Tôhoku Math. J.
45 (1993), 259–269

BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER NONLINEAR ORDINARY DIFFERENTIAL SYSTEMS

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(Received January 6, 1992, revised August 17, 1992)

Abstract. In this paper we study two-point boundary value problems for systems of nonlinear differential equations of order greater than or equal to two. So far the results concerning this class of problems are rare. A new Nagumo condition for systems of higher order differential equations is presented, which is more convenient for applications. Under this new Nagumo condition some existence theorems of solutions for these boundary value problems are proved.

Introduction. Since Nagumo's paper [1] was published, the differential inequality technique has become a powerful tool of studying boundary value problems for nonlinear differential equations. Using this technique many authors had made a lot of valuable works (for example see [2]–[29]). But works related with boundary value problems for higher order differential systems are rare.

In discussion of different kinds of boundary value problems several Nagumo conditions have been introduced for example by [11], [12], [15], [23], [27]. However these classical Nagumo conditions are difficult to apply. Particularly these conditions fail to be applicable to the following differential-difference equations (or systems)

$$y''=f(t, y(t-\tau), y'(t-\tau), y, y'),$$

where $y, f \in R^n$, $n \geqslant 1$, because they cannot restrict $\| y'(t-\tau) \|$.

Here we present a new Nagumo condition, which has a simple form similar to the original Nagumo condition and is very convenient for applications. Particularly, it is suitable to the above differential-difference equations (or systems) (see [34]).

In this paper we study the following boundary value problems for higher order nonlinear differential systems

$$x^{(m)}=f(t, x, x', \dots, x^{(m-1)}), \tag{1.1}$$

$$x^{(j)}(0)=A^j, \quad j=0, 1, \dots, m-2, \quad x^{(m-2)}(1)=B, \tag{1.2}$$

where $m \geqslant 3$ is a given integer, $x, f \in R^n$, $A^j, j=0, 1, \dots, m-2$ and B are all n-dimensional constant vectors. It is difficult to treat the boundary value problems (1.1), (1.2) by the classical Nagumo conditions. However using our new Nagumo condition we can turn (1.1), (1.2) into boundary value problems for second order differential systems, which

1991 *Mathematics Subject Classification*. Primary 34B15.

consists of n equations independent of one another by means of differential inequalities. Hence by Nagumo's theorem [1] and the Schauder fixed point theorem we obtain the existence theorem for solutions of (1.1), (1.2). In Section 1, we prove other existence theorems for solutions of (1.1), (1.2) as well. We exhibit three specific examples in Section 2 as an application of the theorems mentioned in Section 1.

1. Main results. For simplicity we introduce the notation. For $x=(x_1, x_2, \dots, x_n)$, $y=(y_1, y_2, \dots, y_n)$,

$$x \geqslant y \quad \text{means} \quad x_i \geqslant y_i\,, \quad i=1,2,\dots,n\,,$$

$$x[y_i]_i=(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\,,$$

$$x[0]_i=(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)\,,$$

$$|x|=(|x_1|, |x_2|, \dots, |x_n|)\,,$$

$$\|x\| = \max_{1 \leqslant i \leqslant n} |x_i|\,,$$

$$\vec{N}=(N, N, \dots, N) \in R^n, \qquad \text{for} \quad N \in R\,.$$

For $\varphi(t) \in C([0, 1], R^n)$, we define $\|\varphi(t)\|_0$ by $\|\varphi(t)\|_0=\max_{1 \leqslant i \leqslant n}\{\max_{0 \leqslant t \leqslant 1}|\varphi_i(t)|\}$. For $\varphi(t) \in C^m([0, 1], R^n)$

$$\|\varphi(t)\|_m=\max\{\|\varphi^{(j)}(t)\|_0,\ j=0, 1, \dots, m\}\,.$$

DEFINITION 1. Suppose vector-valued functions $\bar{\omega}(t), \underline{\omega}(t) \in C^m([0, 1], R^n)$ satisfy the inequalities

$$\underline{\omega}^{(j)}(t) \leqslant \bar{\omega}^{(j)}(t)\,, \quad j=0, 1, \dots, m-2\,, \quad 0 \leqslant t \leqslant 1\,,$$

$$\underline{\omega}^{(j)}(0) \leqslant A^{(j)} \leqslant \bar{\omega}^{(j)}(0)\,, \quad j=0, 1, \dots, m-2\,,$$

$$\underline{\omega}^{(m-2)}(1) \leqslant B \leqslant \bar{\omega}^{(m-2)}(1)$$

and for any function $\varphi(t) \in C^m([0, 1], R^n)$, $\underline{\omega}^{(j)}(t) \leqslant \varphi^{(j)}(t) \leqslant \bar{\omega}^{(j)}(t)$, $0 \leqslant t \leqslant 1$, $j=0, 1, \dots, m-2$,

$$\bar{\omega}_i^{(m)}(t) \leqslant f_i(t, \varphi, \varphi', \dots, \varphi^{(m-2)}[\bar{\omega}_i^{(m-2)}]_i, \varphi^{(m-1)}[\bar{\omega}_i^{(m-1)}]_i)\,,$$
$$0 \leqslant t \leqslant 1\,, \quad i=1, 2, \dots, n\,,$$

$$\underline{\omega}_i^{(m)}(t) \geqslant f_i(t, \varphi, \varphi', \dots, \varphi^{(m-2)}[\underline{\omega}_i^{(m-2)}]_i, \varphi^{(m-1)}[\underline{\omega}_i^{(m-1)}]_i)\,,$$
$$0 \leqslant t \leqslant 1\,, \quad i=1, 2, \dots, n\,.$$

Then we say that $\bar{\omega}(t)$ is an upper solution of (1.1), (1.2) and $\underline{\omega}(t)$ is a lower solution of (1.1), (1.2).

DEFINITION 2. Suppose for any real numbers $r_j > 0, j=0, 1, \dots, m-2$, there exists a vector-valued function $H(s) \in C([0, \infty)^n, (0, \infty)^n)$ nondecreasing in every s_i, such that

$$(1.3)\qquad |f(t, x, x', \ldots, x^{(m-1)})| \leqslant H(|x^{(m-1)}|)$$

for $0 \leqslant t \leqslant 1$, $\| x^{(j)} \| \leqslant r_j$, $j=0, 1, \ldots, m-2$ and there exists a real number $N_0 > 0$, such that

$$(1.4)\qquad \int_{2r_{m-2}}^{N} [s_i / h_i(\vec{N}[s_i]_i)]\, ds_i > 2r_{m-2}, \qquad i=1, 2, \ldots, n$$

for any $N \geqslant N_0$, where h_i is the i-th component of H. Then we say that the function f satisfies the Nagumo condition.

THEOREM 1. *Assume that the following* (i), (ii), (iii) *are satisfied*:

(i) $f(t, x, x', \ldots, x^{(m-1)}) \in C([0, 1] \times R^{mn}, R^n)$ *satisfies the Nagumo condition.*

(ii) f_i *is strictly increasing in* $x_i^{(m-2)}$ *as the other variables are fixed,* $i=1, 2, \ldots, n$.

(iii) *The boundary value problem* (1.1), (1.2) *has an upper solution* $\bar{\omega}(t)$ *and a lower solution* $\underline{\omega}(t)$.

Then (1.1), (1.2) *has a solution* $x(t)$ *satisfying the inequality*

$$(1.5)\qquad \underline{\omega}^{(j)}(t) \leqslant x^{(j)}(t) \leqslant \bar{\omega}^{(j)}(t), \qquad 0 \leqslant t \leqslant 1, \quad j=0, 1, \ldots, m-2.$$

PROOF. Let $r_j = \max\{\| \bar{\omega}^{(j)}(t) \|_0, \| \underline{\omega}^{(j)}(t) \|_0\}$, $j=0, 1, 2, \ldots, m-2$. Then from condition (i) there exists a vector-valued function $H(s) \in C([0, \infty)^n, (0, \infty)^n)$ such that the inequality (1.3) holds for $0 \leqslant t \leqslant 1$, $\| x^{(j)} \| \leqslant r_j$, $j=0, 1, \ldots, m-2$ and there exists a real number $N_0 > 0$ such that (1.4) holds for any $N \geqslant N_0$. Choose an $N > N_0$ and let

$$\bar{B} = \{\varphi(t) : \varphi(t) \in C^{m-1}([0, 1], R^n), \underline{\omega}^{(j)}(t) \leqslant \varphi^{(j)}(t) \leqslant \bar{\omega}^{(j)}(t),$$
$$0 \leqslant t \leqslant 1, j=0, 1, \ldots, m-2, \| \varphi^{(m-1)}(t) \|_0 \leqslant N\}.$$

(1) We prove that for each $\varphi(t) \in \bar{B}$, the corresponding boundary value problem

$$(1.6)\qquad y_i'' = f_i(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(m-2)}(t)[y_i]_i, \varphi^{(m-1)}(t)[y_i']_i), \qquad i=1, 2, \ldots, n,$$

$$(1.7)\qquad y(0) = A^{m-2}, \qquad y(1) = B$$

has a unique solution $y(t)$ satisfying the inequalities

$$(1.8)\qquad \underline{\omega}^{(m-2)}(t) \leqslant y(t) \leqslant \bar{\omega}^{(m-2)}(t), \qquad 0 \leqslant t \leqslant 1.$$

$$(1.9)\qquad \| y'(t) \|_0 \leqslant N.$$

Indeed (1.6) consists of n equations independent of one another and $|f_i(t, \varphi, \varphi', \ldots, \varphi^{(m-2)}[y_i]_i, \varphi^{(m-1)}[y_i']_i)| \leqslant h_i(|\varphi^{(m-1)}[y_i']_i|) \leqslant h_i(\vec{N}[|y_i'|]_i)$. So that from Theorem 1 in [6] or Theorem 1.3 in [2] we can immediately conclude that the boundary value problem (1.6), (1.7) has a solution $y(t)$ satisfying (1.8) and (1.9). In addition $y(t)$ is a unique solution of (1.6), (1.7) by the condition (ii).

(2) We show that the boundary value problem (1.1), (1.2) has a solution $x(t)$ satisfying the inequality (1.5) by means of the Schauder fixed point theorem. From (1)

for each $\varphi(t)\in\bar{B}$ (1.6), (1.7) has a unique solution $y(t)$, which satisfies (1.8) and (1.9). Let

$$x^{(m-2)}(t)\equiv y(t)\,,\qquad 0\leqslant t\leqslant 1\,,$$

$$x^{(j)}(t)\equiv A^j+\int_0^t x^{(j+1)}(\xi)d\xi\,,\quad 0\leqslant t\leqslant 1,\quad j=m-3,m-4,\ldots,1,0\,.$$

Then $x(t)$ is a solution of the corresponding boundary value problem

$$x_i^{(m)}=f_i(t,\varphi(t),\varphi'(t),\ldots,\varphi^{(m-2)}(t)[x_i^{(m-2)}]_i,\varphi^{(m-1)}(t)[x_i^{(m-1)}]_i)\,,\qquad i=1,2,\ldots,n\,, \tag{1.10}$$

$$x^{(j)}(0)=A^j\,,\quad j=0,1,\ldots,m-2\,,\quad x^{(m-2)}(1)=B \tag{1.11}$$

and $x(t)$ satisfies the inequality (1.5). Furthermore $x(t)$ is a unique solution of (1.10), (1.11). Indeed, if $\hat{x}(t)$ is also a solution of (1.10), (1.11). Let $z(t)=x(t)-\hat{x}(t)$ and $\hat{y}(t)=\hat{x}^{(m-2)}(t)$. Then $z^{(j)}(0)=0$, $j=0,1,\ldots,m-2$, and $\hat{y}(t)$ is also a solution of (1.6), (1.7). From (1) $\hat{y}(t)\equiv y(t)$, $0\leqslant t\leqslant 1$, i.e., $z^{(m-2)}(t)\equiv 0$, $0\leqslant t\leqslant 1$. Hence $z^{(j)}(t)\equiv 0$, $0\leqslant t\leqslant 1$, $j=m-3,m-4,\ldots,1,0$. Particularly $z(t)\equiv 0$, $0\leqslant t\leqslant 1$, i.e., $x(t)$ is a unique solution of (1.10), (1.11).

Define a mapping $T\colon \bar{B}\to\bar{B}$ by

$$T\colon\ \varphi(t)\to x(t)\,,$$

where $\varphi(t)\in\bar{B}$, $x(t)$ is the unique solution of the corresponding boundary value problem (1.10), (1.11). Obviously $\bar{B}$ is a bounded closed convex subset of the Banach space $C^{m-1}([0,1],R^n)$ with norm $\|\cdot\|_{m-1}$. In addition, for $\varphi(t)\in\bar{B}$ the solution $y(t)$ of the corresponding boundary value problem (1.6), (1.7) satisfies the integral equation

$$y_i(t)=a_i^{m-2}+(b_i-a_i^{m-2})t+\int_0^1 G(t,s)f_i(s,\varphi(s),\varphi'(s),\ldots,\varphi^{(m-2)}(s)[y_i(s)]_i,\varphi^{(m-1)}(s)[y_i'(s)]_i)ds\,,$$

where $i=1,2,\ldots,n$, $A^{m-2}=(a_1^{m-2},a_2^{m-2},\ldots,a_n^{m-2})$, $B=(b_1,b_2,\ldots,b_n)$,

$$G(t,s)=\begin{cases}(1-t)s\,, & 0\leqslant s\leqslant t\leqslant 1\\ (1-s)t\,, & 0\leqslant t\leqslant s\leqslant 1\,.\end{cases}$$

Hence it is not difficult to show that T is a continuous mapping.

Assume the $\{x^k(t)\}\subset T(\bar{B})$. Then $\|(x^k(t))^{(j)}\|_0\leqslant r_j$, $j=0,1,\ldots,m-2$, $\|(x^k(t))^{(m-1)}\|_0\leqslant N$. Let $\Omega=\{(t,x,x',\ldots,x^{(m-1)})\colon 0\leqslant t\leqslant 1,\ \|x^{(j)}\|\leqslant r_j,\ j=0,1,\ldots,m-2,\ \|x^{(m-1)}\|\leq N\}$ and $M=\max_{1\leqslant i\leqslant n}\{\max_\Omega|f_i(t,x,x',\ldots,x^{(m-1)})|\}$. Then $\|(x^k(t))^{(m)}\|\leqslant M$. Consequently the $\{(x^k(t))^{(j)}\}$, $j=0,1,\ldots,m-1$, are all equicontinuous and uniformly bounded sequences on $[0,1]$. Hence by the Ascoli-Arzela theorem there exist subsequences $\{(x^{k_l}(t))^{(j)}\}$ uniformly convergent on $[0,1]$, $j=0,1,\ldots,m-1$, such that

$$\lim_{l\to\infty}(x^{k_l}(t))^{(j)}=(x^*(t))^{(j)}\,, \qquad j=0,1,\dots,m-1\,.$$

Furthermore, $x^*(t)\in\bar{B}$. This shows that T is a completely continuous mapping. Thus from the Schauder fixed point theorem T has a fixed point $x^*(t)$ in $\bar{B}$. $x^*(t)$ is just a solution of (1.1), (1.2) and $x^*(t)$ satisfies (1.5). The proof is completed.

THEOREM 2. *Assume that* $f(t,x,x',\dots,x^{(m-1)})\in C([0,1]\times R^{mn},R^n)$, f_i *satisfies the Lipschitz condition with respect to* $x^{(m-1)}$, $i=1,2,\dots,n$, *and the conditions* (ii), (iii) *in Theorem* 1 *hold. Then the boundary value problem* (1.1), (1.2) *has a solution* $x(t)$ *satisfying* (1.5).

PROOF. By assumption for any numbers $r_k>0$, $k=0,1,\dots,m-2$, let M_i be the supremum of $|f_i(t,x,x',\dots,x^{(m-2)},0)|$ on $0\leqslant t\leqslant 1$, $\|x^{(k)}\|\leqslant r_k$, $k=0,1,\dots,m-2$. Then

$$\begin{aligned}|f_i(t,x,x',\dots,x^{(m-1)})|&\le|f_i(t,x,x',\dots,x^{(m-1)})-f_i(t,x,x',\dots,x^{(m-2)},0)|\\&\quad+|f_i(t,x,x',\dots,x^{(m-2)},0)|\\&\leqslant L_i\sum_{j=1}^{n}|x_j^{(m-1)}|+M_i\,, \qquad i=1,2,\dots,n\end{aligned}$$

for $0\leqslant t\leqslant 1$, $\|x^{(k)}\|\leqslant r_k$, $k=0,1,\dots,m-2$, where L_i is the Lipschitz constant. We define $H(s)=(h_1(s),h_2(s),\dots,h_n(s))\in C([0,\infty)^n,(0,\infty)^n)$ by

$$h_i(s)=L_i\sum_{j=1}^{n}s_j+M_i\,, \qquad i=1,2,\dots,n\,.$$

Since

$$\begin{aligned}\int_{2r_{m-2}}^{N}\frac{s_i}{L_is_i+(n-1)L_iN+M_i}ds_i&=\frac{1}{L_i}\int_{2r_{m-2}}^{N}\left(1-\frac{(n-1)L_iN+M_i}{L_is_i+(n-1)L_iN+M_i}\right)ds_i\\&=\frac{1}{L_i}\left[N-2r_{m-2}-\left((n-1)N+\frac{M_i}{L_i}\right)\log\frac{nL_iN+M_i}{(n-1)L_iN+M_i+2r_{m-2}L_i}\right],\end{aligned}$$

let us consider the function $f(\theta)\in C([2r_{m-2},\infty),R)$:

$$f(\theta)=\theta-2r_{m-2}-\left[(n-1)\theta+\frac{M_i}{L_i}\right]\log\frac{nL_i\theta+M_i}{(n-1)L_i\theta+M_i+2r_{m-2}L_i}\,.$$

It is easy to see that

$$\lim_{\theta\to\infty}\frac{f(\theta)}{\theta}=1-(n-1)\log\frac{n}{n-1}=:k>0\,,$$

$$\lim_{\theta\to\infty}(f(\theta)-k\theta)=-2r_{m-2}-\frac{M_i}{L_i}\log\frac{n}{n-1}+\frac{2nL_ir_{m-2}+M_i}{nL_i}=:b\,.$$

Hence $w=k\theta+b$ is the asymptotic line of $f(\theta)$ as $\theta\to\infty$. Hence $f(\theta)\to\infty$ $(\theta\to\infty)$. Consequently there exists a real number $N_0>0$ such that $f(\theta)>2L_ir_{m-2}$ for $\theta>N_0$. This shows that

$$\int_{2r_{m-2}}^{N}\frac{s_i}{L_is_i+(n-1)L_iN+M_i}ds_i>2r_{m-2}$$

for any $N>N_0$.

From this we conclude that $f(t, x, x', \ldots, x^{(m-1)})$ satisfies the Nagumo condition. Hence by Theorem 1 we know that Theorm 2 holds.

For an $n\times n$ matrix $A=(a_{ij})$, we let $\| A \| :=\max_{1\leqslant i,j\leqslant n}\{|a_{ij}|\}$. Similarly we can prove the following:

Corollary 3. *Assume that* $f(t, x, x', \ldots, x^{(m-1)})\in C^1([0, 1]\times R^{mn}, R^n)$, $\partial f/\partial x^{(m-1)}$ *is bounded and the conditions* (ii), (iii) *in Theorem* 1 *hold. Then the boundary value problem* (1.1), (1.2) *has a solution* $x(t)$ *satisfying* (1.5).

Corollary 4. *Assume that* $f(t, x, x', \ldots, x^{(m-1)})=A(t)x^{(m-1)}+g(t, x, x', \ldots, x^{(m-2)})$, $A(t)$ *is a continuous* $n\times n$ *matrix on* $[0, 1]$, $g\in C([0, 1]\times R^{(m-1)n}, R^n)$ *and the conditions* (ii), (iii) *in Theorem* 1 *hold. Then the boundary value problem* (1.1), (1.2) *has a solution* $x(t)$ *satisfying* (1.5).

Theorem 5. *Assume that* $f(t, x, x', \ldots, x^{(m-1)})\in C^1([0, 1]\times R^{mn}, R^n)$, $\partial f/\partial x^{(m-1)}$ *is bounded,* $\partial f_i/\partial x_i^{(m-2)}\geqslant l_i>0$ *and* $f_i(t, x, x', \ldots, x^{(m-2)}[0]_i, x^{(m-1)}[0]_i)$ *is bounded,* $i=1, 2, \ldots, n$. *Then the boundary value problem* (1.1), (1.2) *has a solution.*

Proof. Since $\partial f/\partial x^{(m-1)}$ is bounded, the function f satisfies the Nagumo condition. Hence we need only to show that the boundary value problem (1.1), (1.2) has upper and lower solutions. Assume that $A^j=(a_1^j, a_2^j, \ldots, a_n^j)$, $j=0, 1, \ldots, m-2$, $B=(b_1, b_2, \ldots, b_n)$, $\partial f_i/\partial x_i^{(m-1)}\geqslant m_i$,

$$|f_i(t, x, x', \ldots, x^{(m-2)}[0]_i, x^{(m-1)}[0]_i)|\leqslant M_i\,, \qquad i=1, 2, \ldots, n\,,$$

and let

$$\bar{\omega}_i^{(m-2)}(t)=(|a_i^{m-2}|+|b_i|)e^{\lambda_i t}+\frac{M_i}{l_i}\,, \qquad 0\leqslant t\leqslant 1\,,$$

$$\underline{\omega}_i^{(m-2)}(t)=-\bar{\omega}_i^{(m-2)}(t)\,, \qquad 0\leqslant t\leqslant 1\,,$$

$$\bar{\omega}_i^{(j)}(t)=\sum_{k=0}^{m-3-j}|a_i^{j+k}|\frac{t^k}{k!}+\frac{1}{(m-3-j)!}\int_0^t(t-\xi)^{m-3-j}\bar{\omega}_i^{(m-2)}(\xi)\,d\xi\,, \qquad 0\leqslant t\leqslant 1\,,$$

$$\underline{\omega}_i^{(j)}(t)=-\bar{\omega}_i^{(j)}(t)\,, \qquad 0\leqslant t\leqslant 1\,,$$

where $\lambda_i=(m_i+\sqrt{m_i^2+4l_i})/2$, $i=1, 2, \ldots, n$, $j=m-3, m-4, \ldots, 1, 0$. Then it is not difficult to verify that

$$\begin{aligned}
&\underline{\omega}^{(j)}(t)\leqslant\bar{\omega}^{(j)}(t)\,, \qquad 0\leqslant t\leqslant 1\,,\quad j=0,1,\dots,m-2\,,\\
&\underline{\omega}^{(j)}(0)\leqslant A^j\leqslant\bar{\omega}^{(j)}(0)\,, \qquad\qquad j=0,1,\dots,m-2\,,\\
&\underline{\omega}^{(m-2)}(1)\leqslant B\leqslant\bar{\omega}^{(m-2)}(1)\,.
\end{aligned}$$

Furthermore for any $\varphi(t)\in\bar{B}$,

$$\begin{aligned}
&f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[\bar{\omega}_i^{(m-2)}]_i\,,\varphi^{(m-1)}[\bar{\omega}_i^{(m-1)}]_i)-\bar{\omega}_i^{(m)}\\
&\quad=f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[\bar{\omega}_i^{(m-2)}]_i,\varphi^{(m-1)}[\bar{\omega}_i^{(m-1)}]_i)\\
&\qquad-f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[\bar{\omega}_i^{(m-2)}]_i,\varphi^{(m-1)}[0]_i)\\
&\qquad+f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[\bar{\omega}_i^{(m-2)}]_i,\varphi^{(m-1)}[0]_i)\\
&\qquad-f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[0]_i,\varphi^{(m-1)}[0]_i)\\
&\qquad+f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[0]_i,\varphi^{(m-1)}[0]_i)-\bar{\omega}_i^{(m)}\\
&\quad=\int_0^1\frac{\partial}{\partial x_i^{(m-1)}}f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[\bar{\omega}_i^{(m-2)}]_i,\varphi^{(m-1)}[\theta\bar{\omega}_i^{(m-1)}]_i)\,d\theta\cdot\bar{\omega}_i^{(m-1)}\\
&\qquad+\int_0^1\frac{\partial}{\partial x_i^{(m-2)}}f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[\theta\bar{\omega}_i^{(m-2)}]_i,\varphi^{(m-1)}[0]_i)\,d\theta\cdot\bar{\omega}_i^{(m-2)}\\
&\qquad+f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[0]_i,\varphi^{(m-1)}[0]_i)-\bar{\omega}_i^{(m)}\\
&\quad\geqslant m_i\bar{\omega}_i^{(m-1)}+l_i\bar{\omega}_i^{(m-2)}-M_i-\bar{\omega}_i^{(m)}=0\,,\qquad 0\leqslant t\leqslant 1\,,\quad i=1,2,\dots,n\,.
\end{aligned}$$

Analogously we have

$$f_i(t,\varphi,\varphi',\dots,\varphi^{(m-2)}[\underline{\omega}_i^{(m-2)}]_i,\varphi^{(m-1)}[\underline{\omega}_i^{(m-1)}]_i)-\underline{\omega}_i^{(m)}\leqslant 0\,,$$
$$0\leqslant t\leqslant 1\,,\quad i=1,2,\dots,n\,.$$

This shows that $\bar{\omega}(t)$ and $\underline{\omega}(t)$ are upper and lower solutions of (1.1), (1.2), respectively. Thus from Theorem 1, (1.1), (1.2) has a solution.

2. Examples. As applications of our main results obtained in Section 1 we now exhibit three examples.

EXAMPLE 1. Consider the following boundary value problem:

$$\begin{cases} x'''=x''+\exp[-(y'')^2]+f_1(t,x,y,x',y') \\ y'''=\sin x''\cdot\cos y''+f_2(t,x,y,x',y')\,, \end{cases} \tag{2.1}$$

$$x^{(j)}(0)=y^{(j)}(0)=0\,,\quad j=0,1\,,\quad x'(1)=y'(1)=0\,, \tag{2.2}$$

where $x,y\in R$, $f_1,f_2\in C^1([0,1]\times R^4,R)$, $\partial f_1/\partial x'\geqslant l_1>0$, $\partial f_2/\partial y'\geqslant l_2>0$, $f_1(t,x,y,0,y')$ and $f_2(t,x,y,x',0)$ are all bounded.

Let

$$F_1 = x'' + \exp[-(y'')^2] + f_1(t, x, y, x', y'),$$
$$F_2 = \sin x'' \cdot \cos y'' + f_2(t, x, y, x', y').$$

Since

$$\frac{\partial F_1}{\partial x''} = 1, \quad \left|\frac{\partial F_1}{\partial y''}\right| = |-2y'' \exp[-(y'')^2]| < 1,$$
$$\left|\frac{\partial F_2}{\partial x''}\right| = |\cos x'' \cdot \cos y''| \leqslant 1, \quad \left|\frac{\partial F_2}{\partial y''}\right| = |-\sin x'' \cdot \sin y''| \leqslant 1,$$

from Theorem 5 the boundary value problem (2.1), (2.2) has a solution.

EXAMPLE 2. Consider the following boundary value problem:

$$(2.3) \qquad \begin{cases} x''' = [2 + \sin(x+y)]x'' + \dfrac{y''}{1+(y'')^2} + x' + \arctan y' + e^t \\ y''' = |x''|^{1/2} \cdot y'' + y' \cosh x' + e^{-(x^2+y^2)} + \log(1+t), \end{cases}$$

$$(2.4) \qquad x^{(j)}(0) = y^{(j)}(0) = 0, \qquad j = 0, 1, \quad x'(1) = y'(1) = 0.$$

First we shall prove that functions on the right hand side of (2.3) satisfy the Nagumo condition. For any real numbers $r_0, r_1 > 0$,

$$\left|[2 + \sin(x+y)]x'' + \frac{y''}{1+(y'')^2} + x' + \arctan y' + e^t\right|$$
$$\leqslant 3|x''| + |y''| + M_1 =: h_1(|x''|, |y''|),$$

where M_1 is the maximum of $|x' + \arctan y' + e^t|$ on $0 \leqslant t \leqslant 1$, $|x'|, |y'| \leqslant r_1$. Similarly to the proof of Theorem 2, we know that there exists a real number $N_1 > 0$ such that

$$\int_{2r_1}^{N} [s_1/h_1(s_1, N)]\, ds_1 > 2r_1$$

for any $N > N_1$. On the other hand,

$$\big||x''|^{1/2} \cdot y'' + y' \cosh x' + e^{-(x^2+y^2)} + \log(1+t)\big| \leqslant |x''|^{1/2} \cdot |y''| + M_2 =: h_2(|x''|, |y''|),$$

where M_2 is the maximum of $|y' \cosh x' + e^{-(x^2+y^2)} + \log(1+t)|$ on $0 \leqslant t \leqslant 1$, $|x|, |y| \leqslant r_0$, $|x'|, |y'| \leqslant r_1$. It is not difficult to show that there exists a real number $N_2 > 0$ such that

$$\int_{2r_1}^{N} [s_2/h_2(N, s_2)]\, ds_2 > 2r_1$$

for any $N > N_2$, because of

$$\int_{2r_1}^{N} [s_2/h_2(N, s_2)]\,ds_2 = \int_{2r_1}^{N} \frac{s_2}{\sqrt{N}\,s_2 + M_2}\,ds_2$$

$$= \sqrt{N} - \frac{2r_1}{\sqrt{N}} - \frac{M_2}{N}\log\frac{N^{3/2}+M_2}{2r_1\sqrt{N}+M_2} \to \infty\ (N\to\infty)\,.$$

Let $N_0 = \max\{N_1, N_2\}$. Then we see that the functions on the right hand side of (2.3) satisfy the Nagumo condition.

Second we show that the boundary value problem (2.3), (2.4) has upper and lower solutions. Let

$$\bar{\omega}_1(t) = \frac{2}{1+\sqrt{5}}\exp\left(\frac{1+\sqrt{5}}{2}t\right) + \left(\frac{\pi}{2} - \frac{1}{2}\right)t\,, \qquad 0 \leqslant t \leqslant 1\,,$$

$$\underline{\omega}_1(t) = -\frac{2}{1+\sqrt{5}}\exp\left(\frac{1+\sqrt{5}}{2}t\right) - \left(\frac{1}{2} + \frac{\pi}{2} + e\right)t\,, \qquad 0 \leqslant t \leqslant 1\,,$$

$$\bar{\omega}_2(t) = e^t\,, \qquad 0 \leqslant t \leqslant 1\,,$$

$$\underline{\omega}_2(t) = -e^t - (1+\log 2)t\,, \qquad 0 \leqslant t \leqslant 1\,.$$

Then

$$\underline{\omega}_1^{(j)}(t) < 0 < \bar{\omega}_1^{(j)}(t)\,, \qquad 0 \leqslant t \leqslant 1\,, \quad j = 0, 1\,,$$

$$\underline{\omega}_2^{(j)}(t) < 0 < \bar{\omega}_2^{(j)}(t)\,, \qquad 0 \leqslant t \leqslant 1\,, \quad j = 0, 1\,.$$

For any $g(t) = (g_1(t), g_2(t)) \in C^3([0, 1], R^2)$, $\underline{\omega}_i^{(j)}(t) \leqslant g_i^{(j)}(t) \leqslant \bar{\omega}_i^{(j)}(t)$, $0 \leqslant t \leqslant 1$, $i = 1, 2$, $j = 0, 1$, we have

$$[2+\sin(\bar{\omega}_1 + g_2)]\bar{\omega}_1'' + \frac{g_2''}{1+(g_2'')^2} + \bar{\omega}_1' + \arctan g_2' + e^t - \bar{\omega}_1'''$$

$$\geqslant \bar{\omega}_1'' + \bar{\omega}_1' + \left(1 - \frac{1}{2} - \frac{\pi}{2}\right) - \bar{\omega}_1''' = 0\,, \qquad 0 \leqslant t \leqslant 1\,,$$

$$|g_1''|^{1/2}\cdot\bar{\omega}_2'' + \bar{\omega}_2'\cosh g_1' + e^{-(g_1^2+\bar{\omega}_2^2)} + \log(1+t) - \bar{\omega}_2''' \geqslant \bar{\omega}_2' - \bar{\omega}_2''' = 0\,, \qquad 0 \leqslant t \leqslant 1\,.$$

Similarly we have

$$[2+\sin(\underline{\omega}_1 + g_2)]\underline{\omega}_1'' + \frac{g_2''}{1+(g_2'')^2} + \underline{\omega}_1' + \arctan g_2' + e^t - \underline{\omega}_1''' \leqslant 0\,, \qquad 0 \leqslant t \leqslant 1\,,$$

$$|g_1''|^{1/2}\cdot\underline{\omega}_2'' + \underline{\omega}_2'\cosh g_1' + e^{-(g_1^2+\underline{\omega}_2^2)} + \log(1+t) - \underline{\omega}_2''' \leqslant 0\,, \qquad 0 \leqslant t \leqslant 1\,.$$

This shows that $\bar{\omega}(t) = (\bar{\omega}_1(t), \bar{\omega}_2(t))$ and $\underline{\omega}(t) = (\underline{\omega}_1(t), \underline{\omega}_2(t))$ are upper and lower solutions of (2.3), (2.4), respectively. Consequently from Theorem 1 we know that (2.3), (2.4) has a solution.

EXAMPLE 3. Consider the boundary value problem:

$$\text{(2.5)}\qquad \begin{cases} x'''=(1+x^2+y^2)x''+x'\log[1+(y'')^2]+x'+f_1(t) \\ y'''=\exp[-(x'')^2]+[(x')^4+1]y''+(1+x^2)y'+\sin x'+f_2(t)\,, \end{cases}$$

$$\text{(2.6)}\qquad x^{(j)}(0)=y^{(j)}(0)=0\,,\qquad j=0,1\,,\quad x'(1)=y'(1)=0\,,$$

where $x, y\in R$, $f_1(t), f_2(t)\in C([0, 1], R)$.

First of all for any real numbers $r_0, r_1>0$, we have

$$|(1+x^2+y^2)x''+x'\log[1+(y'')^2]+x'+f_1(t)|\leqslant(1+2r_0^2)|x''|+r_1|y''|+M_1\,,$$

$$|\exp[-(x'')^2]+[(x')^4+1]y''+(1+x^2)y'+\sin x'+f_2(t)|\leqslant 1+(r_1^4+1)|y''|+M_2\,,$$

for $0\leqslant t\leqslant 1$, $|x|, |y|\leqslant r_0$, $|x'|, |y'|\leqslant r_1$, where M_1 is the maximum of $|x'+f_1(t)|$ on $|x'|\leqslant r_1$, $0\leqslant t\leqslant 1$ and M_2 is the maximum of $|(1+x^2)y'+\sin x'+f_2(t)|$ on $|x|\leqslant r_0$, $|x'|, |y'|\leqslant r_1$, $0\leqslant t\leqslant 1$. Similarly to the proof of Theorem 2, we conclude that the functions on the right hand side of (2.5) satisfy the Nagumo condition. Similarly to the proof of Theorem 5, we conclude that (2.5), (2.6) has upper and lower solutions. Thus from Theorem 1 we conclude that (2.5), (2.6) has a solution.

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