Tôhoku Math. J.
45 (1993), 285–295

CONCENTRATION COMPACTNESS OF A SPACE OF NONLINEAR p-HARMONIC FUNCTIONS

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(Received January 20, 1992, revised October 16, 1992)

Abstract. We prove concentration compactness of a space of nonlinear p-harmonic functions.

In this note we are concerned with *nonlinear p-harmonic functions*, i.e. solutions of a degenerate nonlinear elliptic equation

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u)+C_0|u|^{q-2}u=0 \qquad (2\leq p<n) \tag{1}$$

on a domain Ω of $\boldsymbol{R}^n$, where $q:=np/(n-p)$. The equation of the above type is the Euler-Lagrange equation of the p-energy functional

$$\mathscr{F}(u)=\frac{1}{p}\int_{\Omega}\|\nabla u\|^p-\frac{C_0}{q}\int_{\Omega}|u|^q .$$

Then u is called a *weak solution* of the equation (1) (on Ω) if the following two conditions hold:

(1) $u\in L^{1,p}(\Omega)$, i.e., u, $\nabla u\in L^p(\Omega)$. (Then the Sobolev inequality implies $u\in L^q(\Omega)$.)

(2) The function u satisfies

$$-\int_{\Omega}\|\nabla u\|^{p-2}\nabla u\cdot\nabla\varphi+C_0\int_{\Omega}|u|^{q-2}u\varphi=0$$

for any $\varphi\in C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all C^∞-functions with compact support on Ω.

The equation (1) for $p=2$

$$\Delta u+C_0|u|^{2^*-2}u=0 \qquad \left(2^*:=\frac{2n}{n-2}\right)$$

is of Yamabe type, and has been studied from various viewpoints. (See Lee-Parker [4], Bahri [1], Struwe [10], etc. and their references.) Lions [6], [7], Takakuwa [11] showed a concentration phenomenon of the L^{2^*}-norm in a sequence of solutions (or an approximating sequence) of this equation. In this note we give the following

* Partly supported by the Grants-in-Aid for Scientific Research, the Ministry of Education, Science and Culture, Japan.

1991 *Mathematics Subject Classification*. Primary 35J60; Secondary 35J20.

generalization of their results to the case of general p $(2 \leq p < n)$.

THEOREM 1. *Let* u_j $(j=1, 2, \dots)$ *be weak solutions of the equation* (1). *Assume that*

$$\|u_j\|_{L^{1,p}(\Omega)} := \|u_j\|_{L^p(\Omega)} + \|\nabla u_j\|_{L^p(\Omega)} \leq C < \infty ,$$

where C is a constant independent of j. Then there exist

(i) *a subsequence of* $\{u_j\}$ (*we use the same notation* $\{u_j\}$ *below for this subsequence*),

(ii) *a set* $\mathcal{S}$ *of points* $x_1, \dots, x_k$ *of* Ω,

and

(iii) *positive numbers* $a_1, \dots, a_k$

satisfying the following two conditions:

(1) u_j *is continuous on* Ω, *and* $\{u_j\}$ *converges to a function w uniformly on any compact set of* $\Omega - \mathcal{S}$, *where w is a weak solution of* (1) *on* Ω. *Furthermore for any compact set K in* $\Omega - \mathcal{S}$, *there exists* $\alpha > 0$ *such that* u_j *has a uniformly bounded* $C^{1,\alpha}$*-norm on K.*

(2) *The measure* $|u_j|^q dx$ *converges weakly to* $|w|^q dx + \sum_{i=1}^{k} a_i \delta_{x_i}$ *as* $j \to \infty$, *where dx denotes the volume element on* Ω, *and* δ_{x_i} *denotes the Dirac mass supported at* x_i.

The exponent q is critical; q is the critical exponent of the Sobolev embedding $L^{1,p} \to L^q$. In the case of subcritical exponents, we have $\mathcal{S} = \emptyset$. (See Theorem 2 in § 3.) The example in § 1 shows that Theorem 1 is optimal. This is a typical example, which gives a motivation for our theorem. The $C^{1,\alpha}$-estimate is optimal for $p > 2$, since the equation (1) is degenerate elliptic. (cf. Ural'ceva [14], Uhlenbeck [13], Evans [2], Lewis [5] etc.) In case $p = 2$, the C^∞-estimate follows from the $C^{1,\alpha}$-estimate by the bootstrap argument in the theory of elliptic equations; hence the above subsequence $\{u_j\}$ converges in the C^∞-topology on $\Omega - \mathcal{S}$.

Our method is different from Lions' theory [6], [7] of *concentration compactness.* The property (2) in Theorem 1 can be proved also by the method of Lions using a concentration function, except that $\mathcal{S}$ consists of only a finite number of points. Our proof is along Schoen's argument [9] for harmonic maps. (See also Takakuwa [11], Pacard [8].) We use a mean-value estimate (cf. Proposition) and a simple standard argument. In our proof of the mean-value estimate, we use Moser's iteration technique. This estimate says that if the L^q-norm is sufficiently small around a point, we obtain a local C^0-estimate, hence a local $C^{1,\alpha}$-estimate which follows from regularity arguments for p-harmonic functions, $\operatorname{div}(\|\nabla u\|^{p-2} \nabla u) = 0$. The assumption of the boundedness of the L^q-norm implies that such an estimate holds except at a finite number of points of Ω.

ACKNOWLEDGEMENT. The author would like to thank Professor Shoichiro Takakuwa for many helpful conversations. The author is also grateful to the referee for his valuable advice.

1. An example. As mentioned in the introduction, we describe a typical example. Consider a radially symmetric function

$$u_\lambda(x) := \left\{ \frac{\tilde{C}\lambda^{1/(p-1)}}{\lambda^{p/(p-1)} + \|x\|^{p/(p-1)}} \right\}^{(n-p)/p} \qquad (\lambda > 0)$$

on $\Omega = \boldsymbol{R}^n$, where

$$\tilde{C} := \left\{ \frac{n}{C_0} \left(\frac{n-p}{p-1} \right)^{p-1} \right\}^{1/p}.$$

Then u_λ satisfies the equation

$$\operatorname{div}(\|\nabla u_\lambda\|^{p-2}\nabla u_\lambda) + C_0 |u_\lambda|^{q-2} u_\lambda = 0 .$$

We see that

$$\int_{\boldsymbol{R}^n} u_\lambda^q dx = \left(\frac{\tilde{C}}{\lambda} \right)^n \int_{\boldsymbol{R}^n} \frac{dx}{\{1 + (\|x\|/\lambda)^{p/(p-1)}\}^n} = \tilde{C}^n \omega_{n-1} \int_0^\infty \frac{r^{n-1} dr}{(1 + r^{p/(p-1)})^n} ,$$

which is a finite constant independent of λ, where ω_{n-1} denotes the volume of the $(n-1)$-dimensional unit sphere.

The sequence of the measures $|u_\lambda|^q dx$ converges to the Dirac measure supported at the origin as λ tends decreasingly to 0. These solutions look like solitons with one peak, and as λ tends to 0, the slope becomes steeper and the L^q-energy density is attracted to the origin.

2. Proof of Theorem 1. As mentioned in the introduction, the following estimate plays a key role in our proof.

PROPOSITION (a mean-value estimate). *There exist positive numbers ε^* and C^*, depending only on n, p, C_0 and Ω, satisfying the following property*:

Let u be any weak solution of the equation (1) *on Ω. Let $x \in \Omega$ and let* $0 < \rho < \min\{d(x, \partial\Omega), 1\}$, *where $d(x, \partial\Omega)$ denotes the distance between x and $\partial\Omega$. If*

$$\int_{B_\rho(x)} |u|^q < \varepsilon^* ,$$

then

$$\sup_{B_{\rho/2}(x)} |u|^q \le \frac{C^*}{\rho^n} \int_{B_\rho(x)} |u|^q .$$

We collect here basic notation. Let C_Ω denote the Sobolev constant:

$$\left\{ \int_\Omega |\phi|^q \right\}^{p/q} \le C_\Omega \left\{ \int_\Omega \|\nabla\phi\|^p + \int_\Omega |\phi|^p \right\} \tag{2}$$

for any $\phi \in L^{1,p}(\Omega)$. All positive constants $C_1, C_2, C_3, \dots$ depend only on n, p, C_0 and C_Ω. Let $B_\rho(x)$ denote the open ball of radius ρ centered at x. Let $0<\rho_1<\rho_2<\rho$. Let $\eta \in C^\infty(\Omega)$ be a cutoff function such that

$$\begin{aligned} &\eta=1 && \text{on} \quad B_{\rho_1}(x)\,, \\ &\eta \in [0,1] && \text{on} \quad B_{\rho_2}(x)-B_{\rho_1}(x)\,, \\ &\eta=0 && \text{on} \quad \Omega-B_{\rho_2}(x)\,, \end{aligned}$$

and that $\|\nabla\eta\|^2 \le C_1/(\rho_2-\rho_1)^2$. The equation (1) implies the following inequality, which will be used in each iteration step later.

LEMMA 1. *There exist positive constants C_2 and C_3 satisfying*

$$\left(1-\frac{1}{s}\right)\left\{\int_\Omega (|u|^s\eta)^q\right\}^{p/q} \le \frac{C_2}{(\rho_2-\rho_1)^p}\int_{B_{\rho_2}(x)} |u|^{ps}+C_3 s^{p-1}\int_\Omega |u|^{ps+pq/n}\eta^p\,, \tag{3}$$

for any s (≥ 1).

PROOF. The equation (1) implies that

$$\int_\Omega |u|^{ps-p}u\eta^p \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)+C_0\int_\Omega |u|^{ps-p+q}\eta^p=0\,. \tag{4}$$

We assume, for simplicity, that $|u|^{ps-p}u\eta^p$ is a legitimate test function in the definition of the weak solution of (1). In the general situation, we can use a standard approximation argument. See Gilbarg-Trudinger [3, pp. 189–190]. Note that $1 \le ps-p+1 \le ps$. We see

$$\begin{aligned} (5)\quad &\int_\Omega |u|^{ps-p}u\eta^p \operatorname{div}(\|\nabla\|^{p-2}\nabla u) \\ &= -\int_\Omega \|\nabla u\|^{p-2}\nabla u\cdot\nabla(|u|^{ps-p}u\eta^p) \\ &= -(ps-p+1)\int_\Omega \|\nabla u\|^p|u|^{ps-p}\eta^p - p\int_\Omega \|\nabla u\|^{p-2}|u|^{ps-p}u\eta^{p-1}\nabla u\cdot\nabla\eta \\ &\le -\frac{ps-p+1}{s^p}\int_\Omega \|\nabla|u|^s\|^p\eta^p+\frac{p}{s^{p-1}}\int_\Omega \big|\,\|\nabla|u|^s\|^{p-2}\eta^{p-1}|u|^s\nabla|u|^s\cdot\nabla\eta\,\big|\,. \end{aligned}$$

Applying Young's inequality

$$|A\cdot B| \le \frac{p-1}{p}\|A\|^{p/(p-1)}+\frac{1}{p}\|B\|^p$$

for $A=\|\nabla|u|^s\|^{p-2}\eta^{p-1}\nabla|u|^s/(p-1)^{(p-1)/p}$, $B=(p-1)^{(p-1)/p}|u|^s\nabla\eta$, we obtain

$$\int_\Omega |\,\|\nabla |u|^s\|^{p-2}\eta^{p-1}|u|^s\nabla|u|^s\cdot\nabla\eta\,| \le \frac{1}{p}\int_\Omega \|\nabla|u|^s\|^p\eta^p + \frac{(p-1)^{p-1}}{p}\int_\Omega |u|^{ps}\|\nabla\eta\|^p .$$

Hence by (5), we have

$$\text{(6)}\qquad \int_\Omega |u|^{ps-p}u\eta^p \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$$

$$\le -\frac{(p-1)(s-1)}{s^p}\int_\Omega \|\nabla|u|^s\|^p\eta^p + \frac{(p-1)^{p-1}}{s^{p-1}}\int_\Omega |u|^{ps}\|\nabla\eta\|^p .$$

Note the inequality $\|A+B\|^p \le 2^{p-1}(\|A\|^p+\|B\|^p)$, i.e., $-\|A\|^p \le -2^{-(p-1)}\|A+B\|^p + \|B\|^p$. Using this inequality for $A=\eta\nabla|u|^s$, $B=|u|^s\nabla\eta$, we have

$$\text{(7)}\qquad -\int_\Omega \|\nabla|u|^s\|^p\eta^p \le \frac{1}{2^{p-1}}\int_\Omega \|\nabla(|u|^s\eta)\|^p + \int_\Omega |u|^{ps}\|\nabla\eta\|^p .$$

Then by (2), (6), (7), we have

$$\text{(8)}\qquad \int_\Omega |u|^{ps-p}u\eta^p \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$$

$$\le -\frac{(p-1)(s-1)}{2^{p-1}C_\Omega s^p}\left\{\int_\Omega (|u|^s\eta)^q\right\}^{p/q} + \frac{(p-1)(s-1)}{2^{p-1}s^p}\int_\Omega |u|^{ps}\eta^p$$

$$+\left\{\frac{(p-1)(s-1)}{s^p} + \frac{(p-1)^{p-1}}{s^{p-1}}\right\}\int_\Omega |u|^{ps}\|\nabla\eta\|^p$$

$$\le -\frac{C_4}{s^{p-1}}\left(1-\frac{1}{s}\right)\left\{\int_\Omega (|u|^s\eta)^q\right\}^{p/q} + \frac{C_5}{s^{p-1}(\rho_2-\rho_1)^p}\int_{B_{\rho_2}(x)} |u|^{ps} ,$$

since $0<\rho_2-\rho_1<1$. Lemma 1 follows from (4), (8), since $ps-p+q=ps+pq/n$. □

We prove the Proposition. Define

$$\varepsilon^* := \left\{\frac{p}{2nC_3}\left(\frac{p}{q}\right)^{p-1}\right\}^{n/p} .$$

Suppose $\int_{B_\rho(x)}|u|^q<\varepsilon^*$. Under this assumption, we prove the following lemma and Lemma 3.

LEMMA 2.

$$\left\{\int_{B_{(\sigma_1+\sigma_2)/2}(x)} |u|^{q^2/p}\right\}^{p/q^2} \le \frac{C_6}{(\sigma_2-\sigma_1)^{p/q}}\left\{\int_{B_{\sigma_2}(x)} |u|^q\right\}^{1/q} \le \frac{C_7}{(\sigma_2-\sigma_1)^{p/q}}$$

with $0<\sigma_1<\sigma_2\le\rho$.

PROOF. Let $s=q/p$ (>1). Let $\rho_1=(\sigma_1+\sigma_2)/2$ and $\rho_2=\sigma_2$. By Hölder's inequality, we have

$$\int_{\Omega} |u|^{ps+pq/n}\eta^p \leq \left\{\int_{B_{\sigma_2}(x)} |u|^q\right\}^{p/n} \left\{\int_{\Omega} (|u|^{ps}\eta^p)^{q/p}\right\}^{p/q}$$
$$\leq (\varepsilon^*)^{p/n}\left\{\int_{\Omega} (|u|^s\eta)^q\right\}^{p/q} = \frac{1}{2C_3 s^{p-1}}\left(1-\frac{1}{s}\right)\left\{\int_{\Omega} (|u|^s\eta)^q\right\}^{p/q},$$

since $1-1/s=1-p/q=p/n$. Then (3) implies

$$\left\{\int_{\Omega} (|u|^s\eta)^q\right\}^{p/q} \leq \frac{C_8}{(\sigma_2-\sigma_1)^p}\int_{B_{\sigma_2}(x)} |u|^{ps}.$$

Since $qs=q^2/p$ and $ps=q$, we have Lemma 2. □

LEMMA 3.

$$\Phi(qs, \sigma_1) \leq \frac{(C_9 s^n)^{1/ps}}{(\sigma_2-\sigma_1)^{1/s}}\,\Phi(ps, \sigma_2) \qquad (0<\sigma_1<\sigma_2\leq\rho)$$

for any s $(\geq q)$, *where*

$$\Phi(s, \rho) := \left\{\int_{B_\rho(x)} |u|^s\right\}^{1/s}.$$

PROOF. Let $\gamma=n/(n-p)=q/p$ and define $a=n\gamma/p=nq/p^2$, $b=\gamma^2$, $c=n/p$. Note $1/a+1/b+1/c=1$ and $\gamma/b+1/c=1$. Let $\rho_1=\sigma_1$ and $\rho_2=(\sigma_1+\sigma_2)/2$. Then

$$\int_{\Omega} |u|^{ps+pq/n}\eta^p = \int_{B_{(\sigma_1+\sigma_2)/2}(x)} |u|^{pq/n}(|u|^{ps}\eta^p)^{\gamma/b}(|u|^{ps}\eta^p)^{1/c}$$
$$\leq \left\{\int_{B_{(\sigma_1+\sigma_2)/2}(x)} |u|^{pqa/n}\right\}^{1/a}\left\{\int_{\Omega} (|u|^{ps}\eta^p)^{\gamma}\right\}^{1/b}\left\{\int_{\Omega} |u|^{ps}\eta^p\right\}^{1/c}$$
$$= \left\{\int_{B_{(\sigma_1+\sigma_2)/2}(x)} |u|^{q^2/p}\right\}^{p^2/nq}\left\{\int_{\Omega} (|u|^s\eta)^q\right\}^{1/b}\left\{\int_{\Omega} |u|^{ps}\eta^p\right\}^{1/c}$$
$$\leq \frac{C_{10}}{(\sigma_2-\sigma_1)^{p^2/n}}\left\{\int_{\Omega} (|u|^s\eta)^q\right\}^{1/b}\left\{\int_{\Omega} |u|^{ps}\eta^p\right\}^{1/c} \qquad \text{(by Lemma 2)}$$
$$\leq \frac{1}{2C_3 s^{p-1}}\left(1-\frac{1}{s}\right)\left\{\int_{\Omega} (|u|^s\eta)^q\right\}^{p/q} + \frac{C_{11}s^{n(p-1)/q}}{\left(1-\frac{1}{s}\right)^{n/q}(\sigma_2-\sigma_1)^p}\int_{\Omega} |u|^{ps}\eta^p,$$

since $AB\leq\varepsilon A^\gamma+B^c/\varepsilon^{n/q}$. Hence (3) implies

$$\frac{1}{2}\left(1-\frac{1}{s}\right)\left\{\int_{\Omega} (|u|^s\eta)^q\right\}^{p/q} \leq \left\{\frac{C_2}{(\rho_2-\rho_1)^p} + \frac{C_{12}s^{n(p-1)/q+p-1}}{\left(1-\frac{1}{s}\right)^{n/q}(\sigma_2-\sigma_1)^p}\right\}\int_{B_{\sigma_2}(x)} |u|^{ps}$$

$$\leq \frac{C_{13}s^{n-1}}{(s-1)^{n/q}(\sigma_2-\sigma_1)^p}\int_{B_{\sigma_2}(x)}|u|^{ps},$$

since $np/q+p-1=n-1$. Therefore

$$\left\{\int_{\Omega}(|u|^s\eta)^q\right\}^{p/q}\leq \frac{C_{14}s^n}{(s-1)^{n/p}(\sigma_2-\sigma_1)^p}\int_{B_{\sigma_2}(x)}|u|^{ps}.$$

Since $1/(s-1)^{n/p}\leq 1/(q-1)^{n/p}$, we have Lemma 3. □

Let $r^{(j)}:=q\gamma^j=p\gamma^{j+1}$ $(\gamma=q/p>1)$, $\rho^{(j)}:=(1+1/2^j)\rho/2$ $(j=0, 1, 2, \dots)$. Then Lemma 3 implies

$$\Phi(r^{(j)}, \rho^{(j)})\leq \frac{C_{15}^{j/\gamma^j}}{\rho^{1/\gamma^j}}\,\Phi(r^{(j-1)}, \rho^{(j-1)}).$$

Hence by iterating the above inequality, we have

$$\Phi(r^{(j)}, \rho^{(j)})\leq \frac{C_{16}}{\rho^{(n-p)/p}}\,\Phi(r^{(0)}, \rho^{(0)})=\frac{C_{16}}{\rho^{n/q}}\left\{\int_{B_\rho(x)}|u|^q\right\}^{1/q}.$$

Letting $j\to\infty$, we have the Proposition.

Proof of Theorem 1. Let $\underline{\mathscr{S}}$, $\overline{\mathscr{S}}$ denote the subsets of Ω defined by

$$\underline{\mathscr{S}}:=\bigcap_{\rho>0}\left\{x\in\Omega;\ \liminf_{j\to\infty}\int_{B_\rho(x)}|u_j|^q\geq\frac{\varepsilon^*}{2}\right\},$$

$$\overline{\mathscr{S}}:=\bigcap_{\rho>0}\left\{x\in\Omega;\ \limsup_{j\to\infty}\int_{B_\rho(x)}|u_j|^q\geq\frac{\varepsilon^*}{2}\right\},$$

where ε^* denotes the constant in the Proposition.

We show that the cardinality $\sharp(\underline{\mathscr{S}})$ of $\mathscr{S}$ is bounded by such a constant as C_{17}/ε^*. Take any finite subset $\{x_1, \dots, x_k\}$ of $\underline{\mathscr{S}}$. Let

$$\rho:=\min\{d(x_i, x_j), d(x_j, \partial\Omega);\ i\neq j,\ 1\leq i, j\leq k\}>0,$$

where $d(\ ,\ )$ denotes the standard distance in $\boldsymbol{R}^n$. Take a sufficiently large j such that

$$\int_{B_\rho(x)}|u_j|^q\geq\frac{\varepsilon^*}{4}.$$

Since the open balls $B_\rho(x_j)$ $(j=1, \dots, k)$ are mutually disjoint, we see that

$$k\frac{\varepsilon^*}{4}\leq\sum_{j=1}^k\int_{B_\rho(x_j)}|u|^q\leq\int_\Omega|u|^q\leq C_{18}, \qquad \text{i.e.,}\quad k\leq\frac{C_{19}}{\varepsilon^*}.$$

Hence $\sharp(\underline{\mathscr{S}})\leq C_{19}/\varepsilon^*$.

Note that $\underline{\mathscr{S}} \subset \overline{\mathscr{S}}$ and that $\underline{\mathscr{S}}$, $\overline{\mathscr{S}}$ depend on the choice of the sequence $\{u_j\}_{j=1}^{\infty}$. We show that there exists a subsequence satisfying $\underline{\mathscr{S}} = \overline{\mathscr{S}}$. Suppose $\underline{\mathscr{S}} \neq \overline{\mathscr{S}}$. Take any $x_0 \in \overline{\mathscr{S}} - \underline{\mathscr{S}}$. Then we can find a subsequence such that $\liminf_{j\to\infty} \int_{B_\rho(x_0)} |u_j|^q \geq \varepsilon^*/2$. The number $\sharp(\underline{\mathscr{S}})$ for this new sequence is greater than that for the old one, since x_0 belongs to the new $\underline{\mathscr{S}}$, but not to the old one. We can iterate this step inductively and the number $\sharp(\underline{\mathscr{S}})$ increases as long as $\underline{\mathscr{S}} \neq \overline{\mathscr{S}}$. Since $\sharp(\underline{\mathscr{S}})$ is bounded from above by the constant C_{19}/ε^*, we have, after a finite number of steps, a subsequence such that $\underline{\mathscr{S}} = \overline{\mathscr{S}}$.

We prove the property (1) in Theorem 1. Let $\{u_j\}_{j=1}^{\infty}$ be a subsequence such that $\underline{\mathscr{S}} = \overline{\mathscr{S}}$ $(=: \mathscr{S})$. Take any $x \in \Omega - \mathscr{S}$. Then it follows from the definition of $\mathscr{S}$ that for some $\rho > 0$,

$$\int_{B_\rho(x)} |u_j|^q \leq \varepsilon^* .$$

Applying the Proposition, we have

$$\sup_{B_{\rho/2}(x)} |u_j|^q \leq \frac{C^*}{\rho^n} \int_{B_\rho(x)} |u_j|^q \leq \frac{\varepsilon^* C^*}{\rho^n} .$$

Hence we see that u_j is uniformly bounded on $B_{\rho/2}(x)$. Then by arguments similar to those in the proof of the regularity for p-harmonic functions (see Evans [2]), there exists $\alpha > 0$ such that the $C^{1,\alpha}$-norms are locally bounded above by a constant independent of j. Note that the term $C_0 |u|^{q-2} u$ in (1) is locally bounded there. Then a subsequence of $\{u_j\}$ converges uniformly to a continuous function on $B_{\rho/2}(x)$ as $j \to \infty$. Hence for any compact set K in $\Omega - \mathscr{S}$, there exists a subsequence of u_j uniformly convergent on K. We take an exhaustion of $\Omega - \mathscr{S}$ by compact sets. By Cantor's diagonal argument, we can find a subsequence (also denoted by $\{u_j\}$) converging in the C^0-topology to a continuous function w on $\Omega - \mathscr{S}$. We can verify that w is a weak solution of (1) on $\Omega - \mathscr{S}$, since a subsequence of $\{u_j\}$ converges to w in $L^{1,p}_{\mathrm{loc}}(\Omega - \mathscr{S})$. Furthermore w is a weak solution on Ω. Indeed, for $\mathscr{S} = \{x_1, \cdots, x_k\}$, we take a cutoff function $\eta_m \in C^\infty(R^n)$ for sufficiently large m such that

$$\begin{aligned} \eta_m(x) &= 0 && \text{if} \quad \|x - x_j\| \leq 1/m , \\ \eta_m(x) &\in [0, 1] && \text{if} \quad 1/m \leq \|x - x_j\| \leq 2/m , \\ \eta_m(x) &= 1 && \text{if} \quad \|x - x_j\| \geq 2/m , \end{aligned}$$

for $j = 1, \cdots, k$, and that $\|\nabla \eta_m\| \leq 2m$. Since w is a weak solution on $\Omega - \mathscr{S}$, we have

$$-\int_\Omega \|\nabla w\|^{p-2} \nabla w \cdot \nabla(\varphi \eta_m) + C_0 \int_\Omega |w|^{q-2} w \varphi \eta_m = 0 ,$$

i.e.

$$(9)\qquad -\int_\Omega \|\nabla w\|^{p-2}\eta_m \nabla w\cdot\nabla\varphi + C_0\int_\Omega |w|^{q-2}w\varphi\eta_m - \int_\Omega \|\nabla w\|^{p-2}\varphi\nabla w\cdot\nabla\eta_m = 0$$

for any $\varphi\in C_0^\infty(\Omega)$. By Lebesgue's convergence theorem, we have

$$(10)\qquad \int_\Omega \|\nabla w\|^{p-2}\eta_m\nabla w\cdot\nabla\varphi \xrightarrow{m\to\infty} \int_\Omega \|\nabla w\|^{p-2}\nabla w\cdot\nabla\varphi\,,$$

$$(11)\qquad \int_\Omega |w|^{q-2}w\varphi\eta_m \xrightarrow{m\to\infty} \int_\Omega |w|^{q-2}w\varphi\,.$$

We see

$$\left|\int_\Omega \|\nabla w\|^{p-2}\varphi\nabla w\cdot\nabla\eta_m\right| \le 2m\int_{A_m}\|\nabla u\|^{p-1} \le 2m\left\{\int_{A_m}\|\nabla u\|^p\right\}^{(p-1)/p}\mathrm{Vol}(A_m)^{1/p}\,,$$

where $A_m := \{x\in\Omega;\ 1/m < \|x-x_0\| < 2/m\}$, and $\mathrm{Vol}(A_m)$ denotes the volume of the annular domain A_m. Since $\mathrm{Vol}(A_m)\le C_{20}/m^m$, we have

$$(12)\qquad \int_\Omega \|\nabla w\|^{p-2}\varphi\nabla w\cdot\nabla\eta_m \xrightarrow{m\to\infty} 0\,.$$

By (9)–(12), we obtain

$$-\int_\Omega \|\nabla w\|^{p-2}\nabla w\cdot\nabla\varphi + C_0\int_\Omega |w|^{q-2}w\varphi = 0$$

for any $\varphi\in C_0^\infty(\Omega)$, i.e., w is weak solution on Ω.

We show the property (2) in Theorem 1. Our assumption says that the sequence of measures $\{|u_j|^q dx\}_{j=1}^\infty$ has a uniformly finite total mass; hence so does the sequence of signed measures $\{(|u_j|^q - |w|^q)dx\}_{j=1}^\infty$. Then we can find a subsequence of signed measures converging weakly to a signed measure μ, whose support is contained in $\mathscr{S}$. Since $\mathscr{S}$ is a finite set of points $x_1,\dots,x_k$, the signed measure μ is written as $\mu=\sum_{j=1}^k a_j\delta_{x_j}$ ($a_j\in R$). Take any x_j. For any $\rho>0$, we see that

$$\varepsilon^*/2 \le \liminf_{j\to\infty}\int_{B_\rho(x_j)} |u_j|^q \le a_j + \int_{B_\rho(x_j)} |w|^q\,.$$

Letting ρ tend to zero, we have $a_j\ge\varepsilon^*/2>0$. This is the property (2). □

REMARK 1. For each $y\in\mathscr{S}$, an appropriate scale-change leads us to showing that a renormalized function

$$\hat u_j(x) = \rho_j^{(n-p)/p}u_j(\rho_j x + y_j)\qquad (\rho_j\to 0,\ y_j\to y \text{ as } j\to\infty)$$

converges to a weak solution of (1) on $\boldsymbol{R}^n$.

REMARK 2. Theorem 1 (and Theorem 2 below) holds also for solutions of more general equations such as

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u)+f(x, u)=0\ ,$$

where f satisfies

$$|f(x, u)|\leq C|u|^{q-1}\ .$$

REMARK 3. We can extend our theorems to results on any Riemannian manifold M, on which the Sobolev constant C_M of $L^{1,p}\to L^q$ is positive and finite.

3. Subcritical case. In the subcritical case, an argument similar to that in §2 leads us to the following:

THEOREM 2. *Let u_j $(j=1, 2, \dots)$ be a weak solutions of the equation* (1). *Assume that there exists $\varepsilon>0$ such that*

$$\|u_j\|_{L^{q+\varepsilon}(\Omega)}\leq C<\infty\ ,$$

where C is a constant independent of j. Then u_j is continuous on Ω, and $\{u_j\}$ converges in Ω uniformly on any compact set. Furthermore for any compact set K in Ω, there exists $\alpha>0$ such that u_j has a uniformly bounded $C^{1,\alpha}$-norm on K.

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