ i \text{ even}, \\ (aN)_{q+1}^i + (a(yx+\varepsilon_{i+1}\varepsilon_{q+1}))_{q+1}^{i+1} & \text{if } q \text{ odd}, \ i \text{ odd}, \\ (a(x-1))_{q+1}^i + (a(y-\varepsilon_{i+1}\varepsilon_{q+1}))_{q+1}^{i+1} & \text{if } q \text{ odd}, \ i \text{ even}. \end{cases}$$

If A and B are right ZD_{2n}-modules, so is $A \otimes B$ via the diagonal action and we have a cochain cup product pairing

$$C_A^* \otimes C_B^* \to C_{A\otimes B}^*$$

arising from our diagonal approximation Δ (Theorem 2.2) which induces the standard cohomology cup product pairing

$$H^*(D_{2n}; A) \otimes H^*(D_{2n}; B) \to H^*(D_{2n}; A \otimes B).$$

For $\alpha \in C_A^s$ and $\beta \in C_B^t$, the above cochain cup product $\alpha\beta \in C_{A\otimes B}^{s+t}$ is characterized by $(\alpha\beta)(u) = (\alpha \otimes \beta)(\Delta u)$ for $u \in C_{s+t}$ where $(\alpha\otimes\beta)(v \otimes w) = (-1)^{st}\alpha(v) \otimes \beta(w)$. Our next task is to determine the cochain cup products $a_s^u b_t^v$ for $a \in A, b \in B$, $1 \le u \le s+1$, and $1 \le v \le t+1$.

Let C_s^u denote the ZD_{2n}-submodule of C_s generated by c_s^u. Then $C \otimes C$ is the direct sum of the $C_s^u \otimes C_t^v$. If $z \in C \otimes C$, we write $z_{s,t}^{u,v}$ for the $C_s^u \otimes C_t^v$-component of z with respect to this direct sum decomposition, and $\Delta_{s,t}^{u,v}$ for the $C_s^u \otimes C_t^v$-component of Δ.

In the lemma below, the $A_i(q, k)$ and $B_i(q, k)$ are as in the proof of Theorem 3.4.

LEMMA 4.2. *Suppose* $1 \le u \le s+1$ *and* $1 \le v \le t+1$. *Then the* $A_i(s+t, k)_{s,t}^{u,v}$ *for* k *even,* $0 \le k \le s+t-1$, *and the* $B_i(s+t, k)_{s,t}^{u,v}$ *for* k *odd,* $-1 \le k \le s+t-1$, *are all* 0 *except for the following cases*:

(a) $A_1(s+t, s+t-u-v+1)_{s,t}^{u,v} = (-1)^{(t+1)(s-u+1)/2} c_s^u \otimes c_t^v$
for s *even,* u *odd, and* $t-v$ *even*;

(b) $A_2(s+t, s+t-u-v)_{s,t}^{u,v} = (-1)^{t(s-u)/2} c_s^u \otimes c_t^v$
for s *even,* u *even, and* $t-v$ *even*;

(c) $A_3(s+t, s+t-u-v+1)_{s,t}^{u,v} = (-1)^{t(s-u)/2} c_s^u \otimes c_t^v y$
for s *even,* u *even, and* $t-v$ *odd*;

(d) $A_4(s+t, s+t-u-v)_{s,t}^{u,v} = (-1)^{t(s-u)/2}(1/2)(s-u) c_s^u \otimes c_t^v N$
for s *even,* u *even, and* $t-v$ *even*;

(e) $A_5(s+t, s+t-u-v+1)_{s,t}^{u,v} = (-1)^{t(s-u)/2} c_s^u \otimes c_t^v x$
for s *odd,* u *odd, and* $t-v$ *odd*;

(f) $A_6(s+t, s+t-u-v)_{s,t}^{u,v} = (-1)^{t(s-u)/2} c_s^u \otimes c_t^v yx$
for s *odd,* u *odd, and* $t-v$ *even*;

(g) $A_7(s+t, s+t-u-v+1)_{s,t}^{u,v} = (-1)^{(t+1)(s-u+1)/2} c_s^u x \otimes c_t^v yx$
for s *odd,* u *even, and* $t-v$ *even*;

(h) $A_8(s+t, s+t-u-v)_{s,t}^{u,v} = (-1)^{t(s-u)/2}(1/2)(s-u) c_s^u \otimes c_t^v yN$
for s *odd,* u *odd, and* $t-v$ *even*;

(i) $B_1(s+t, s+t-u-v+1)_{s,t}^{u,v} = (-1)^{t(s-u+1)/2} c_s^u \otimes c_t^v$
for s *even,* u *odd, and* $t-v$ *odd*;

(j) $B_2(s+t, s+t-u-v)_{s,t}^{u,v} = (-1)^{t(s-u+1)/2}(1/2)(s-u+1) c_s^u \otimes c_t^v yN$
for s *even,* u *odd, and* $t-v$ *even*;

(k) $B_3(s+t, s+t-u-v+1)_{s,t}^{u,v} = \sum_{j=1}^{n-1} (-1)^{(t+1)(s-u+2)/2} c_s^u N_j x^{-j} \otimes c_t^v yx^{-j}$
for s *even,* u *even, and* $t-v$ *even*;

(l) $B_4(s+t, s+t-u-v+1)_{s,t}^{u,v} = (-1)^{t(s-u+1)/2} c_s^u \otimes c_t^v y$
for s *odd,* u *even, and* $t-v$ *odd*;

(m) $B_5(s+t, s+t-u-v)_{s,t}^{u,v} = (-1)^{t(s-u+1)/2}(1/2)(s-u+1) c_s^u \otimes c_t^v N$
for s *odd,* u *even, and* $t-v$ *even*;

(n) $B_6(s+t, s+t-u-v+1)^{u,v}_{s,t} = \sum_{j=1}^{n-1} (-1)^{(t+1)(s-u+2)/2} c^u_s N_j \otimes c^v_t x^j$
for s odd, u odd, and t−v even.

PROOF. Suppose k is even and $0 \leq k \leq s+t-1$. Then term indexed by i and r in the summation defining $A_1(s+t, k)$ is $(-1)^{r(s+t+1)} c_i^{i+1-2r} \otimes c^{s+t-i+2r-k}_{s+t-i}$. Then latter will contribute to $A_1(s+t, k)^{u,v}_{s,t}$ if and only if $i=s$, $i+1-2r=u$, $s+t-i=t$, $s+t-i+2r-k=v$, i is even, and $s+t-i+2r-v$ is even. Thus, the only possible contribution occurs when s is even, u is odd, $r=(s-u+1)/2$, $t-v$ is even, and $k=s+t-u-v+1$. The assertion about the $A_1(s+t, k)^{u,v}_{s,t}$ now follows. The other parts are similar. □

PROPOSITION 4.3. *Let $n \geq 2$ and suppose A and B are right ZD_{2n}-modules. Then for $a \in A$, $b \in B$, $1 \leq u \leq s+1$, $1 \leq v \leq t+1$, the cochain cup product $a^u_s b^v_t \in C^{s+t}_{A \otimes B}$ derived from Theorem 2.2 is as follows*:

(a) $(-1)^{(t+1)(s-u+2)/2} (\sum_{j=1}^{n-1} aN_j x^{-j} \otimes byx^{-j})^{u+v-1}_{s+t}$
$+(-1)^{t(s-u)/2} (a \otimes b + (1/2)(s-u) a \otimes bN)^{u+v}_{s+t}$ *for s even, u even, and t−v even*;

(b) $(-1)^{t(s-u)/2} (a \otimes by)^{u+v-1}_{s+t}$ *for s even, u even, and t−v odd*;

(c) $(-1)^{(t+1)(s-u+1)/2} (a \otimes b)^{u+v-1}_{s+t} + (-1)^{t(s-u+1)/2} (1/2)(s-u+1)(a \otimes byN)^{u+v}_{s+t}$
for s even, u odd, and t−v even;

(d) $(-1)^{t(s-u+1)/2} (a \otimes b)^{u+v-1}_{s+t}$ *for s even, u odd, and t−v odd*;

(e) $(-1)^{t+(t+1)(s-u+1)/2} (ax \otimes byx)^{u+v-1}_{s+t} + (-1)^{t(s-u-1)/2} (1/2)(s-u+1)(a \otimes bN)^{u+v}_{s+t}$
for s odd, u even, and t−v even;

(f) $(-1)^{t(s-u-1)/2} (a \otimes by)^{u+v-1}_{s+t}$ *for s odd, u even, and t−v odd*;

(g) $(-1)^{t+(t+1)(s-u+2)/2} (\sum_{j=1}^{n-1} aN_j \otimes bx^j)^{u+v-1}_{s+t}$
$+(-1)^{t(s-u+2)/2} (a \otimes byx + (1/2)(s-u) a \otimes byN)^{u+v}_{s+t}$ *for s odd, u odd, and t−v even*;

(h) $(-1)^{t(s-u+2)/2} (a \otimes bx)^{u+v-1}_{s+t}$ *for s odd, u odd, and t−v odd.*

PROOF. We have

$$a^u_s b^v_t = \sum_{k=1}^{s+t+1} ((a^u_s b^v_t)(c^k_{s+t}))^k_{s+t} = \sum_{k=1}^{s+t+1} ((a^u_s \otimes b^u_t)(\Delta^{u,v}_{s,t}(c^k_{s+t})))^k_{s+t}.$$

It follows from Lemma 4.2 that the $\Delta^{u,v}_{s,t}(c^k_{s+t})$ are all 0, except possibly for $k=u+v-1$ or $u+v$.

Suppose s, u, and $t-v$ are all even. Scanning Lemma 4.2, we see that only case (k) contributes to $\Delta^{u,v}_{s,t}(c^{u+v-1}_{s+t})$, and only cases (b) and (d) contribute to $\Delta^{u,v}_{s,t}(c^{u+v}_{s+t})$. Thus

$$\begin{aligned}(a^u_s \otimes b^v_t)(\Delta^{u,v}_{s,t}(c^{u+v-1}_{s+t})) &= (a^u_s \otimes b^v_t)(\textstyle\sum_{j=1}^{n-1} (-1)^{(t+1)(s-u+2)/2} c^u_s N_j x^{-j} \otimes c^v_t yx^{-j}) \\ &= (-1)^{st} \sum_{j=1}^{n-1} (-1)^{(t+1)(s-u+2)/2} a^u_s(c^u_s N_j x^{-j}) \otimes b^v_t(c^v_t yx^{-j}) \\ &= \sum_{j=1}^{n-1} (-1)^{(t+1)(s-u+2)/2} aN_j x^{-j} \otimes byx^{-j},\end{aligned}$$

and

$$\begin{aligned}(a_s^u\otimes b_t^v)(\Delta_{s,t}^{u,v}(c_{s+t}^{u+v})) &= (a_s^u\otimes b_t^v)((-1)^{t(s-u)/2}c_s^u\otimes c_t^v+(-1)^{t(s-u)/2}(1/2)(s-u)c_s^u\otimes c_t^vN)\\ &=(-1)^{st}(-1)^{t(s-u)/2}(a_s^u(c_s^u)\otimes b_t^v(c_t^v)+(1/2)(s-u)a_s^u(c_s^u)\otimes b_t^v(c_t^vN))\\ &=(-1)^{t(s-u)/2}(a\otimes b+(1/2)(s-u)a\otimes bN).\end{aligned}$$

Part (a) now follows. The other parts are similar. □

5. Cohomology products. We first calculate the cohomology rings $H^*(D_{2n}; \boldsymbol{Z})$ and $H^*(D_{2n}; \boldsymbol{Z}/2\boldsymbol{Z})$, where the coefficients are simple. Although these rings have previously been known, it is useful to describe them in terms of cocycles coming from the Wall-Hamada resolution in order to use Proposition 4.3 to describe $H^*(D_{2n}; A)$ as a module over $H^*(D_{2n}; \boldsymbol{Z})$ (or over $H^*(D_{2n}; \boldsymbol{Z}/2\boldsymbol{Z})$ if A is a $(\boldsymbol{Z}/2\boldsymbol{Z})$-module) for certain non-trivial $\boldsymbol{Z}D_{2n}$-modules A. We accomplish the latter when A is any of the non-trivial $\boldsymbol{Z}D_{2n}$-modules whose underlying $\boldsymbol{Z}$-module is $\boldsymbol{Z}$.

For notational convenience, we write ι instead of 1 in $\boldsymbol{Z}$ when we regard $\boldsymbol{Z}$ as a trivial $\boldsymbol{Z}D_{2n}$-module. Under the canonical identification $\boldsymbol{Z}\otimes\boldsymbol{Z}=\boldsymbol{Z}$, we identify $\iota\otimes\iota$ with ι. Thus, using the notation of §4, $C_{\boldsymbol{Z}}^q$ is the free $\boldsymbol{Z}$-module with basis $\{\iota_q^i \mid 1\le i\le q+1\}$ for $q\ge 0$. From Proposition 4.1 we obtain

$$(5.1)\qquad \delta^q(\iota_q^i)=\begin{cases} -(1-\varepsilon_{i+1}\varepsilon_{q+1})\iota_{q+1}^{i+1} & \text{if } q \text{ even}, \quad i \text{ odd},\\ n\iota_{q+1}^i-(1-\varepsilon_{i+1}\varepsilon_{q+1})\iota_{q+1}^{i+1} & \text{if } q \text{ even}, \quad i \text{ even},\\ n\iota_{q+1}^i+(1+\varepsilon_{i+1}\varepsilon_{q+1})\iota_{q+1}^{i+1} & \text{if } q \text{ odd}, \quad i \text{ odd},\\ (1-\varepsilon_{i+1}\varepsilon_{q+1})\iota_{q+1}^{i+1} & \text{if } q \text{ odd}, \quad i \text{ even}.\end{cases}$$

If z is a cocycle, let $[z]$ denote its cohomology class.

If n is even, it follows from (5.1) that we have the following cohomology classes: $a_2=[\iota_2^3]\in H^2(D_{2n}; \boldsymbol{Z})$, $b_2=[(n/2)\iota_2^1+\iota_2^2]\in H^2(D_{2n}; \boldsymbol{Z})$, $c_3=[\iota_3^2]\in H^3(D_{2n}; \boldsymbol{Z})$, and $d_4=[\iota_4^1]\in H^4(D_{2n}; \boldsymbol{Z})$.

THEOREM 5.2. *Let $n\ge 2$ be even. Then $H^*(D_{2n}; \boldsymbol{Z})=\boldsymbol{Z}[a_2, b_2, c_3, d_4]/I$ where I is the ideal generated by $2a_2$, $2b_2$, $2c_3$, nd_4, $(b_2)^2+a_2b_2+(n^2/4)d_4$, and $(c_3)^2+a_2d_4$.*

PROOF. Using (5.1), one finds that $H^q(D_{2n}; \boldsymbol{Z})$ for positive q is as follows:

If $q\equiv 1 \pmod 4$, then $H^q(D_{2n}; \boldsymbol{Z})=(\boldsymbol{Z}/2\boldsymbol{Z})^{(q-1)/2}$ (a direct sume of $(q-1)/2$ copies of $\boldsymbol{Z}/2\boldsymbol{Z}$) with generators $[\iota_q^{4i}]$ for $1\le i\le(q-1)/4$, and $[(n/2)\iota_q^{4i+2}-\iota_q^{4i+3}]$ for $0\le i\le(q-5)/4$.

If $q\equiv 2 \pmod 4$, then $H^q(D_{2n}; \boldsymbol{Z})=(\boldsymbol{Z}/2\boldsymbol{Z})^{(q+2)/2}$ with generators $[\iota_q^{4i+3}]$ for $0\le i\le(q-2)/4$, and $[(n/2)\iota_q^{4i+1}+\iota_q^{4i+2}]$ for $0\le i\le(q-2)/4$.

If $q\equiv 3 \pmod 4$, then $H^q(D_{2n}; \boldsymbol{Z})=(\boldsymbol{Z}/2\boldsymbol{Z})^{(q-1)/2}$ with generators $[\iota_q^{4i+2}]$ for $0\le i\le(q-3)/4$, and $[(n/2)\iota_q^{4i}-\iota_q^{4i+1}]$ for $1\le i\le(q-3)/4$.

If $q\equiv 0 \pmod 4$, then $H^q(D_{2n}; \boldsymbol{Z})=(\boldsymbol{Z}/n\boldsymbol{Z})\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{q/2}$; $[\iota_q^1]$ generates the $\boldsymbol{Z}/n\boldsymbol{Z}$ summand, and the generators of the $\boldsymbol{Z}/2\boldsymbol{Z}$ summands are $[\iota_q^{4i+1}]$ for $1\le i\le q/4$, and

$[(n/2)\iota_q^{4i+3}+\iota_q^{4i+4}]$ for $0\leq i\leq(q-4)/4$.

We proceed to show that a_2, b_2, c_3, and d_4 multiplicatively generate $H^*(D_{2n};\boldsymbol{Z})$. Using Proposition 4.3, we obtain $(a_2)^2=[\iota_2^3\iota_2^3]=[\iota_4^5]$, $a_2b_2=[(n/2)\iota_2^3\iota_2^1+\iota_2^3\iota_2^2]=[(n/2)\iota_4^3+\iota_4^4]$, $a_2c_3=[\iota_2^3\iota_3^2]=[\iota_5^4]$, $-b_2c_3=[-(n/2)\iota_2^1\iota_3^2-\iota_2^2\iota_3^2]=[(n/2)\iota_5^2-\iota_5^3]$, $a_2d_4=[\iota_2^3\iota_4^1]=[\iota_6^3]$, $a_2(a_2)^2=[\iota_2^3\iota_4^5]=[\iota_6^7]$, $b_2d_4=[(n/2)\iota_2^1\iota_4^1+\iota_2^2\iota_4^1]=[(n/2)\iota_6^1+\iota_6^2]$, $(a_2)^2b_2=[(n/2)\iota_4^5\iota_2^1+\iota_4^5\iota_2^2]=[(n/2)\iota_6^5+\iota_6^6]$, $c_3d_4=[\iota_3^2\iota_4^1]=[\iota_7^2]$, $(a_2)^2c_3=[\iota_4^5\iota_3^2]=[\iota_7^6]$, and $-(a_2b_2)c_3=[-(n/2)\iota_4^3\iota_3^2-\iota_4^4\iota_3^2]=[(n/2)\iota_7^4-\iota_7^5]$, which shows that a_2, b_2, c_3, and d_4 multiplicatively generate $H^*(D_{2n};\boldsymbol{Z})$ in grades ≤ 7. Let $q>7$ and suppose, inductively, that a_2, b_2, c_3, and d_4 multiplicatively generate $H^*(D_{2n};\boldsymbol{Z})$ through grade $q-1$. Using Proposition 4.3, we obtain the following:

If $q\equiv 0 \pmod 4$, then
$[\iota_q^1]=d_4[\iota_{q-4}^1]$,
$[\iota_q^{4i+1}]=a_2[\iota_{q-2}^{4i-1}]$ for $1\leq i\leq q/4$, and
$[(n/2)\iota_q^{4i+3}+\iota_q^{4i+4}]=a_2[(n/2)\iota_{q-2}^{4i+1}+\iota_{q-2}^{4i+2}]$ for $0\leq i\leq (q-4)/4$.

If $q\equiv 1 \pmod 4$, then
$[(n/2)\iota_q^{4i+2}-\iota_q^{4i+3}]=a_2[(n/2)\iota_{q-2}^{4i}-\iota_{q-2}^{4i+1}]$ for $1\leq i\leq (q-5)/4$,
$[(n/2)\iota_q^2-\iota_q^3]=[(n/2)\iota_{q-4}^2-\iota_{q-4}^3]d_4$, and
$[\iota_q^{4i}]=a_2[\iota_{q-2}^{4i-2}]$ for $1\leq i\leq (q-1)/4$.

If $q\equiv 2 \pmod 4$, then
$[\iota_q^{4i+3}]=a_2[\iota_{q-2}^{4i+1}]$ for $0\leq i\leq (q-2)/4$,
$[(n/2)\iota_q^{4i+1}+\iota_q^{4i+2}]=a_2[(n/2)\iota_{q-2}^{4i-1}+\iota_{q-2}^{4i}]$ for $1\leq i\leq (q-2)/4$, and
$[(n/2)\iota_q^1+\iota_q^2]=[(n/2)\iota_{q-4}^1+\iota_{q-4}^2]d_4$.

If $q\equiv 3 \pmod 4$, then
$[\iota_q^{4i+2}]=a_2[\iota_{q-2}^{4i}]$ for $1\leq i\leq (q-3)/4$,
$[\iota_q^2]=[\iota_{q-4}^2]d_4$, and
$[(n/2)\iota_q^{4i}-\iota_q^{4i+1}]=a_2[(n/2)\iota_{q-2}^{4i-2}-\iota_{q-2}^{4i-1}]$ for $1\leq i\leq (q-3)/4$.

It follows that a_2, b_2, c_3, and d_4 multiplicatively generate $H^*(D_{2n};\boldsymbol{Z})$ through grade q, completing the induction.

The additive orders of a_2, b_2, c_3, and d_4 are implicit in the above. We next check that the relations $(b_2)^2+a_2b_2+(n^2/4)d_4=0$ and $(c_3)^2+a_2d_4=0$ hold. Equivalently, since the additive orders of a_2 and b_2 are both 2, it suffices to check that $-(b_2)^2+a_2b_2+(n^2/4)d_4=0$ and $(c_3)^2-a_2d_4=0$. We have $-(b_2)^2+a_2b_2+(n^2/4)d_4=-[(n^2/4)\iota_2^1\iota_2^1+(n/2)\iota_2^1\iota_2^2+(n/2)\iota_2^2\iota_2^1+\iota_2^2\iota_2^2]+[(n/2)\iota_2^3\iota_2^1+\iota_2^3\iota_2^2]+(n^2/4)[\iota_4^1]$ and $(c_3)^2-a_2d_4=[\iota_3^2\iota_3^2]-[\iota_2^3\iota_4^1]$. We apply the appropriate parts of Proposition 4.3 to calculate the above cochain products: From part (d) we obtain $\iota_2^1\iota_2^1=\iota_4^1$, $\iota_2^3\iota_2^1=\iota_4^3$, and $\iota_2^3\iota_4^1=\iota_6^3$; from part (c) we obtain $\iota_2^1\iota_2^2=-\iota_4^2+n\iota_4^3$ and $\iota_2^3\iota_2^2=\iota_4^4$; from part (b) we obtain $\iota_2^2\iota_2^1=\iota_4^2$; from part (a) we obtain $\iota_2^2\iota_2^2=-\sum_{j=1}^{n-1} j\iota_4^3+\iota_4^4=-(n-1)(n/2)\iota_4^3+\iota_4^4$; from part (f) we obtain $\iota_3^2\iota_3^2=\iota_6^3$. The two desired relations now follow easily.

Thus if A, B, C, D are abstract symbols of grades 2, 2, 3, and 4, respectively, the map of algebras $\boldsymbol{Z}[A,B,C,D]\to H^*(D_{2n};\boldsymbol{Z})$ which sends A, B, C, D to a_2, b_2, c_3, d_4, respectively, induces a surjective map of graded algebras $\boldsymbol{Z}[A,B,C,D]/J=R\to$

$H^*(D_{2n}; \boldsymbol{Z})$ where J is the ideal generated by $2A$, $2B$, $2C$, nD, $B^2+AB+(n^2/4)D$, and C^2+AD. Since each $H^q(D_{2n}; \boldsymbol{Z})$ is finite for $q>0$, it remains only to check that the order of R^q is at most the order of $H^q(D_{2n}; \boldsymbol{Z})$ for each $q>0$. By abuse of notation we write A, B, C, D for their images in R under the canonical projection.

If q is odd, then R^q is additively generated by the D^iA^jC where $i, j \geq 0$ and $4i+2j+3=q$, and the D^iA^jBC where $i, j \geq 0$ and $4i+2j+5=q$. All these elements have order dividing 2, and one checks that there are $(q-1)/2$ such elements. Thus the order of R^q is at most $2^{(q-1)/2}$, which is the order of $H^q(D_{2n}; \boldsymbol{Z})$.

If q is even, then R^q is additively generated by the D^iA^j where $i, j \geq 0$ and $4i+2j=q$, and the D^iA^jB where $i, j \geq 0$ and $4i+2j+2=q$. If $q \equiv 2 \pmod 4$, all these elements have order dividing 2, and one checks that there are $(q+2)/2$ such elements. If $q \equiv 0 \pmod 4$, $D^{q/4}$ has order dividing n, while the rest of these elements have order dividing 2. One checks that there are $q/2$ of the latter. In either case one sees that the order of R^q is at most the order of $H^q(D_{2n}; \boldsymbol{Z})$, completing the proof. □

If n is odd, it follows from (5.1) that we have the cohomology classes $a_2 = [\iota_2^3] \in H^2(D_{2n}; \boldsymbol{Z})$ and $d_4 = [\iota_4^1] \in H^4(D_{2n}; \boldsymbol{Z})$.

THEOREM 5.3. *Let $n \geq 3$ be odd. Then $H^*(D_{2n}; \boldsymbol{Z}) = \boldsymbol{Z}[a_2, d_4]/I$ where I is the ideal generated by $2a_2$ and nd_4.*

PROOF. Using (5.1) one finds that $H^q(D_{2n}; \boldsymbol{Z})$ for positive q is as follows:

If q is odd, then $H^q(D_{2n}; \boldsymbol{Z}) = 0$.

If $q \equiv 2 \pmod 4$, then $H^q(D_{2n}; \boldsymbol{Z}) = \boldsymbol{Z}/2$ with generator $[\iota_q^{q+1}]$.

If $q \equiv 0 \pmod 4$, then $H^q(D_{2n}; \boldsymbol{Z}) = (\boldsymbol{Z}/n\boldsymbol{Z}) \oplus (\boldsymbol{Z}/2\boldsymbol{Z})$; $[\iota_q^1]$ generates the $\boldsymbol{Z}/n\boldsymbol{Z}$ summand, and $[\iota_q^{q+1}]$ generates the $\boldsymbol{Z}/2\boldsymbol{Z}$ summand.

From Proposition 4.3(d), $\iota_2^3\iota_q^{q+1} = \iota_{q+2}^{q+3}$ for q even, and $\iota_4^1\iota_q^1 = \iota_{q+4}^1$ for $q \equiv 0 \pmod 4$. It follows by induction on q that $[\iota_q^{q+1}] = (a_2)^{q/2}$ for q even, and $[\iota_q^1] = (d_4)^{q/4}$ for $q \equiv 0 \pmod 4$. The assertion now follows. □

We denote the generator of $\boldsymbol{Z}/2\boldsymbol{Z}$ by λ. From Proposition 4.1 we obtain, for $1 \leq i \leq q+1$,

$$\delta^q(\lambda_q^i) = \begin{cases} 0 & \text{if } q+i \text{ is odd}, \\ n\lambda_{q+1}^i & \text{if } q+i \text{ is even}. \end{cases} \tag{5.4}$$

If n is even, it follows from (5.4) that $\delta^q(\lambda_q^i) = 0$ for all q and i. Thus for $q \geq 0$, $H^q(D_{2n}; \boldsymbol{Z}/2\boldsymbol{Z})$ has a $(\boldsymbol{Z}/2\boldsymbol{Z})$-basis consisting of the $[\lambda_q^i]$ for $1 \leq i \leq q+1$. In particular, we have the cohomology classes $u_1 = [\lambda_1^1] \in H^1(D_{2n}; \boldsymbol{Z}/2\boldsymbol{Z})$, $v_1 = [\lambda_1^2] \in H^1(D_{2n}; \boldsymbol{Z}/2\boldsymbol{Z})$, and $w_2 = [\lambda_2^1] \in H^2(D_{2n}; \boldsymbol{Z}/2\boldsymbol{Z})$.

THEOREM 5.5. *Let $n \geq 2$ be even. Then $H^*(D_{2n}; \boldsymbol{Z}/2\boldsymbol{Z}) = (\boldsymbol{Z}/2)[u_1, v_1, w_2]/I$ where I is the ideal generated by $(u_1)^2 + u_1v_1 + (n/2)w_2$.*

PROOF. Under the canonical identification $(\boldsymbol{Z}/2\boldsymbol{Z}) \otimes (\boldsymbol{Z}/2\boldsymbol{Z}) = \boldsymbol{Z}/2\boldsymbol{Z}$, $\lambda \otimes \lambda$ is identified with λ. We first check that the relation $(u_1)^2+u_1v_1+(n/2)w_2=0$ holds. Using Proposition 4.3, $(u_1)^2=[\lambda_1^1\lambda_1^1]=[(n-1)(n/2)\lambda_2^1+\lambda_2^2]=[(n/2)\lambda_2^1+\lambda_2^2]$ by part (g), and $u_1v_1=[\lambda_1^1\lambda_1^2]=[\lambda_2^2]$ by part (h). The relation now follows.

We next check that u_1, v_1 and w_2 multiplicatively generate $H^*(D_{2n}; \mathbf{Z}/2\mathbf{Z})$. As noted above, $u_1v_1=[\lambda_2^2]$. By Proposition 4.2(f), $(v_1)^2=[\lambda_1^2\lambda_1^2]=[\lambda_2^3]$ and so u_1, v_1, and w_2 multiplicatively generate $H^*(D_{2n}; \mathbf{Z}/2\mathbf{Z})$ through dimension 2. Let $q>2$ and suppose, inductively, that u_1, v_1, and w_2 multiplicatively generate $H^*(D_{2n}; \mathbf{Z}/2\mathbf{Z})$ through dimension $q-1$. If $q-i$ is odd, $2\le i\le q+1$, then by Proposition 4.3(f), $[\lambda_q^i]=[\lambda_1^2\lambda_{q-1}^{i-1}]=v_1[\lambda_{q-1}^{i-1}]$. If $q-i$ is even, $1\le i\le q$, then by Proposition 4.3(h), $[\lambda_q^i]=[\lambda_1^1\lambda_{q-1}^i]=u_1[\lambda_{q-1}^i]$. If q is even, then by Proposition 4.3(d), $[\lambda_q^1]=[\lambda_2^1\lambda_{q-2}^1]=w_2[\lambda_{q-2}^1]$. It follows that u_1, v_1, and w_2 multiplicatively generate $H^q(D_{2n}; \mathbf{Z}/2\mathbf{Z})$, completing the induction.

Thus if U, V, and W are abstract symbols of grades 1, 1, and 2, respectively, the map of algebras $(\mathbf{Z}/2\mathbf{Z})[U, V, W]\to H^*(D_{2n}; \mathbf{Z}/2\mathbf{Z})$ which sends U, V, and W to u_1, v_1, and w_2, respectively, induces a surjective map of graded algebras $(\mathbf{Z}/2\mathbf{Z})[U, V, W]/J=R\to H^*(D_{2n}; \mathbf{Z}/2\mathbf{Z})$ where J is the ideal generated by $U^2+UV+(n/2)W$. Abusing notation, we write U, V, and W for their images in R under the canonical projection. R is additively generated by the monomials UV^iW^j and V^iW^j, $i, j\ge 0$. An easy counting argument, similar to that used in the proof of Theorem 5.2, shows that for each $q\ge 0$, there are precisely $q+1$ such monomials of grade q. Since $H^q(D_{2n}; \mathbf{Z}/2\mathbf{Z})$ is $(q+1)$-dimensional over $\mathbf{Z}/2\mathbf{Z}$, R is mapped isomorphically onto $H^*(D_{2n}; \mathbf{Z}/2\mathbf{Z})$, completing the proof. □

If n is odd, it follows from (5.4) that we have the cohomology class $v_1=[\lambda_1^2]\in H^1(D_{2n}; \mathbf{Z}/2\mathbf{Z})$.

THEOREM 5.6. *Let $n\ge 3$ be odd. Then $H^*(D_{2n}; \mathbf{Z}/2\mathbf{Z})=(\mathbf{Z}/2\mathbf{Z})[v_1]$.*

PROOF. Let $\langle x\rangle$ denote the subgroup of D_{2n} generated by x. Since $\langle x\rangle$ has odd order, it follows from the Lyndon-Hochschild-Serre spectral sequence of the extension

$$1\to\langle x\rangle\to D_{2n}\xrightarrow{p} D_{2n}/\langle x\rangle\to 1$$

(e.g. [1, Ch. VII] or [2, Ch. 7]) that $p^*: H^*(D_{2n}/\langle x\rangle; \mathbf{Z}/2\mathbf{Z})\to H^*(D_{2n}; \mathbf{Z}/2\mathbf{Z})$ is an isomorphism. Since $D_{2n}/\langle x\rangle$ is cyclic of order 2, it follows that $H^*(D_{2n}; \mathbf{Z}/2\mathbf{Z})$ is a polynomial algebra over $\mathbf{Z}/2\mathbf{Z}$ on a 1-dimensional class. By (5.4), $H^1(D_{2n}; \mathbf{Z}/2\mathbf{Z})$ is generated by v_1. The theorem now follows. □

We next describe $H^*(D_{2n}; M)$ as a module over $H^*(D_{2n}; \mathbf{Z})$ when M is a nontrivial $\mathbf{Z}D_{2n}$-module whose underlying $\mathbf{Z}$-module is free on one generator. x and y can only act via multiplication by ± 1. If n is odd, x can only act as the identity.

Let M_α denote the $\mathbf{Z}D_{2n}$-module where M_α is the free abelian group on one generator α with D_{2n}-action given by $\alpha x=\alpha$, $\alpha y=-\alpha$. From Proposition 4.1 we obtain, for $1\le i\le q+1$,

$$(5.7) \qquad \delta^q(\alpha_q^i)=\begin{cases}(1+\varepsilon_{i+1}\varepsilon_{q+1})\alpha_{q+1}^{i+1} & \text{if } q \text{ even}, \quad i \text{ odd},\\ n\alpha_{q+1}^i+(1+\varepsilon_{i+1}\varepsilon_{q+1})\alpha_{q+1}^{i+1} & \text{if } q \text{ even}, \quad i \text{ even},\\ n\alpha_{q+1}^i-(1-\varepsilon_{i+1}\varepsilon_{q+1})\alpha_{q+1}^{i+1} & \text{if } q \text{ odd}, \quad i \text{ odd},\\ -(1+\varepsilon_{i+1}\varepsilon_{q+1})\alpha_{q+1}^{i+1} & \text{if } q \text{ odd}, \quad i \text{ even}.\end{cases}$$

It follows from (5.7) that we have the cohomology classes $\alpha_1=[\alpha_1^2]\in H^1(D_{2n};\, M_\alpha)$ and $\alpha_2=[\alpha_2^1]\in H^2(D_{2n};\, M_\alpha)$.

THEOREM 5.8. *Let $n\geq 2$ be even. Then $H^*(D_{2n};\, M_\alpha)$ is the free $H^*(D_{2n};\, \boldsymbol{Z})$-module on α_1 and α_2, modulo the $H^*(D_{2n};\, \boldsymbol{Z})$-submodule generated by $2\alpha_1$, $n\alpha_2$, $c_3\alpha_1+a_2\alpha_2$, and $d_4\alpha_1+c_3\alpha_2$.*

PROOF. From (5.7), $H^0(D_{2n};\, M_\alpha)=0$, and for $q>0$, $H^q(D_{2n};\, M_\alpha)$ is as follows:

If $q\equiv 1 \pmod 4$, then $H^q(D_{2n};\, M_\alpha)=(\boldsymbol{Z}/2\boldsymbol{Z})^{(q+1)/2}$ with generators $[\alpha_q^{4i+2}]$ for $0\leq i\leq (q-1)/4$, and $[(n/2)\alpha_q^{4i}+\alpha_q^{4i+1}]$ for $1\leq i\leq (q-1)/4$.

If $q\equiv 2 \pmod 4$, then $H^q(D_{2n};\, M_\alpha)=(\boldsymbol{Z}/n\boldsymbol{Z})\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{(q-2)/2}$. $[\alpha_q^1]$ generates the $\boldsymbol{Z}/n\boldsymbol{Z}$ summand; the generators of the $\boldsymbol{Z}/2\boldsymbol{Z}$ summands are $[\alpha_q^{4i+1}]$ for $1\leq i\leq (q-2)/4$ and $[(n/2)\alpha_q^{4i+3}-\alpha_q^{4i+4}]$ for $0\leq i\leq (q-6)/4$.

If $q\equiv 3 \pmod 4$, then $H^q(D_{2n};\, M_\alpha)=(\boldsymbol{Z}/2\boldsymbol{Z})^{(q+1)/2}$ with generators $[\alpha_q^{4i}]$ for $1\leq i\leq (q+1)/4$, and $[(n/2)\alpha_q^{4i+2}+\alpha_q^{4i+3}]$ for $0\leq i\leq (q-3)/4$.

If $q\equiv 0 \pmod 4$, then $H^q(D_{2n};\, M_\alpha)=(\boldsymbol{Z}/2\boldsymbol{Z})^{q/2}$ with generators $[\alpha_q^{4i+3}]$ for $0\leq i\leq (q-4)/4$, and $[(n/2)\alpha_q^{4i+1}-\alpha_q^{4i+2}]$ for $0\leq i\leq (q-4)/4$.

We proceed to show that α_1 and α_2 generate $H^*(D_{2n};\, M_\alpha)$ as an $H^*(D_{2n};\, \boldsymbol{Z})$-module. Under the canonical identification $\boldsymbol{Z}\otimes M_\alpha=M_\alpha$, we identify $\iota\otimes\alpha$ with α. Using Proposition 4.3 we obtain $b_2\alpha_1=-[(n/2)\alpha_3^2+\alpha_3^3]$, $a_2\alpha_1=[\alpha_3^4]$, $b_2\alpha_2=[(n/2)\alpha_4^1-\alpha_4^2]$, and $a_2\alpha_2=[\alpha_4^3]$ which shows that α_1 and α_2 generate $H^*(D_{2n};\, M_\alpha)$ as an $H^*(D_{2n};\, \boldsymbol{Z})$-module in grades ≤ 4. Let $q>4$ and suppose, inductively, α_1 and α_2 generate $H^*(D_{2n};\, M_\alpha)$ as an $H^*(D_{2n};\, \boldsymbol{Z})$-module in grades $\leq q-1$. Using Proposition 4.3 we obtain the following:

If $q\equiv 1 \pmod 4$, then

$[\alpha_q^2]=d_4[\alpha_{q-4}^2]$,

$[\alpha_q^{4i+2}]=a_2[\alpha_{q-2}^{4i}]$ for $1\leq i\leq (q-1)/4$, and

$[(n/2)\alpha_q^{4i}+\alpha_q^{4i+1}]=a_2[(n/2)\alpha_{q-2}^{4i-2}+\alpha_{q-2}^{4i-1}]$ for $1\leq i\leq (q-1)/4$.

If $q\equiv 2 \pmod 4$, then

$[\alpha_q^1]=d_4[\alpha_{q-4}^1]$,

$[\alpha_q^{4i+1}]=a_2[\alpha_{q-2}^{4i-1}]$ for $1\leq i\leq (q-2)/4$, and

$[(n/2)\alpha_q^{4i+3}-\alpha_q^{4i+4}]=a_2[(n/2)\alpha_{q-2}^{4i+1}-\alpha_{q-2}^{4i+2}]$ for $0\leq i\leq (q-6)/4$.

If $q\equiv 3 \pmod 4$, then

$[\alpha_q^{4i}]=a_2[\alpha_{q-2}^{4i-2}]$ for $1\leq i\leq (q+1)/4$,

$[(n/2)\alpha_q^2+\alpha_q^3]=d_4[(n/2)\alpha_{q-4}^2+\alpha_{q-4}^3]+2na_2[\alpha_{q-2}^2]$, and

$[(n/2)\alpha_q^{4i+2}+\alpha_q^{4i+3}]=a_2[(n/2)\alpha_{q-2}^{4i}+\alpha_{q-2}^{4i+1}]$ for $1\leq i\leq (q-3)/4$.

If $q\equiv 0 \pmod 4$, then

$[\alpha_q^{4i+3}] = a_2[\alpha_{q-2}^{4i+1}]$ for $0 \le i \le (q-4)/4$,
$[(n/2)\alpha_q^1 - \alpha_q^2] = d_4[(n/2)\alpha_{q-4}^1 - \alpha_{q-4}^2] - 2nd_4[\alpha_{q-4}^3]$, and
$[(n/2)\alpha_q^{4i+1} - \alpha_q^{4i+2}] = a_2[(n/2)\alpha_{q-2}^{4i-1} - \alpha_{q-2}^{4i}]$ for $1 \le i \le (q-4)/4$.

It follows that α_1 and α_2 generate $H^*(D_{2n}; M_\alpha)$ as an $H^*(D_{2n}; \boldsymbol{Z})$-module through grade q, completing the induction.

The additive orders of α_1 and α_2 are implicit in the above. By Proposition 4.3, $c_3\alpha_1 + a_2\alpha_2 = [\iota_3^2\alpha_1^2 + \iota_2^3\alpha_2^1] = [-\alpha_4^3 + \alpha_4^3] = 0$, and $d_4\alpha_1 + c_3\alpha_2 = [\iota_4^1\alpha_1^2 + \iota_3^2\alpha_2^1] = [\alpha_5^2 - \alpha_5^2] = 0$, establishing the stated relations.

Thus if F is the free graded $H^*(D_{2n}; \boldsymbol{Z})$-module on two generators A_1 and A_2 of grades 1 and 2, respectively, the $H^*(D_{2n}; \boldsymbol{Z})$-module homomorphism $F \to H^*(D_{2n}; M_\alpha)$ which sends A_i to α_i, $i = 1, 2$, induces a surjection of $H^*(D_{2n}; \boldsymbol{Z})$-modules $F/R \to H^*(D_{2n}; M_\alpha)$ where R is the submodule generated by $2A_1$, nA_2, $c_3A_1 + a_2A_2$, and $d_4A_1 + c_3A_2$. We will be done if we show that for each $q > 0$, the order of $(F/R)^q$ is at most the order of $H^q(D_{2n}; M_\alpha)$. Abusing notation, we write A_1 and A_2 for their images in F/R under the canonical projection. In view of the relations above and the structure of $H^*(D_{2n}; \boldsymbol{Z})$ as given by Theorem 5.2, F/R is additively generated by the $(a_2)^i(d_4)^jA_1$, $(a_2)^i(d_4)^jA_2$, $b_2(a_2)^i(d_4)^jA_1$, and $b_2(a_2)^i(d_4)^jA_2$ for $i, j \ge 0$. An easy counting argument, similar to that used in the proof of Theorem 5.2, shows that if $q > 0$, then:

If q is odd, precisely $(q+1)/2$ of the $(a_2)^i(d_4)^jA_1$ and $b_2(a_2)^i(d_4)^jA_1$ have grade q, and all of these have additive order dividing 2.

If q is even, precisely $q/2$ of the $(a_2)^i(d_4)^jA_2$ and $b_2(a_2)^i(d_4)^jA_2$ have grade q, and that all of these have additive order dividing 2, with the possible exception of $(d_4)^{(q-2)/2}A_2$ (when $q \equiv 2 \pmod 4$) which has additive order dividing n.

In all cases, it follows easily that the order of $(F/R)^q$ is at most the order of $H^*(D_{2n}; M_\alpha)$, completing the proof. □

THEOREM 5.9. *Let $n \ge 3$ be odd. Then $H^*(D_{2n}; M_\alpha)$ is the free $H^*(D_{2n}; \boldsymbol{Z})$-module on α_1 and α_2, modulo the $H^*(D_{2n}; \boldsymbol{Z})$-submodule generated by $2\alpha_1$ and $n\alpha_2$.*

PROOF. It follows from (5.7) that for $q \ge 0$, $H^q(D_{2n}; M_\alpha)$ is as follows:
0 if $q \equiv 0 \pmod 4$;
$\boldsymbol{Z}/n\boldsymbol{Z}$ with generator $[\alpha_q^1]$ if $q \equiv 2 \pmod 4$;
$\boldsymbol{Z}/2\boldsymbol{Z}$ with generator $[\alpha_q^{q+1}]$ if q is odd.

From Proposition 4.3(d), $a_2[\alpha_q^{q+1}] = [\iota_2^3\alpha_q^{q+1}] = [\alpha_{q+2}^{q+3}]$ for q odd, and $d_4[\alpha_q^1] = [\iota_4^1\alpha_q^1] = [\alpha_{q+4}^1]$ if $q \equiv 2 \pmod 4$. It follows by induction on q that $[\alpha_q^{q+1}] = (a_2)^{(q-1)/2}\alpha_1$ for q odd, and $[\alpha_q^1] = (d_4)^{(q-2)/4}\alpha_2$ for $q \equiv 2 \pmod 4$. In view of the structure of $H^*(D_{2n}; \boldsymbol{Z})$ as given by Theorem 5.3, the result now follows. □

For the remainder of this section we assume $n \ge 2$ is even.

Let M_β denote the $\boldsymbol{Z}D_{2n}$-module where M_β is the free abelian group on one generator β with D_{2n}-action given by $\beta x = -\beta$, $\beta y = \beta$. From Proposition 4.1 we obtain, for $1 \le i \le q+1$,

(5.10)
$$\delta^q(\beta_q^i)=\begin{cases} 2\beta_{q+1}^i-(1-\varepsilon_{i+1}\varepsilon_{q+1})\beta_{q+1}^{i+1} & \text{if } q \text{ even}, \quad i \text{ odd},\\ (1+\varepsilon_{i+1}\varepsilon_{q+1})\beta_{q+1}^{i+1} & \text{if } q \text{ even}, \quad i \text{ even},\\ -(1-\varepsilon_{i+1}\varepsilon_{q+1})\beta_{q+1}^{i+1} & \text{if } q \text{ odd}, \quad i \text{ odd},\\ -2\beta_{q+1}^i+(1-\varepsilon_{i+1}\varepsilon_{q+1})\beta_{q+1}^{i+1} & \text{if } q \text{ odd}, \quad i \text{ even}.\end{cases}$$

It follows from (5.10) that we have the cohomology classes $\beta_1=[\beta_1^1]\in H^1(D_{2n};M_\beta)$, $\beta_2=[\beta_2^2-\beta_2^3]\in H^2(D_{2n};M_\beta)$, and $\beta_3=[\beta_3^1-\beta_3^2]\in H^3(D_{2n};M_\beta)$.

THEOREM 5.11. (a) *Suppose* $n\equiv 0 \pmod 4$, $n\geq 4$. *Then* $H^*(D_{2n};M_\beta)$ *is the free* $H^*(D_{2n};\mathbf{Z})$*-module on* β_1, β_2, *and* β_3, *modulo the* $H^*(D_{2n};\mathbf{Z})$*-submodule generated by* $2\beta_1$, $2\beta_2$, $2\beta_3$, $b_2\beta_1+a_2\beta_1$, $b_2\beta_2$, $b_2\beta_3$, $c_3\beta_2+a_2\beta_3$, *and* $d_4\beta_2+c_3\beta_3$.

(b) *Suppose* $n\equiv 2 \pmod 4$, $n\geq 2$. *Then* $H^*(D_{2n};M_\beta)$ *is the free* $H^*(D_{2n};\mathbf{Z})$*-module on* β_1 *and* β_2, *modulo the* $H^*(D_{2n};\mathbf{Z})$*-submodule generated by* $2\beta_1$, $2\beta_2$, $c_3\beta_1+b_2\beta_2$, *and* $a_2b_2\beta_1+(a_2)^2\beta_1+c_3\beta_2$.

PROOF. From (5.10), $H^0(D_{2n};M_\beta)=0$, and for $q>0$, $H^q(D_{2n};M_\beta)$ is as follows:

If $q\equiv 1 \pmod 4$, then $H^q(D_{2n};M_\beta)=(\mathbf{Z}/2\mathbf{Z})^{(q+1)/2}$ with generators $[\beta_q^{4i+1}]$ for $0\leq i\leq (q-1)/4$, and $[\beta_q^{4i+3}-\beta_q^{4i+4}]$ for $0\leq i\leq (q-5)/4$.

If $q\equiv 2 \pmod 4$, then $H^q(D_{2n};M_\beta)=(\mathbf{Z}/2\mathbf{Z})^{q/2}$ with generators $[\beta_q^{4i}]$ for $1\leq i\leq (q-2)/4$, and $[\beta_q^{4i+2}-\beta_q^{4i+3}]$ for $0\leq i\leq (q-2)/4$.

If $q\equiv 3 \pmod 4$, then $H^q(D_{2n};M_\beta)=(\mathbf{Z}/2\mathbf{Z})^{(q+1)/2}$ with generators $[\beta_q^{4i+3}]$ for $0\leq i\leq (q-3)/4$, and $[\beta_q^{4i+1}-\beta_q^{4i+2}]$ for $0\leq i\leq (q-3)/4$.

If $q\equiv 0 \pmod 4$, then $H^q(D_{2n};M_\beta)=(\mathbf{Z}/2\mathbf{Z})^{q/2}$ with generators $[\beta_q^{4i+2}]$ for $0\leq i\leq (q-4)/4$, and $[\beta_q^{4i}-\beta_q^{4i+1}]$ for $1\leq i\leq q/4$.

We proceed to show that β_1, β_2, and β_3 generate $H^*(D_{2n};M_\beta)$ as an $H^*(D_{2n};\mathbf{Z})$-module. Under the canonical identification $\mathbf{Z}\otimes M_\beta=M_\beta$, we identify $\iota\otimes\beta$ with β. Using Proposition 4.3 we obtain $a_2\beta_1=[\beta_3^3]$, $a_2\beta_2=[\beta_4^4-\beta_4^5]$, and $c_3\beta_1=[\beta_4^2]$ which shows that β_1, β_2, and β_3 generate $H^*(D_{2n};M_\beta)$ as an $H^*(D_{2n};\mathbf{Z})$-module in grades ≤ 4. Let $q>4$ and suppose, inductively, β_1, β_2, and β_3 generate $H^*(D_{2n};M_\beta)$ as an $H^*(D_{2n};\mathbf{Z})$-module in grades $\leq q-1$. Using Proposition 4.3 we obtain the following:

If $q\equiv 1 \pmod 4$, then

$[\beta_q^1]=d_4[\beta_{q-4}^1]$,

$[\beta_q^{4i+1}]=a_2[\beta_{q-2}^{4i-1}]$ for $1\leq i\leq (q-1)/4$, and

$[\beta_q^{4i+3}-\beta_q^{4i+4}]=a_2[\beta_{q-2}^{4i+1}-\beta_{q-2}^{4i+2}]$ for $0\leq i\leq (q-5)/4$.

If $q\equiv 2 \pmod 4$, then

$[\beta_q^{4i}]=a_2[\beta_{q-2}^{4i-2}]$ for $1\leq i\leq (q-2)/4$,

$[\beta_q^2-\beta_q^3]=d_4[\beta_{q-4}^2-\beta_{q-4}^3]$, and

$[\beta_q^{4i+2}-\beta_q^{4i+3}]=a_2[\beta_{q-2}^{4i}-\beta_{q-2}^{4i+1}]$ for $1\leq i\leq (q-2)/4$.

If $q\equiv 3 \pmod 4$, then

$[\beta_q^{4i+3}]=a_2[\beta_{q-2}^{4i+1}]$ for $0\leq i\leq (q-3)/4$,

$[\beta_q^1-\beta_q^2]=d_4[\beta_{q-4}^1-\beta_{q-4}^2]$, and

$[\beta_q^{4i+1}-\beta_q^{4i+2}]=a_2[\beta_{q-2}^{4i-1}-\beta_{q-2}^{4i}]$ for $1\leq i\leq (q-3)/4$.

If $q \equiv 0 \pmod 4$, then

$$[\beta_q^2] = d_4[\beta_{q-4}^2],$$
$$[\beta_q^{4i+2}] = a_2[\beta_{q-2}^{4i}] \text{ for } 1 \le i \le (q-4)/4, \text{ and}$$
$$[\beta_q^{4i} - \beta_q^{4i+1}] = a_2[\beta_{q-2}^{4i-2} - \beta_{q-2}^{4i-1}] \text{ for } 1 \le i \le q/4.$$

It follows that β_1, β_2, and β_3 generate $H^*(D_{2n}; M_\beta)$ as an $H^*(D_{2n}; \boldsymbol{Z})$-module through grade q, completing the induction.

It is implicit in the above that $2\beta_i = 0$ for $i = 1$, 2, and 3. By Proposition 4.3,

$$\begin{aligned} -a_2\beta_1 + b_2\beta_1 &= [-\iota_2^3\beta_1^1 + (n/2)\iota_2^1\beta_1^1 + \iota_2^2\beta_1^1] \\ &= \left[-\beta_3^3 + (n/2)\beta_3^1 + \sum_{j=1}^{n-1}(-1)^j j\beta_3^2 + \beta_3^3\right] \\ &= [(n/2)\beta_3^1 - (n/2)\beta_3^2] = (n/2)\beta_3. \end{aligned}$$

Thus,

$$a_2\beta_1 + b_2\beta_1 = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4, \\ \beta_3 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

In particular, β_3 is superfluous as an $H^*(D_{2n}; \boldsymbol{Z})$-module generator if $n \equiv 2 \pmod 4$. The remaining relations are similarly checked using Proposition 4.3.

Let F be a free graded $H^*(D_{2n}; \boldsymbol{Z})$-module on generators B_i of grade i, $i = 1, 2, 3$ if $n \equiv 0 \pmod 4$, $i = 1, 2$ if $n \equiv 2 \pmod 4$. The map of $H^*(D_{2n}; \boldsymbol{Z})$-modules $F \to H^*(D_{2n}; M_\beta)$ which sends B_i to β_i induces a surjection of $H^*(D_{2n}; \boldsymbol{Z})$-modules $F/R \to H^*(D_{2n}; M_\beta)$ where R is the $H^*(D_{2n}; \boldsymbol{Z})$-submodule of F generated as follows: by $2B_1$, $2B_2$, $2B_3$, $b_2B_1 + a_2B_1$, b_2B_2, b_2B_3, $c_3B_2 + a_2B_3$, and $d_4B_2 + c_3B_3$ if $n \equiv 0 \pmod 4$; by $2B_1$, $2B_2$, $c_3B_1 + b_2B_2$, and $a_2b_2B_1 + (a_2)^2B_1 + c_3B_2$ if $n \equiv 2 \pmod 4$. For positive q, $(F/R)^q$ is a vector space over $\boldsymbol{Z}/2\boldsymbol{Z}$, and it remains only to check that its dimension does not exceed the dimension of $H^q(D_{2n}; M_\beta)$ over $\boldsymbol{Z}/2\boldsymbol{Z}$. Abusing notation, write B_i for its image in F/R under the canonical projection for each i.

Suppose $n \equiv 0 \pmod 4$. From the definition of R and the structure of $H^*(D_{2n}; \boldsymbol{Z})$ as given by Theorem 5.2, F/R is additively generated by the $(a_2)^i(d_4)^jB_1$, $(a_2)^ic_3(d_4)^jB_1$, $(a_2)^i(d_4)^jB_2$, and $(a_2)^i(d_4)^jB_3$, $i, j \ge 0$. An easy counting argument shows that if q is odd, exactly $(q+1)/2$ of the $(a_2)^i(d_4)^jB_1$ and $(a_2)^i(d_4)^jB_3$ have grade q; if q is even, exactly $q/2$ of the $(a_2)^ic_3(d_4)^jB_1$ and $(a_2)^i(d_4)^jB_2$ have grade q. In each case, the number of additive generators of $(F/R)^q$ is the dimension of $H^q(D_{2n}; M_\beta)$ over $\boldsymbol{Z}/2\boldsymbol{Z}$.

Suppose $n \equiv 2 \pmod 4$. From the definition of R and the structure of $H^*(D_{2n}; \boldsymbol{Z})$ as given by Theorem 5.2, F/R is additively generated by the $(a_2)^i(d_4)^jB_1$, $(a_2)^ib_2(d_4)^jB_1$, $(a_2)^i(d_4)^jB_2$, and $(a_2)^ib_2(d_4)^jB_2$, $i, j \ge 0$. An easy counting argument shows that if q is odd, exactly $(q+1)/2$ of the $(a_2)^i(d_4)^jB_1$ and $(a_2)^ib_2(d_4)^jB_1$ have grade q; if q is even, exactly $q/2$ of the $(a_2)^i(d_4)^jB_2$ and $(a_2)^ib_2(d_4)^jB_2$ have grade q. In each case, the number of additive generators of $(F/R)^q$ is the dimension of $H^q(D_{2n}; M_\beta)$ over $\boldsymbol{Z}/2\boldsymbol{Z}$, completing the proof. □

Let M_γ denote the ZD_{2n}-module where M_γ is the free abelian group on one generator γ with D_{2n}-action given by $\gamma x=\gamma y=-\gamma$. From Proposition 4.1 we obtain, for $1\le i\le q+1$,

$$(5.12)\qquad \delta^q(\gamma_q^i)=\begin{cases} 2\gamma_{q+1}^i+(1+\varepsilon_{i+1}\varepsilon_{q+1})\gamma_{q+1}^{i+1} & \text{if } q \text{ even}, \quad i \text{ odd},\\ -(1-\varepsilon_{i+1}\varepsilon_{q+1})\gamma_{q+1}^{i+1} & \text{if } q \text{ even}, \quad i \text{ even},\\ (1+\varepsilon_{i+1}\varepsilon_{q+1})\gamma_{q+1}^{i+1} & \text{if } q \text{ odd}, \quad i \text{ odd},\\ -2\gamma_{q+1}^i-(1+\varepsilon_{i+1}\varepsilon_{q+1})\gamma_{q+1}^{i+1} & \text{if } q \text{ odd}, \quad i \text{ even}.\end{cases}$$

It follows from (5.12) that we have the cohomology classes $\gamma_1=[\gamma_1^1+\gamma_1^2]\in H^1(D_{2n};M_\gamma)$, $\gamma_2=[\gamma_2^2]\in H^2(D_{2n};M_\gamma)$, and $\gamma_3=[\gamma_3^1]\in H^3(D_{2n};M_\gamma)$.

THEOREM 5.13. (a) *Suppose* $n\equiv 0 \pmod 4$, $n\ge 4$. *Then* $H^*(D_{2n};M_\gamma)$ *is the free* $H^*(D_{2n};Z)$*-module on* γ_1, γ_2, *and* γ_3, *modulo the* $H^*(D_{2n};Z)$*-submodule generated by* $2\gamma_1$, $2\gamma_2$, $2\gamma_3$, $b_2\gamma_1$, $a_2\gamma_2+b_2\gamma_2$, $a_2\gamma_3+b_2\gamma_3$, $c_3\gamma_2+a_2\gamma_3$, *and* $d_4\gamma_2+c_3\gamma_3$.

(b) *Suppose* $n\equiv 2 \pmod 4$, $n\ge 2$. *Then* $H^*(D_{2n};M_\gamma)$ *is the free* $H^*(D_{2n};Z)$*-module on* γ_1 *and* γ_2, *modulo the* $H^*(D_{2n};Z)$*-submodule generated by* $2\gamma_1$, $2\gamma_2$, $c_3\gamma_1+a_2\gamma_2+b_2\gamma_2$, *and* $a_2b_2\gamma_1+c_3\gamma_2$.

PROOF. From (5.12), $H^0(D_{2n};M_\gamma)=0$, and for $q>0$, $H^q(D_{2n};M_\gamma)$ is as follows:

If $q\equiv 1 \pmod 4$, then $H^q(D_{2n};M_\gamma)=(Z/2Z)^{(q+1)/2}$ with generators $[\gamma_q^{4i+3}]$ for $0\le i\le (q-5)/4$, and $[\gamma_q^{4i+1}+\gamma_q^{4i+2}]$ for $0\le i\le (q-1)/4$.

If $q\equiv 2 \pmod 4$, then $H^q(D_{2n};M_\gamma)=(Z/2Z)^{q/2}$ with generators $[\gamma_q^{4i+2}]$ for $0\le i\le (q-2)/4$, and $[\gamma_q^{4i}+\gamma_q^{4i+1}]$ for $1\le i\le (q-2)/4$.

If $q\equiv 3 \pmod 4$, then $H^q(D_{2n};M_\gamma)=(Z/2Z)^{(q+1)/2}$ with generators $[\gamma_q^{4i+1}]$ for $0\le i\le (q-3)/4$, and $[\gamma_q^{4i+3}+\gamma_q^{4i+4}]$ for $0\le i\le (q-3)/4$.

If $q\equiv 0 \pmod 4$, then $H^q(D_{2n};M_\gamma)=(Z/2Z)^{q/2}$ with generators $[\gamma_q^{4i}]$ for $1\le i\le q/4$, and $[\gamma_q^{4i+2}+\gamma_q^{4i+3}]$ for $0\le i\le (q-4)/4$.

We proceed to show that γ_1, γ_2, and γ_3 generate $H^*(D_{2n};M_\gamma)$ as an $H^*(D_{2n};Z)$-module. Under the canonical identification $Z\otimes M_\gamma=M_\gamma$, we identify $\iota\otimes\gamma$ with γ. Using Proposition 4.3 we obtain $a_2\gamma_1=[\gamma_3^3+\gamma_3^4]$, $a_2\gamma_2=[\gamma_4^4]$, and $c_3\gamma_1=[\gamma_4^2+\gamma_4^3]$ which shows that γ_1, γ_2, and γ_3 generate $H^*(D_{2n};M_\gamma)$ as an $H^*(D_{2n};Z)$-module in grades ≤ 4. Let $q>4$ and suppose, inductively, γ_1, γ_2, and γ_3 generate $H^*(D_{2n};M_\gamma)$ as an $H^*(D_{2n};Z)$-module in grades $\le q-1$. Using Proposition 4.3 we obtain the following:

If $q\equiv 1 \pmod 4$, then

$[\gamma_q^{4i+3}]=a_2[\gamma_{q-2}^{4i+1}]$ for $0\le i\le (q-5)/4$,
$[\gamma_q^1+\gamma_q^2]=d_4[\gamma_{q-4}^1+\gamma_{q-4}^2]$, and
$[\gamma_q^{4i+1}+\gamma_q^{4i+2}]=a_2[\gamma_{q-2}^{4i-1}+\gamma_{q-2}^{4i}]$ for $1\le i\le (q-1)/4$.

If $q\equiv 2 \pmod 4$, then

$[\gamma_q^2]=d_4[\gamma_{q-4}^2]$,
$[\gamma_q^{4i+2}]=a_2[\gamma_{q-2}^{4i}]$ for $1\le i\le (q-2)/4$, and
$[\gamma_q^{4i}+\gamma_q^{4i+1}]=a_2[\gamma_{q-2}^{4i-2}+\gamma_{q-2}^{4i-1}]$ for $1\le i\le (q-2)/4$.

If $q \equiv 3 \pmod 4$, then

$[\gamma_q^1] = d_4[\gamma_{q-4}^1]$,

$[\gamma_q^{4i+1}] = a_2[\gamma_{q-2}^{4i-1}]$ for $1 \le i \le (q-3)/4$, and

$[\gamma_q^{4i+3} + \gamma_q^{4i+4}] = a_2[\gamma_{q-2}^{4i+1} + \gamma_{q-2}^{4i+2}]$ for $0 \le i \le (q-3)/4$.

If $q \equiv 0 \pmod 4$, then

$[\gamma_q^{4i}] = a_2[\gamma_{q-2}^{4i-2}]$ for $1 \le i \le q/4$,

$[\gamma_q^2 + \gamma_q^3] = d_4[\gamma_{q-4}^2 + \gamma_{q-4}^3]$, and

$[\gamma_q^{4i+2} + \gamma_q^{4i+3}] = a_2[\gamma_{q-2}^{4i} + \gamma_{q-2}^{4i+1}]$ for $1 \le i \le (q-4)/4$.

It follows that γ_1, γ_2, and γ_3 generate $H^*(D_{2n}; M_\gamma)$ as an $H^*(D_{2n}; \boldsymbol{Z})$-module through grade q, completing the induction.

It is implicit in the above that $2\gamma_i = 0$ for $i = 1$, 2, and 3. By Proposition 4.3,

$$\begin{aligned} b_2\gamma_1 &= [(n/2)\iota_2^1\gamma_1^1 + (n/2)\iota_2^1\gamma_1^2 + \iota_2^2\gamma_1^1 + \iota_2^2\gamma_1^2] \\ &= \left[(n/2)\gamma_3^1 - (n/2)\gamma_3^2 + \sum_{j=1}^{n-1}(-1)^{j+1} j\gamma_3^2 + \gamma_3^3 - \gamma_3^3\right] \\ &= [(n/2)\gamma_3^1 - (n/2)\gamma_3^2 + (n/2)\gamma_3^2] = (n/2)\gamma_3 . \end{aligned}$$

Thus,

$$b_2\gamma_1 = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4, \\ \gamma_3 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

In particular, γ_3 is superfluous as an $H^*(D_{2n}; \boldsymbol{Z})$-module generator if $n \equiv 2 \pmod 4$. The remaining relations are similarly checked using Proposition 4.3.

The remainder of the proof is formally identical to that of Theorem 5.11 with the β_i replaced by γ_i. □

Using Propositon 4.3, other cup products resulting from pairings among the $\boldsymbol{Z}D_{2n}$-modules we have considered can be computed as needed. For example, if $n \ge 2$ is even, $M_\alpha \otimes M_\beta$ can be identified with M_γ as a $\boldsymbol{Z}D_{2n}$-module where we identify $\alpha \otimes \beta = \gamma$. Under the cup product pairing

$$H^*(D_{2n}; M_\alpha) \otimes H^*(D_{2n}; M_\beta) \to H^*(D_{2n}; M_\gamma)$$

we have $\alpha_1\beta_1 = \gamma_2$, $\alpha_1\beta_2 = a_2\gamma_1$, etc.

6. Proofs of Lemmas 3.5–3.10.

The proofs proceed by direct application of Theorem 3.3 and the definition of U. It is useful to note the following consequence of Theorem 3.3:

If $i \ge 0$, $1 \le a \le i+1$, and $0 \le b \le n-1$, then

$$(**) \qquad T_i(c_i^a x^b) = \begin{cases} c_{i+1}^1 N_b & \text{if } i \text{ even}, a=1, \text{ and } 1 \le b \le n-1, \\ c_{i+1}^1 & \text{if } i \text{ odd}, a=1, \text{ and } b=n-1, \\ 0 & \text{otherwise}. \end{cases}$$

PROOF OF LEMMA 3.5. We have

$$U_q(A_1(q,k)yx) = \sum_{\substack{i\text{ even}\\ r\geq 0}} (-1)^{r(q+1)} T_i(c_i^{i+1-2r}yx) \otimes c_{q-i}^{q-i+2r-k}yx + c_0^1 \otimes T_q(c_q^{q-k}yx)$$

$$= \sum_{\substack{i\text{ even}\\ r\geq 0}} (-1)^{r(q+1)} c_{i+1}^{i+2-2r}x \otimes c_{q-i}^{q-i+2r-k}yx - \sum_{r\geq 0} (-1)^{r(q+1)} \varepsilon_{2r} c_{2r+1}^1 \otimes c_{q-2r}^{q-k}yx + c_0^1 \otimes c_{q+1}^{q+1-k}$$

(the second summation arising from the $i=2r$ terms)

$$= \sum_{\substack{i\text{ odd}\\ r\geq 0}} (-1)^{r(q+1)} c_i^{i+1-2r}x \otimes c_{q+1-i}^{q+1-i+2r-k}yx + \sum_{r\geq 0} (-1)^{rq} c_{2r+1}^1 \otimes c_{q-2r}^{q-k}yx + c_0^1 \otimes c_{q+1}^{q+1-k}$$

(since $\varepsilon_{2r} = (-1)^{r+1}$), which yields part (a).

For $t=2$, 3, and 4, $A_t(q,k)yx$ has the form

$$\sum_{\substack{i\text{ even}\\ r\geq 0}} a_{r,q} c_i^{i-2r}yx \otimes c_{q-i}^{q-i+2r-k+\delta}w$$

where $a_{r,q} \in \boldsymbol{Z}$, $w \in \boldsymbol{Z}D_{2n}$, and δ is either 0 or 1. We have

$$U_q\left(\sum_{\substack{i\text{ even}\\ r\geq 0}} a_{r,q} c_i^{i-2r}yx \otimes c_{q-i}^{q-i+2r-k+\delta}w\right) = \sum_{\substack{i\text{ even}\\ r\geq 0}} a_{r,q} T_i(c_i^{i-2r}yx) \otimes c_{q-i}^{q-i+2r-k+\delta}w$$

$$= \sum_{\substack{i\text{ even}\\ r\geq 0\\ i+1-2r\geq 3}} a_{r,q} c_{i+1}^{i+1-2r} \otimes c_{q-i}^{q-i+2r-k+\delta}w = \sum_{\substack{i\text{ odd}\\ r\geq 0\\ i-2r\geq 3}} a_{r,q} c_i^{i-2r} \otimes c_{q+1-i}^{q+1-i+2r-k+\delta}w$$

$$= \sum_{\substack{i\text{ odd}\\ r\geq 0}} a_{r,q} c_i^{i-2r} \otimes c_{q+1-i}^{q+1-i+2r-k+\delta}w - \sum_{r\geq 0} a_{r,q} c_{2r+1}^1 \otimes c_{q-2r}^{q-k+\delta}w.$$

Parts (b), (c), and (d) now follow.

For $t=5$, 6 and 8, $A_t(q,k)yx$ has the form

$$\sum_{\substack{i\text{ odd}\\ r\geq 0}} a_{r,q} c_i^{i-2r}yx \otimes c_{q-i}^{q-i+2r-k+\delta}w$$

where $a_{r,q} \in \boldsymbol{Z}$, $w \in \boldsymbol{Z}D_{2n}$, and δ is either 0 or 1. We have

$$U_q\left(\sum_{\substack{i\text{ odd}\\ r\geq 0}} a_{r,q} c_i^{i-2r}yx \otimes c_{q-i}^{q-i+2r-k+\delta}w\right) = \sum_{\substack{i\text{ odd}\\ r\geq 0}} a_{r,q} T_i(c_i^{i-2r}yx) \otimes c_{q-i}^{q-i+2r-k+\delta}w$$

$$= \sum_{\substack{i\text{ odd}\\ r\geq 0}} a_{r,q} c_{i+1}^{i+1-2r} \otimes c_{q-i}^{q-i+2r-k+\delta}w = \sum_{\substack{i\text{ even}\\ r\geq 0}} a_{r,q} c_i^{i-2r} \otimes c_{q+1-i}^{q+1-i+2r-k+\delta}w.$$

Parts (e), (f), and (h) now follow.

We have

$$U_q(A_7(q,k)yx)=\sum_{\substack{i\,\text{odd}\\ r\geq 0}}(-1)^{rq}T_i(c_i^{i+1-2r}y)\otimes c_{q-i}^{q-i+2r-k}$$

$$=\sum_{\substack{i\,\text{odd}\\ r\geq 0\\ i+2-2r\geq 3}}(-1)^{rq}c_{i+1}^{i+2-2r}\otimes c_{q-i}^{q-i+2r-k}=\sum_{\substack{i\,\text{even}\\ r\geq 0\\ i+1-2r\geq 3}}(-1)^{rq}c_i^{i+1-2r}\otimes c_{q+1-i}^{q+1-i+2r-k}$$

$$=\sum_{\substack{i\,\text{even}\\ r\geq 0}}(-1)^{rq}c_i^{i+1-2r}\otimes c_{q+1-i}^{q+1-i+2r-k}-\sum_{r\geq 0}(-1)^{rq}c_{2r}^{1}\otimes c_{q+1-2r}^{q+1-k}.$$

Part (g) now follows. □

PROOFS OF LEMMAS 3.6 AND 3.8. It follows by inspection, using (**), that the $A_t(q,k)$ for $1\leq t\leq 8$ and the $B_t(q,k)$ for $1\leq t\leq 6$ are linear combinations over $\boldsymbol{Z}$ of terms which are annihilated by U_q. □

PROOF OF LEMMA 3.7. From the definition of U and (**),

$$U_q(A_1(q,k)N)=\sum_{\substack{i\,\text{even}\\ r\geq 0}}\sum_{j=0}^{n-1}(-1)^{r(q+1)}U_q(c_i^{i+1-2r}x^j\otimes c_{q-i}^{q-i+2r-k}x^j)$$

$$=\sum_{\substack{i\,\text{even}\\ r\geq 0}}\sum_{j=0}^{n-1}(-1)^{r(q+1)}T_i(c_i^{i+1-2r}x^j)\otimes c_{q-i}^{q-i+2r-k}x^j+\sum_{j=0}^{n-1}c_0^1\otimes T_q(c_q^{q-k}x^j)$$

$$=\sum_{r\geq 0}\sum_{j=0}^{n-1}(-1)^{r(q+1)}T_{2r}(c_{2r}^1x^j)\otimes c_{q-2r}^{q-k}x^j=\sum_{r\geq 0}\sum_{j=0}^{n-1}(-1)^{r(q+1)}c_{2r+1}^1N_j\otimes c_{q-2r}^{q-k}x^j.$$

Since $N_0=0$, part (a) follows.

For $t=5$, 6, and 8, $A_t(q,k)N$ has the form

$$\sum_{\substack{i\,\text{odd}\\ r\geq 0}}\sum_{j=0}^{n-1}a_{r,q}c_i^{i-2r}x^j\otimes c_{q-i}^{q-i+2r-k+\delta}wx^j$$

where $a_{r,q}\in\boldsymbol{Z}$, $w\in\boldsymbol{Z}D_{2n}$, and δ is either 0 or 1. We have

$$U_q\left(\sum_{\substack{i\,\text{odd}\\ r\geq 0}}\sum_{j=0}^{n-1}a_{r,q}c_i^{i-2r}x^j\otimes c_{q-i}^{q-i+2r-k+\delta}wx^j\right)=\sum_{\substack{i\,\text{odd}\\ r\geq 0}}\sum_{j=0}^{n-1}a_{r,q}T_i(c_i^{i-2r}x^j)\otimes c_{q-i}^{q-i+2r-k+\delta}wx^j$$

$$=\sum_{r\geq 0}\sum_{j=0}^{n-1}a_{r,q}T_{2r+1}(c_{2r+1}^1x^j)\otimes c_{q-1-2r}^{q-1-k+\delta}wx^j=\sum_{r\geq 0}a_{r,q}c_{2r+2}^1\otimes c_{q-1-2r}^{q-1-k+\delta}wx^{n-1}$$

$$=\sum_{r>0}a_{r-1,q}c_{2r}^1\otimes c_{q+1-2r}^{q-1-k+\delta}wx^{n-1}.$$

The only non-zero contributions to these last summations come from the $j=n-1$ terms.

Parts (b), (c), and (d) now follow.

It follows by inspection, using (**), that the $A_t(q, k)N$ for $t=2$, 3, 4, and 7 are linear combinations over $\boldsymbol{Z}$ of terms which are annihilated by U_q, yielding part (e).

□

PROOF OF LEMMA 3.9. For $t=1$ and 2, $B_t(q, k)y$ has the form

$$\sum_{\substack{i \text{ even} \\ r\geq 0}} a_{r,q} c_i^{i+1-2r} y \otimes c_{q-i}^{q-i+2r-k-\delta} w$$

where $a_{r,q} \in \boldsymbol{Z}$, $w \in \boldsymbol{Z}D_{2n}$, and $\delta=0$ or 1. We have

$$\begin{aligned} U_q(B_t(q, k)y) &= \sum_{\substack{i \text{ even} \\ r\geq 0}} a_{r,q} T_i(c_i^{i+1-2r} y) \otimes c_{q-i}^{q-i+2r-k-\delta} w + a_{0,q} c_0^1 \otimes T_q(c_q^{q-k-\delta} w) \\ &= \sum_{\substack{i \text{ even} \\ r\geq 0}} a_{r,q} c_{i+1}^{i+2-2r} \otimes c_{q-i}^{q-i+2r-k-\delta} w + a_{0,q} c_0^1 \otimes T_q(c_q^{q-k-\delta} w) \\ &= \sum_{\substack{i \text{ odd} \\ r\geq 0}} a_{r,q} c_i^{i+1-2r} \otimes c_{q+1-i}^{q+1-i+2r-k-\delta} w + a_{0,q} c_0^1 \otimes T_q(c_q^{q-k-\delta} w). \end{aligned}$$

For $t=1$ we have $a_{0,q}=1$, $w=y$, $\delta=0$ and $T_q(c_q^{q-k} y)=c_{q+1}^{q+1-k}$. For $t=2$ we have $a_{0,q}=0$. Parts (a) and (b) now follow.

Noting that $N_j x^{-j} y = yxN_j$, we have

$$\begin{aligned} U_q(B_3(q, k)y) &= \sum_{\substack{i \text{ even} \\ r\geq 0}} \sum_{j=1}^{n-1} (-1)^{(r+1)(q+1)} T_i(c_i^{i-2r} yxN_j) \otimes c_{q-i}^{q-i+2r-k+1} x^j \\ &= \sum_{\substack{i \text{ even} \\ r\geq 0 \\ i+1-2r\geq 3}} \sum_{j=1}^{n-1} (-1)^{(r+1)(q+1)} c_{i+1}^{i+1-2r} N_j \otimes c_{q-i}^{q-i+2r-k+1} x^j \\ &= \sum_{\substack{i \text{ odd} \\ r\geq 0 \\ i-2r\geq 3}} \sum_{j=1}^{n-1} (-1)^{(r+1)(q+1)} c_i^{i-2r} N_j \otimes c_{q+1-i}^{q+1-i+2r-k+1} x^j \\ &= \sum_{\substack{i \text{ odd} \\ r\geq 0}} \sum_{j=1}^{n-1} (-1)^{(r+1)(q+1)} c_i^{i-2r} N_j \otimes c_{q+1-i}^{q+1-i+2r-k+1} x^j \\ &\quad - \sum_{r\geq 0} \sum_{j=1}^{n-1} (-1)^{(r+1)(q+1)} c_{2r+1}^1 N_j \otimes c_{q-2r}^{q+1-k} x^j. \end{aligned}$$

Part (c) now follows.

For $t=4$ and 5, $B_t(q, k)y$ has the form

$$\sum_{\substack{i\,\mathrm{odd}\\ r\geq 0}} a_{r,q}c_i^{i+1-2r}y\otimes c_{q-i}^{q-i+2r-k-\delta}w$$

where $a_{r,q}\in \mathbf{Z}$, $w\in \mathbf{Z}D_{2n}$, and $\delta=0$ or 1. We have

$$U_q\left(\sum_{\substack{i\,\mathrm{odd}\\ r\geq 0}} a_{r,q}c_i^{i+1-2r}y\otimes c_{q-i}^{q-i+2r-k-\delta}w\right)=\sum_{\substack{i\,\mathrm{odd}\\ r\geq 0}} a_{r,q}T_i(c_i^{i+1-2r}y)\otimes c_{q-i}^{q-i+2r-k-\delta}w$$

$$=\sum_{\substack{i\,\mathrm{odd}\\ r\geq 0\\ i+2-2r\geq 3}} a_{r,q}c_{i+1}^{i+2-2r}\otimes c_{q-i}^{q-i+2r-k-\delta}w=\sum_{\substack{i\,\mathrm{even}\\ r\geq 0\\ i+1-2r\geq 3}} a_{r,q}c_i^{i+1-2r}\otimes c_{q+1-i}^{q+1-i+2r-k-\delta}w$$

$$=\sum_{\substack{i\,\mathrm{even}\\ r\geq 0}} a_{r,q}c_i^{i+1-2r}\otimes c_{q+1-i}^{q+1-i+2r-k-\delta}w-\sum_{r\geq 0} a_{r,q}c_{2r}^{1}\otimes c_{q+1-2r}^{q+1-k-\delta}w.$$

Parts (d) and (e) now follow.

We have

$$U_q(B_6(q,k)y)=\sum_{\substack{i\,\mathrm{odd}\\ r\geq 0}}\sum_{j=1}^{n-1}(-1)^{(r+1)q}T_i(c_i^{i-2r}N_jy)\otimes c_{q-i}^{q-i+2r-k+1}x^jy$$

$$=\sum_{\substack{i\,\mathrm{odd}\\ r\geq 0\\ i-2r\geq 3}}\sum_{j=1}^{n-1}(-1)^{(r+1)q}T_i(c_i^{i-2r}yN_jx^{-j+1})\otimes c_{q-i}^{q-i+2r-k+1}yx^{-j}$$

$$+\sum_{r\geq 0}\sum_{j=1}^{n-1}(-1)^{(r+1)q}T_{2r+1}(c_{2r+1}^{1}yN_jx^{-j+1})\otimes c_{q-1-2r}^{q-k}yx^{-j}$$

$$=\sum_{\substack{i\,\mathrm{odd}\\ r\geq 0\\ i+1-2r\geq 4}}\sum_{j=1}^{n-1}(-1)^{(r+1)q}c_{i+1}^{i+1-2r}N_jx^{-j}\otimes c_{q-i}^{q-i+2r-k+1}yx^{-j}$$

$$+\sum_{r\geq 0}\sum_{j=1}^{n-1}(-1)^{(r+1)q}(c_{2r+2}^{2}N_jx^{-j}-\varepsilon_{2r+1}c_{2r+2}^{1})\otimes c_{q-1-2r}^{q-k}yx^{-j}$$

$$=\sum_{\substack{i\,\mathrm{even}\\ r\geq 0}}\sum_{j=1}^{n-1}(-1)^{(r+1)q}c_i^{i-2r}N_jx^{-j}\otimes c_{q+1-i}^{q+1-i+2r-k+1}yx^{-j}$$

$$-\sum_{r\geq 0}(-1)^{(r+1)q}(-1)^r c_{2r+2}^{1}\otimes\left(\sum_{j=1}^{n-1}c_{q-1-2r}^{q-k}yx^{-j}\right)$$

$$=B_3(q+1,k)+\sum_{r>0}(-1)^{r(q+1)}c_{2r}^{1}\otimes c_{q+1-2r}^{q-k}y(N-1).$$

Part (f) now follows. □

PROOF OF LEMMA 3.10. For $t=1$ and 2, $B_t(q,k)x$ has the form

$$\sum_{\substack{i\text{ even}\\ r\geq 0}} a_{r,q}c_i^{i+1-2r}x\otimes c_{q-i}^{q-i+2r-k-\delta}w$$

where $a_{r,q}\in \mathbf{Z}$, $w\in \mathbf{Z}D_{2n}$, and $\delta=0$ or 1. We have

$$\begin{aligned} U_q\Bigg(\sum_{\substack{i\text{ even}\\ r\geq 0}} a_{r,q}c_i^{i+1-2r}x\otimes c_{q-i}^{q-i+2r-k-\delta}w\Bigg) &= \sum_{\substack{i\text{ even}\\ r\geq 0}} a_{r,q}T_i(c_i^{i+1-2r}x)\otimes c_{q-i}^{q-i+2r-k-\delta}w+a_{0,q}c_0^1\otimes T_q(c_q^{q-k-\delta}w)\\ &= \sum_{r\geq 0} a_{r,q}T_{2r}(c_{2r}^1x)\otimes c_{q-2r}^{q-k-\delta}w+a_{0,q}c_0^1\otimes T_q(c_q^{q-k-\delta}w)\\ &= \sum_{r\geq 0} a_{r,q}c_{2r+1}^1\otimes c_{q-2r}^{q-k-\delta}w+a_{0,q}c_0^1\otimes T_q(c_q^{q-k-\delta}w). \end{aligned}$$

If $t=1$, then $w=x$, $q-k-\delta=q-k>1$ and thus $T_q(c_q^{q-k-\delta}w)=0$. If $t=2$, then $a_{0,q}=0$. Parts (a) and (b) now follow.

We have

$$\begin{aligned} U_q(B_6(q,k)x) &= \sum_{\substack{i\text{ odd}\\ r\geq 0}}\sum_{j=1}^{n-1}(-1)^{q(r+1)}T_i(c_i^{i-2r}N_jx)\otimes c_{q-i}^{q-i+2r-k+1}x^{j+1}\\ &= \sum_{r\geq 0}\sum_{j=1}^{n-1}(-1)^{q(r+1)}T_{2r+1}(c_{2r+1}^1N_jx)\otimes c_{q-1-2r}^{q-k}x^{j+1}\\ &= \sum_{r\geq 0}(-1)^{q(r+1)}c_{2r+2}^1\otimes c_{q-1-2r}^{q-k}=\sum_{r>0}(-1)^{rq}c_{2r}^1\otimes c_{q+1-2r}^{q-k} \end{aligned}$$

(only the $j=n-1$ terms contribute to these last summations), proving part (c).

It follows by inspection, using (**), that the $B_t(q,k)x$ for $t=3$, 4, and 5 are linear combinations over $\mathbf{Z}$ of terms which are annihilated by U_q, yielding part (d). □

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