Tôhoku Math. J.
45 (1993), 1–12

HARMONIC MAPS OF NONORIENTABLE SURFACES TO FOUR-DIMENSIONAL MANIFOLDS

TORU ISHIHARA*

(Received June 17, 1991, revised June 22, 1992)

Abstract. We construct explicit harmonic maps of the projective plane or a quotient space of a hyperelliptic Riemann surface into the unit 4-sphere.

1. Introduction. Harmonic maps of nonorientable surfaces are not studied so much (see, for example, [EeL1], [EeL3]). The existence problem of harmonic representatives in homotopy classes of maps of nonorientable surfaces was studied in [Ee12]. Equivariant minimal immersions of the projective plane into S^n or P^n are determined by Ejiri [Eg]. In the present paper, we will try to construct harmonic maps from nonorientable surfaces into 4-dimensional Riemannian manifolds. We deal with a nonorientable surface $\mathscr{M}$ which is a quotient of a Riemann surface M by the equivalent relation $z \sim w$ if and only if $w = I(z)$, where I is an anti-holomorphic involution of M without fixed points. Especially, we will be concerned with the following nonorientable surfaces. We first identify the unit 2-sphere S^2 with $\boldsymbol{C} \cup \{\infty\}$ and put $M = \boldsymbol{C} \cup \{\infty\}$. The map corresponding to the antipodal map is an involution of M given by $I(z) = -1/\bar{z}$. The quotient space is the projective plane. Next, let T_{l-1} be a hyperelliptic Riemann surface given by

$$T_{l-1} = \{(z, w) \in (\boldsymbol{C} \cup \{\infty\})^2;\ w^2 = \prod_{j=1}^{l} (d_j - z)(\bar{d}_j + z)\}, \tag{1.1}$$

where $d_i \neq d_j$ for any $i \neq j$ and $d_i \neq -\bar{d}_j$ for any $i \neq j$. Let $I(z,w) := (-\bar{z}, -\bar{w})$ for $(z, w) \in T_{l-1}$. Then it is an antiholomorphic involution without fixed points (see [11]). Let $P_l := T_{l-1}/\{I\}$ be the quotient space of T_{l-1} by the equivalence relation given by I. Then P_l is a nonorientable surface of genus l. We may regard P_1 as the projective plane and P_2 as the Klein bottle. Now we return to the general setting. Let M be a Riemann surface with involution I and $\pi: M \to \mathscr{M}$ the natural projection of M to the quotient space. A map h of M into a Riemannian manifold N is factored as $h = \mathscr{h} \cdot \pi$, where $\mathscr{h}$ is a map of $\mathscr{M}$ into N, if and only if $h(I(p)) = h(p)$ for each $p \in M$. Let g be a Riemannian metric compatible with the conformal structure of M. We give a natural Riemannian structure $\mathscr{g}$ on $\mathscr{M}$ such that π is locally isometric. Evidently the assign-

* Partly supported by the Grants-in-Aid for Scientific Research, the Ministry of Education, Science and Culture, Japan.

1991 *Mathematics Subject Classification*. Primary 58E20.

ment $h \mapsto \hat{h}$ is a bijective correspondence between the set of conformal harmonic maps $h: M \to N$ with $h \cdot I = h$ and the set of harmonic maps $\hat{h}: \mathcal{M} \to N$. Hence instead of studying harmonic maps $\hat{h}: \mathcal{M} \to N$, we investigate harmonic maps $h: M \to N$ with $h \cdot I = h$. This method was introduced by Meeks in [M] to study minimal immersions of non-orientable surfaces and developped in [Eg], [O], [I1], [I2].

Let N be a 4-dimensional oriented Riemannian manifold and S its twistor space with almost complex structures J_1 and J_2. In Section 2, we introduce a natural involution I_S of S which is anti-holomorphic with respect to J_1 and J_2. For harmonic maps $h: M \to N$, Eells and Salamon defined the twistor lifts $\tilde{h}: M \to S$ and gave the fundamental correspondence between them. In Section 2, using their results, we will show the following:

THEOREM I. *The assignment $h \mapsto \tilde{h}$ is a bijective correspondence between the set of nonconstant conformal harmonic maps $h: M \to N$ with $h \cdot I = h$ and the set of nonvertical J_2-holomorphic curves $\tilde{h}: M \to S$ with $\tilde{h} \cdot I = I_S \cdot \tilde{h}$.*

Now, let N be the unit 4-sphere S^4. Then its twistor space is the complex projective 3-space $CP^3 = \{{}^t[a_1, a_2, a_3, a_4]\}$ (for details, see Section 2 and [AHS], [B], [EeS], [S]). Bryant [B] proved that a conformal map $h: M \to S^4$ is isotropic and harmonic if and only if the twistor lift $\tilde{h}: M \to CP^3$ is holomorphic and horizontal. Moreover, he showed that for given meromorphic functions f and g on M with g nonconstant,

$$\tilde{h}(f, g) = {}^t\left[1, 2f - g\frac{df}{dg}, g, \frac{df}{dg}\right] \tag{1.2}$$

is horizontal and holomorphic, and that any nonconstant horizontal holomorphic map $M \to CP^3$ arises in this manner for unique meromorphic functions f and g on M or else is contained in a line in CP^3. If we replace f in (1.2) by $f/2$, we get the original formula of Bryant. In the sequel, we will call f and g the *Bryant meromorphic functions for h.* In Section 3, we will show:

THEOREM II. *A conformal isotropic harmonic map $h: M \to S^4$ has the property $h \cdot I = h$ if and only if Bryant meromorphic functions f and g for h satisfy*

$$2fg^* - (gg^* + 1)\frac{df}{dg} = 0\,, \tag{1.3}$$

$$4f \cdot f^* + (1 + g \cdot g^*)^2 = 0\,, \tag{1.4}$$

where we put $f^* = \overline{f \cdot I}$ and $g^* = \overline{g \cdot I}$.

In Section 4, we will construct harmonic maps h of S^2 into S^4 with $h \cdot I = h$. In fact we obtain:

THEOREM III. *Suppose f and g are the Bryant meromorphic functions corresponding to a harmonic map $h: S^2 \to S^4$ with $h \cdot I = h$ and with $f \cdot f^*$ or $g \cdot g^*$ constant. Then* *h gives a harmonic map h of the projective plane into S^4, if and only if f and g are of the form*

$$f(z)=Ak(z)^m, \quad g(z)=Bk(z)^n, \quad k(z)=z^\lambda \frac{\prod_{i=1}^{\rho}(z-a_i)}{\prod_{i=1}^{\rho}(\bar{a}_i z+1)}, \tag{1.5}$$

where both $\lambda+\rho$ and m are odd, $m\neq n$, $m\neq 2n$, $(-1)^n m(2n-m)>0$ and

$$|A|=\left|\frac{n}{2n-m}\right|, \qquad |B|^2=(-1)^n\frac{m}{2n-m}. \tag{1.6}$$

When $\rho=0$, $\lambda=1$, $m=3$, $n=1$, $A=-1$ and $B=\sqrt{3}$, the formula (1.5) yields $f=-z^3$ and $g=\sqrt{3}$, which are the Bryant meromorphic functions corresponding to the Veronese surface in S^4 (see [EeS, §9]). We cannot interpret the condition that $f\cdot f^*$ or $g\cdot g^*$ is constant. Neither can we determine the general Bryant meromorphic functions which satisfy the relations (1.3) and (1.4). We are concerned with harmonic maps of a non-orientable surface P_l into S^4 in Section 5.

THEOREM IV. *Suppose f and g are the Bryant meromorpic functions corresponding to a harmonic map h: $T_{l-1}\to S^4$ with $h\cdot I=h$ and with $f\cdot f^*$ or $g\cdot g^*$ constant. Then h gives a harmonic map $\mathscr{h}$ of a nonorientable surface P_l into S^4 if and only if there exists a meromorphic function k on T_{l-1} such that*

$$f=Ak^m, \quad g=Bk^n, \tag{1.7}$$

where m and n are integers, m is odd, $(-1)^n m(2n-m)>0$ and either (1) *k is given by*

$$k=\frac{\prod_{i=1}^{\mu}(z-a_i)w}{\prod_{i=1}^{\mu}(z+\bar{a}_i)\prod_{j=1}^{l}(z-e_j)}, \qquad e_j=d_j\ (1\leq j\leq l) \quad \text{or} \quad e_j=-\bar{d}_j\ (1\leq j\leq l)$$

and $|A|=|m/(2n-m)|$, $|B|^2=(-1)^n m/(2n-m)$ or (2) *k is given by*

$$k=\frac{\prod_{i=1}^{\mu}(z-a_i)\left(D\prod_{j=1}^{\delta}(z-c_j)+\prod_{i=1}^{\nu}(z-b_i)w\right)}{\prod_{i=1}^{\mu}(z+\bar{a}_i)\prod_{j=1}^{\lambda}(z-e_j)},$$

with

$$(-1)^\delta|D|^2\prod_{j=1}^{\delta}(z-c_j)^2-(-1)^\nu\prod_{j=1}^{\nu}(z-b_j)^2=(-1)^\lambda c\prod_{j=1}^{\lambda}(z-e_j)(z+\bar{e}_j),$$

$$\prod_{i=1}^{\delta}(z-c_i)=\prod_{i=1}^{\delta}(z+\bar{c}_i)\,, \quad \prod_{i=1}^{\nu}(z-b_i)=\prod_{i=1}^{\nu}(z+\bar{b}_i)\,, \quad \bar{D}=(-1)^{\delta+\nu}D\,,$$

and $|A|^2=c^{-m}(n/(2n-m))^2$, $|B|^2=c^{-n}m/(2n-m)$, *c is a negative real number.*

Here $a_1, \dots, a_\mu, b_1, \dots, b_\delta, c_1, \dots, c_\lambda, e_1, \dots, e_l$ *are complex numbers and* $d_1, \dots, d_l$ *are as in the definition of* P_l *as the quotient of* T_{l-1}.

ACKNOWLEDGMENT. The author would like to thank the referee for correcting many mistakes in the previous version of this paper and providing valuable comments.

2. An involution of a twistor space. Let M be a Riemann surface with an anti-holomorphic involution I without fixed points. Let N be a 4-dimensional oriented Riemannian manifold with a Riemannian metric g. Let $\pi: SO(N)\to N$ be the $SO(4)$-principal bundle of oriented orthonormal frames over N, that is,

$$SO(N)=\{(x, e=(e_1, e_2, e_3, e_4)), x\in N\}\,.$$

Let $\pi_2: S\to N$ be the orthogonal twistor bundle over N, where

$$S=\{(x, J), x\in N, J \text{ is an orientation compatible almost complex structure of } T_xN \text{ with } g(JX, JY)=g(X, Y), X, Y\in T_xN\}\,.$$

We also consider the projection

$$\pi_1: SO(N)\to S\,, \quad (x, e=(e_1, e_2, e_3, e_4))\mapsto(x, J_e)\,,$$

where $J_e(e_1)=e_2$ and $J_e(e_3)=e_4$. *Let* $\Theta=(\Theta^\alpha)$ be the $\boldsymbol{R}^4$-valued canonical form on $SO(N)$. We have the structure equation,

$$d\Theta^\alpha=-\sum\Omega^\alpha_\beta\wedge\Theta^\beta\,, \tag{2.1}$$

where $\Omega=(\Omega^\alpha_\beta)$ is the Levi-Civita connection form on $SO(N)$.

Now we define an involution of S by

$$I_S((x, J)):=(x, \bar{J})\,,$$

where for $J(e_1)=e_2$, $J(e_3)=e_4$, $\bar{J}$ is defined by $\bar{J}(e_1)=-e_2$, $\bar{J}(e_3)=-e_4$. The map $\tilde{I}_S$ of $SO(N)$ into itself given by

$$\tilde{I}_S((x, e=(e_1, e_2, e_3, e_4)))=(x, e=(e_1, -e_2, e_3, -e_4))$$

is also an involution satisfying $\pi_1\cdot\tilde{I}_S=I_S\cdot\pi_1$. By definition, we get

$$\tilde{I}^*_S(\Theta^1)=\Theta^1\,, \quad \tilde{I}^*_S(\Theta^2)=-\Theta^2\,, \quad \tilde{I}^*_S(\Theta^3)=\Theta^3\,, \quad \tilde{I}^*_S(\Theta^4)=-\Theta^4\,. \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$\tilde{I}_s^*(\Omega_2^1) = -\Omega_2^1, \quad \tilde{I}_s^*(\Omega_3^1) = \Omega_3^1, \quad \tilde{I}_s^*(\Omega_4^1) = -\Omega_4^1,$$
(2.3)
$$\tilde{I}_s^*(\Omega_3^2) = -\Omega_3^2, \quad \tilde{I}_s^*(\Omega_4^2) = \Omega_4^2, \quad \tilde{I}_s^*(\Omega_4^3) = -\Omega_4^3.$$

Put $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. Let $e = (e_1, e_2, e_3, e_4)$ be a local oriented orthonormal frame. For each j ($j = 1$ or 2), we obtain the almost complex structure J_j on S by assuming that the 1-forms on S which are pulled backs of

$$\Theta^1 + i\Theta^2, \quad \Theta^3 + i\Theta^4, \quad \frac{1}{2}(\Omega_3^1 - \Omega_4^2 + \varepsilon_j i(\Omega_3^2 + \Omega_4^1))$$

following by a local section of $\pi_1: SO(N) \to S$ give a local coframe of (1,0)-forms on S (see, [Y]). Using $\pi_1 \cdot \tilde{I}_S = I_S \cdot \pi_1$, we obtain the following by (2.2) and (2.3):

PROPOSITION 2.1. *The involution I_S of S is anti-holomorphic with respect to J_1 and to J_2.*

Let $h: M \to N$ be a conformal harmonic map. Then h has at most isolated singular points. Hence we can find a Riemannian metric ds_M^2 on M such that $h^*(ds_N^2) = ds_M^2$ except at the singular points. Let $e = (e_1, e_2, e_3, e_4)$ be a Darboux frame along h, that is, a local oriented orthonormal frame in N such that $(e_1 \cdot h, e_2 \cdot h)$ is a local oriented orthonormal frame in (M, ds_M^2) and $e_3 \cdot h$, $e_4 \cdot h$ are normal to M. Hence we have

$$h^*e^*\Theta^3 = 0, \quad h^*e^*\Theta^4 = 0. \tag{2.4}$$

We assume that the Darboux frame e is compatible with the almost complex structure. There is a local 1-form ϕ such that

$$ds_M^2 = \phi\bar{\phi} \quad \text{and} \quad \phi = h^*e^*\theta^1 + ih^*e^*\Theta^2$$

except at the singular points. The conformality of h implies that ϕ is a local (1,0)-form on M.

The twistor lift of h is a map $\tilde{h}: M \to S$ given by

$$\tilde{h}((x, e)) = \pi_1((x, e)),$$

where $e = (e_1, e_2, e_3, e_4)$ is a Darboux frame along h.

Now we assume that the map f satisfies $h \cdot I = h$. Since we have $h*(ds_N^2) = ds_M^2$ at nonsingular points, the involution I is an isometry of (M, ds_M^2) into itself and $I^*(\theta^1 + i\theta^2) = \theta^1 - i\theta^2$ holds. Hence we have

$$\tilde{h}(I(x)) = \pi_1(x, (e_1, -e_2, e_3, -e_4)) = I_S\tilde{h}(x).$$

Conversely, the relation $\tilde{h} \cdot I = I_S \cdot \tilde{h}$ evidently implies $h \cdot I = f$. By the fundamental theorem of Eells and Salamon [EeS], we obtain Theorem I.

It is also shown in [EeS] that a conformal map $h: M \to N$ is isotropic if and only if $\tilde{h}$ is J_1 holomorphic.

3. Harmonic maps into S^4. In the sequel, we assume that N is the unit 4-sphere S^4. The correspondence $(x, (e_1, e_2, e_3, e_4)) \mapsto (x, e_1, e_2, e_3, e_4)$ determines an isomorphism $SO(N) \to SO(5)$. The unit sphere S^4 is isomorphic to the quaternionic projective space $\boldsymbol{HP}^1$. We have the following commutative diagram

$$\begin{array}{ccccc} SP(2) & \xrightarrow[\pi'_1]{} & \boldsymbol{CP}^3 & \xrightarrow[\pi'_2]{} & \boldsymbol{HP}^1 \\ \downarrow{\scriptstyle \Phi^*} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi_*} \\ SO(5) & \xrightarrow[\pi_1]{} & S & \xrightarrow[\pi_2]{} & S^4 , \end{array}$$

where Φ^* is a double covering of $Sp(2)$ to $SO(5)$, Φ, Φ_* are diffeomorphisms and the natural complex structure of $\boldsymbol{CP}^3$ corresponds to the almost complex structure J_1 of the twistor space S. For more details see, for example, [AHS], [EeS], [S].

We identify $\boldsymbol{H}^2$ with $\boldsymbol{C}^4$ by the correspondence $(z_1 + jz_2, z_3 + jz_4) \mapsto (z_1, z_2, z_3, z_4)$. For a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2)$$

with $a = a_1 + ja_2$ and $c = c_1 + jc_2$, the projection π'_1 in the above diagram maps A to ${}^t[a_1, a_2, c_1, c_2] \in \boldsymbol{CP}^3$, where ${}^t[a_1, a_2, c_1, c_2]$ is the complex line containing ${}^t(a_1, a_2, c_1, c_2)$. We also have $\pi'_2 \pi'_1(A) = {}^t[a, c] \in \boldsymbol{HP}^1$.

Put $U := \boldsymbol{C}^4 \cong \boldsymbol{H}^2$. Then U has a unitary base of the form $\{u^1, u^2 = u^1 j, u^3, u^4 = u^3 j\}$. Set

$$\begin{aligned} &v^0 = u^1 \wedge u^2 + u^3 \wedge u^4 , \quad v^1 = u^1 \wedge u^2 - u^3 \wedge u^4 , \quad v^2 = u^1 \wedge u^3 + u^2 \wedge u^4 , \\ &v^3 = i(u^1 \wedge u^3 - u^2 \wedge u^4) , \quad v^4 = u^1 \wedge u^4 - u^2 \wedge u^3 , \quad v^5 = i(u^1 \wedge u^4 + u^2 \wedge u^3) . \end{aligned}$$

Then one checks directly that $\{v^0, v^1, v^2, v^3, v^4, v^5\}$ is a unitary base of $\bigwedge^2 U$ and v^0 is invariant under $Sp(2)$. Let $\bigwedge_0^2 U$ be the subspace spanned by $\{v^1, v^2, v^3, v^4, v^5\}$. For $A \in Sp(2)$, put

$$Av^i = \sum_{j=1}^{5} A_{ij} v_j , \qquad j = 1, \dots, 5 . \tag{3.1}$$

Then $(A_{ij}) \in SO(5)$, and the homomorphism $\Phi^* : Sp(2) \to SO(5)$ is given by $\Phi^*(A) = (A_{ij})$. Since $\Phi_*(\pi'_2 \pi'_1(A)) = (A_{1j}) \in S^4$, we have

$$\Phi_*({}^t[a, c]) = {}^t(x_1, x_2, x_3, x_4, x_5) , \tag{3.2}$$

$$x_1 = |a|^2 - |c|^2 , \quad x_2 = 2(a\bar{c})_3 , \quad x_3 = 2(a\bar{c})_4 , \quad x_4 = 2(a\bar{c})_1 , \quad x_5 = 2(a\bar{c})_2 ,$$

where $|a|^2 + |c|^2 = 1$, $\bar{c} = \bar{c}_1 - jc_2$ and $a\bar{c} = (a\bar{c})_1 + (a\bar{c})_2 i + (a\bar{c})_3 j + (a\bar{c})_4 k$.

Since we have $(a_1 + ja_2, c_1 + jc_2) j = (-\bar{a}_2 + j\bar{a}_1, -\bar{c}_2 + j\bar{c}_1)$, we define an involution I' of $\boldsymbol{CP}^3$ by

$$I'({}^t[a_1, a_2, c_1, c_2]) := [-\bar{a}_2, \bar{a}_1, -\bar{c}_2, \bar{c}_1]\,. \tag{3.3}$$

Then it corresponds to the involution I_S of S. In fact we can show:

LEMMA 3.1. *The involution I' is anti-holomorphic with respect to the natural complex structure and satisfies $I_S\Phi = \Phi I'$.*

PROOF. The anti-holomorphy of I' is evident by the definition of I'. Since $I'(u^1) = u^2$, $I'(u^2) = -u^1$, $I'(u^3) = u^4$, $I'(u^3) = -u^4$, we get $I'(v^1) = v^1$, $I'(v^2) = v^2$, $I'(v^3) = -v^3$, $I'(v^4) = v^4$, $I'(v^5) = -v^5$. This implies the equality $I_S\Phi = \Phi I'$. q.e.d.

The horizontal distribution H on CP^3 is defined to be the othogonal complement to the fiber of $\pi'_2 : CP^3 \to HP^1$ with respect to the Fubini-Study metric. A map $\tilde{h}\colon M \to CP^3$ is said to be horizontal if it is tangent to H. A horizontal map $\tilde{h}$ is J_1-holomorphic if and only if J_2-holomorphic. Hence a conformal map $h\colon M \to S^4$ is isotropic and harmonic if and only if the twistor lift $\tilde{h}\colon M \to CP^3$ is holomorphic and horizontal (cf. [B]). Using Bryant's formula (1.2) and Lemma 3.1 we see that f and g are the Bryant meromorphic functions corresponding to a harmonic map h with $h \cdot I = h$ if and only if f and g satisfy

$$gg^* + \frac{df}{dg}\frac{df^*}{dg^*} = 0\,, \quad 2fg^* - (gg^* + 1)\frac{df}{dg} = 0\,. \tag{3.4}$$

From the second equation, we get

$$2f^*g - (gg^* + 1)\frac{df^*}{dg^*} = 0\,.$$

Thus the conditions (3.4) are equivalent to (1.3) and (1.4), and we get Theorem II.

4. Harmonic maps of S^2 into S^4. In this section, we will consider maps of S^2 to S^4. We identify S^2 with $C \cup \{\infty\}$ and consider its involution I as given in Section 1. Let $h\colon S^2 \to S^4$ be a full conformal isotropic harmonic map with $h \cdot I = h$. We look for the Bryant meromrphic functions f, g under the condition gg^* constant. From (1.4), it follows that this conditon holds if and only if ff^* is also constant, and in this case, we can put

$$f(z) = Az^{\alpha}\frac{\prod_{i=1}^{\mu}(z - a_i)}{\prod_{j=1}^{\mu}(\bar{a}_j z + 1)}\,, \quad g(z) = Bz^{\beta}\frac{\prod_{i=1}^{\nu}(z - b_i)}{\prod_{j=1}^{\nu}(\bar{b}_j z + 1)}\,, \tag{4.1}$$

where $a_i \neq 0$, $b_j \neq 0$. It may happen that $a_i = a_j$ for $i \neq j$. Then $ff^* = (-1)^{\alpha+\mu}|A|^2$ and $gg^* = (-1)^{\beta+\nu}|B|^2$. Since $4ff^* + (1 + gg^*)^2 = 0$, we see that $\alpha + \mu$ is odd and

$$(4.2)\qquad 4|A|^2=(1+(-1)^{\beta+\nu}|B|^2)^2 .$$

Since we have $g^*=(-1)^{\beta+\nu}|B|^2/g$, by (1.3) we get

$$2(-1)^{\beta+\nu}|B|^2\frac{g'}{g}=(1+(-1)^{\beta+\nu}|B|^2)\frac{f'}{f},$$

where $g'=dg/dz$, $f'=df/dz$. Hence we obtain, for some constant C

$$2(-1)^{\beta+\nu}|B|^2\log g=(1+(-1)^{\beta+\nu}|B|^2)\log f+C .$$

Substituting (4.1) into the above equation and comparing functions $\log z$, $\log(z-a_i)$, $\log(\bar{a}_jz+1)$, $\log(z-b_k)$ and $\log(\bar{b}_lz+1)$ of both sides of the equation, we find that there exists a meromorphic function

$$k(z)=z^\lambda\frac{\prod_{i=1}^{\rho}(z-c_i)}{\prod_{j=1}^{\rho}(e_jz+1)}$$

on $\boldsymbol{C}$ such that $f=Ak(z)^m$, $g=Bk(z)^n$, where $(2n-m)(-1)^{n(\lambda+\rho)}|B|^2=m$. Since $\alpha+\mu=m(\lambda+\rho)$ is odd, both m and $\lambda+\rho$ are odd. Thus, $2n\neq m$ and $|B|^2=(-1)^n m/(2n-m)$. From (4.2), it follows $|A|^2=(n/(2n-m))^2$. Since ff^* is constant, kk^* is constant. Hence we may assume $e_j=\bar{c}_j$.

Now, we get the corresponding holomorphic map $\tilde{h}(f,g)$ of $\boldsymbol{C}\cup\{\infty\}$ to S^4 for the Bryant meromorphic functions given by (1.5) as follows:

$$(4.3)\qquad \tilde{h}(f,g)={}^t[nB,(2n-m)ABk^m,nB^2k^n,nAk^{m-n}],\qquad k=z^\lambda\frac{\prod_{i=1}^{\rho}(z-a_i)}{\prod_{j=1}^{\rho}(\bar{a}_jz+1)}.$$

If $m=n$, this is not full. Thus we obtain Theorem III.

Notice that the holomorphic curves given by (4.3) is contained in the quadric $mX_1X_2-(2n-m)X_3X_4=0$ in $\boldsymbol{CP}^3=\{{}^t[X_1,X_2,X_3,X_4]\}$. Using (3.2), we obtain the corresponding conformal isotropic harmonic maps.

THEOREM 4.1. *Let $h\colon S^2\to S^4$ be given by $h(z)=(x_1,x_2,x_3,x_4,x_5)$*

$$x_1=\frac{mn^2((-1)^n(2n-m)(1+|k|^{2m})-m(|k|^{2n}+|k|^{2m-2n}))}{(2n-m)^2t},$$

$$x_2+ix_3=\frac{2nm((-1)^n|k|^{2n}-1)A\bar{B}\,k^{m-n}}{t},$$

$$x_4+ix_5=\frac{2mn^2((-1)^n|k|^{2n}+|k|^{2m})\bar{B}k^{-n}}{(2n-m)t},$$

$$k=z^{\lambda}\frac{\prod_{i=1}^{\rho}(z-a_i)}{\prod_{j=1}^{\rho}(\bar{a}_jz+1)},$$

$$t=\frac{mn^2((-1)^n(2n-m)(1+|k|^{2m})+m(|k|^{2n}+|k|^{2m-2n}))}{(2n-m)^2},$$

where $\lambda+\rho$ and m are odd, $m\neq 2n$, $m\neq n$, $(-1)^n m(2n-m)>0$, $|A|=|n/(2n-m)|$, $|B|^2=(-1)^n m/(2n-m)$. Then h is a conformal isotropic harmonic map with $h\cdot I=h$. Hence h gives a harmonic map of P^2 to S^4.

Unfortunately, in the present paper we cannot determine the general forms of Bryant meromorphic functions on S^2. There seem to exist a lot of Bryant meromorphic functions with gg^* nonconstant. We here give only some examples. Put

$$f=Az\frac{(z-a)^2}{(z-c)^2},\quad g=Bz^{\beta}\frac{(z-b)}{(z-c)},$$

where

$$a=x_1e,\quad b=x_2e,\quad c=x_3e,\quad |e|=1,\quad |B|=1.$$

Then if one of the following conditions (4.4) and (4.5) is satisfied, f and g are the Bryant meromorphic functions with gg^* nonconstant.

(4.4) $\beta=1,\quad |A|=1+\sqrt{3},\quad x_1=\sqrt{(7\pm 3\sqrt{3})/2},\quad x_2=\pm|A|x_1,\quad x_3=|A|x_1.$

(4.5) $\beta=-2,\quad |A|=(\pm 5+3\sqrt{21})/41,\quad x_1=\sqrt{82}/(29\pm\sqrt{21}),$
$x_2=-|A|x_1/2,\quad x_3=\pm 5|A|x_1/2.$

5. Harmonic maps of nonorientable surfaces of genus l into S^4. Let T_{l-1} be a hyperelliptic Riemann surface with an involution I as given in Section 1. Let f and g be the Bryant meromorphic functions given by

$$f=\frac{P_1+Q_1w}{R_1},\quad g=\frac{P_2+Q_2w}{R_2},$$

where P_i, Q_i, R_i are polynomial functions of a variable z and have no common factor for each $i=1$ or 2 (see, for example, [SP, Chapter 10]). Moreover, we can set for $i=1,2$

$$P_i=A_i\prod_{j=1}^{\mu_i}(z-a_{ij}),\quad Q_i=B_i\prod_{j=1}^{\nu_i}(z-b_{ij}),\quad R_i=\prod_{j=1}^{\lambda_i}(z-e_{ij}).$$

We investigate which f and g satisfy the equations (1.3) and (1.4). It is very difficult to determine such f and g in general. Hence we impose the same condition as in Section 4, that is, ff^* and gg^* are constant. In this section, for a polynomial function $P(z)$ of a variable z, we put $P^*(z)=\overline{P(-\bar{z})}$. Put

$$ff^*=c_1\,, \qquad gg^*=c_2\,, \tag{5.1}$$

where c_1 and c_2 are constants. These imply $P_i^*Q_i=P_iQ_i^*$, $i=1, 2$ and

$$P_iP_i^*-Q_iQ_i^*w^2=c_iR_iR_i^*\,, \qquad i=1, 2\,. \tag{5.2}$$

Now, from (1.3) and (5.1), we get $2c_2dg/g=(1+c_2)df/f$. Hence we obtain

$$2c_2\log\left(\frac{P_2+Q_2w}{R_2}\right)=(1+c_2)\left(\log\left(\frac{P_1+Q_1w}{R_1}\right)+C\right), \tag{5.3}$$

This implies that $Q_1=0$ if and only if $Q_2=0$.

LEMMA 5.1. *If f and g satisfy* (5.1), *c_1 and c_2 are real. Moreover, Q_1 and Q_2 do not vanish.*

PROOF. Relacing z by $-\bar{z}$ and taking the complex conjugates of both sides, from the equation (5.2), we get

$$P_iP_i^*-Q_iQ_i^*w^2=\bar{c}_iR_iR_i^*\,, \qquad i=1, 2\,.$$

Hence c_1 and c_2 are real.

If Q_1 vanishes, we have

$$c_1=ff^*=(-1)^{\mu_1-\lambda_1}|A_1|^2\frac{\prod_{i=1}^{\mu_1}(z-a_{1i})(z+\bar{a}_{1i})}{\prod_{j=1}^{\lambda_1}(z-e_{1j})(z+\bar{e}_{1j})}\,.$$

Hence, we get $\mu_1=\lambda_1$, $\prod_{i=1}^{\mu_1}(z-a_{1i})=\prod_{i=1}^{\mu_1}(z+\bar{e}_{1i})$ and $\prod_{i=1}^{\mu_1}(z-e_{1i})=\prod_{i=1}^{\mu_1}(z+\bar{a}_{1i})$. Thus, we have $c_1=|A_1|^2$. Since c_2 is real, this contradicts (1.4). q.e.d.

From (5.3), it follows that $P_1=0$ if and only if $P_2=0$. The equation (5.3) implies that the irreducible factors of the polynomial $P_1+Q_1\omega$ (resp. R_1) of variables z and ω (resp. a variable z) coincide with those of the polynomial $P_2+Q_2\omega$ (resp. R_2). Hence, there exists a meromorphic function

$$k=\frac{P+Qw}{R}$$

such that

$$f=Ak^m\,, \quad g=Bk^n\,,$$

were $P=D\prod_{j=1}^{\mu}(z-a_j)$, $Q=\prod_{j=1}^{\nu}(z-b_j)$ and $R=\prod_{j=1}^{\lambda}(z-e_j)$ have no common factor, and the integers m and n satisfy

$$2c_2 n=(1+c_2)m\,. \tag{5.4}$$

From (5.1), it follows that kk^* is also constant. Hence we have

$$c_1=|A|^2c^m\,,\quad c_2=|B|^2c^n\,,\qquad c=kk^* \tag{5.5}$$

and

$$P^*Q=PQ^*\,,\quad PP^*-QQ^*w^2=cRR^*\,. \tag{5.6}$$

If $Q=0$, then $Q_1=Q_2=0$. Hence we have $Q\neq 0$. Since $c_1<0$, we see that $c<0$ and that m is odd. Hence $P=0$ if and only if $P_1=0$ and $P_2=0$.

We first assume $P=0$. Then, from (5.6), we get

$$-(-1)^{\nu+l}\prod_{j=1}^{\nu}(z-b_j)(z+\bar{b}_j)\prod_{j=1}^{l}(z-d_j)(z+\bar{d}_j)=(-1)^{\lambda}c\prod_{j=1}^{\lambda}(z-e_j)(z+\bar{e}_j)\,.$$

Hence, we have $\lambda=\nu+l$ and $c=-1$. We may put $e_j=-\bar{b}_j$ $(1\leqq j\leqq \nu)$, and $e_j=d_{j-\nu}$ $(\nu+1\leqq j\leqq\lambda)$ or $e_j=-\bar{d}_{j-\nu}$ $(\nu+1\leqq j\leqq\lambda)$. Thus we can set

$$k=\frac{\prod_{i=1}^{\nu}(z-b_j)w}{\prod_{i=1}^{\nu}(z+\bar{b}_i)\prod_{j=1}^{l}(z-e_j)}\,,\qquad e_j=d_j\ (1\leqq j\leqq l)\quad\text{or}\quad e_j=-\bar{d}_j\ (1\leqq j\leqq l).$$

Using (5.4), (5.5) and $4c_1+(1+c_2)^2=0$, we obtain $|B|^2=(-1)^n m/(2n-m)$ and $|A|=|n/(2n-m)|$.

Next, we assume that $P\neq 0$. From (5.6), it follows that

$$(-1)^{\nu}D\prod_{j=1}^{\mu}(z-a_j)\prod_{j=1}^{\nu}(z-\bar{b}_j)=(-1)^{\mu}\bar{D}\prod_{j=1}^{\mu}(z+\bar{a}_j)\prod_{j=1}^{\nu}(z-b_j)\,. \tag{5.7}$$

$$(-1)^{\mu}|D|^2\prod_{j=1}^{\mu}(z-a_j)(z+\bar{a}_j)-(-1)^{\nu}\prod_{j=1}^{\nu}(z-b_j)(z+\bar{b}_j)=(-1)^{\lambda}c\prod_{j=1}^{\lambda}(z-e_j)(z+\bar{e}_j). \tag{5.8}$$

From (5.7), it follows that D is real if $\mu+\nu$ is even and pure imaginary otherwise. Moreover, we can set

$$P=D\prod^{\mu}(z-a_i)\,,\quad Q=\prod_{i=\mu_1+1}^{\mu_2}(z-a_i)\prod_{j=1}^{\nu_1}(z-b_j)\,,\quad R=\prod_{i=\mu_1+1}^{\mu_2}(z+\bar{a}_i)\prod_{j=1}^{\lambda_1}(z-e_j)\,,$$

where $\mu=\mu_1+\mu_2$, $\nu=\nu_1+\mu_2$ and $\lambda=\lambda_1+\mu_2$. Thus (5.8) gives

$$(-1)^{\mu_1}|D|^2\prod_{j=1}^{\mu_1}(z-a_j)^2-(-1)^{\nu_1}\prod_{j=1}^{\nu_1}(z-b_j)^2=(-1)^{\lambda_1}c\prod_{j=1}^{\lambda_1}(z-e_j)(z+\bar{e}_j)\,.$$

$$\prod^{\mu_1}(z-a_i)=\prod^{\mu_1}(z+\bar{a}_i)\,,\quad \prod^{\nu_1}(z-b_i)=\prod^{\nu_1}(z+\bar{b}_i)\,.$$

From (5.4) and (5.5), we get $|B|^2=c^{-n}m/(2n-m)$ and $|A|^2=c^{-m}(n/(2n-m))^2$. Summing up we obtain Theorem IV.

REFERENCES

[AHS] M. F. ATIYAH, N. J. HITHIN AND I. M. SINGER, Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London. Ser. A 362 (1978), 425–461.

[B] R. L. BRYANT, Conformal and minimal immersions of compact surfaces into the 4-sphere, J. Differential Geom. 17 (1982), 455–473.

[EeL1] J. EELLS AND L. LEMAIRE, A report on harmonic maps, Bull. London Math. Soc. 10 (1978), 1–68.

[EeL2] J. EELLS AND L. LEMAIRE, On the construction of harmonic and holomorphic maps between surfaces, Math. Ann. 252 (1980), 27–52.

[EeL3] J. EELLS AND L. LEMAIRE, Another report on harmonic maps, Bull. London Math. Soc. 20 (1988), 385–524.

[EeS] J. EELLS AND S. SALAMON, Twistorial construction of harmonic maps of surfaces into Four-manifolds, Ann. Scuola Norm. Sup. Pisa (4) 12 (1985), 589–640.

[EeW] J. EELLS AND J. C. WOOD, Harmonic maps from surfaces to complex projective spaces, Adv. in Math. 49 (1983), 217–263.

[Eg] N. EJIRI, Equivariant minimal immersions of S^2 into $S^{2m}(1)$, Trans. Amer. Math. Soc. 297 (1986), 105–124.

[I1] T. ISHIHARA, Complete Mobius strips minimally immersed in R^3, Proc. Amer. Math. Soc. 107 (1989), 803–806.

[I2] T. ISHIHARA, Complete non-orientable minimal surfaces, Trans. Amer. Math. Soc., to appear.

[M] W. MEEKS, The classification of complete minimal surfaces in R^3 with total curvature greater than -8π, Duke Math. J. 48 (1981), 523–535.

[O] M. E. OLIVEIRA, Some new examples of nonorientable minimal surfaces, Proc. Amer. Math. Soc. 98 (1986), 629–635.

[S] S. SALAMON, Topics in four-dimensional Riemannian geometry, Geometry Seminar Luigi Bianchi, Lecture Notes in Mathematics 1022, Springer-Verlag, Berlin, Heidelberg, New York and Tokyo, 1983.

[Sp] G. SPRINGER, Introduction to Riemann surfaces, Addison-Wesley, Reading and Massachusetts, 1957.

[Y] K. YANG, Complete and compact minimal surfaces, Kluwer Academic Publlishers, Dordrecht, Boston and London, 1989.

DEPARTMENT OF MATHEMATICS
FACULTY OF INTEGRAL ARTS AND SCIENCES
TOKUSHIMA UNIVERSITY
TOKUSHIMA 770
JAPAN