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ON THE EXISTENCE OF CERTAIN GENERAL EXTREMAL METRICS II

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Abstract. The author was the first to show the important role played by quadratic differentials all of whose structure domains are ring domains manifesting in particular their relation to module problems for multiple curve families and the related problems for linear sums of modules of double-connected domains. The original method used variational techniques. In this paper a treatment is given using directly the method of the extremal metric.

1. The present author was the first to manifest, in the paper [3], the important role played by quadratic differentials on a finite Riemann surface all of whose structure domains are ring domains. The fundamental theorem in that paper relates the solutions of two types of extremal problems with such quadratic differentials and was proved using a variational method. It is now possible to give a simple direct proof using only the concepts and techniques of the method of the extremal metric.

We begin by recalling some terminology which is primarily that found in [3] [4]. We confine our attention to finite Riemann surfaces, understood in the sense of [4]. (In present day terminology the word "oriented" is superfluous.) Such a surface is determined topologically by its connectivity n (number of border components) and genus g. By a free family of homotopy classes $\mathscr{H}_j$, $j=1, \dots, L$, on such a surface $\mathscr{R}$ we mean a family of distinct free homotopy classes which can be represented by disjoint Jordan curves C_j, $j=1, \dots, L$. It should be emphasized that this is unsensed homotopy, i.e., a closed path and its inverse are to be equivalent. A proper doubly-connected domain D can be mapped conformally on a circular ring $r_1<|w|<r_2$ $(0<r_1<r_2<\infty)$. The curves in D corresponding to the concentric circles are called its level curves. The positive quadratic differential induced on D by $-dw^2/(4\pi^2 w^2)$ (which corresponds to the extremal metric determining the module $(1/2\pi)\log(r_2/r_1)$ of D) will be called the basic quadratic differential of D. A doubly-connected domain on $\mathscr{R}$ will be said to be associated with the homotopy class $\mathscr{H}_j$ if its level curves are in $\mathscr{H}_j$. A family D_j of disjoint doubly-connected domains associated with $\mathscr{H}_j$, $j=1, \dots, L$, is called an admissible family of domains. We allow certain domains to be missing and speak of them as degenerate. The structure domains of a positive quadratic differential on a finite Riemann surface are the domains obtained in the Basic Structure Theorem [4] [5].

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We define two types of extremal problems for a finite Riemann surface $\mathscr{R}$ and a free family of homotopy classes $\mathscr{H}_j$, $j=1,\dots,L$, on $\mathscr{R}$.

Problem $P(a_1,\dots,a_L)$. Let a_j, $j=1,\dots,L$, be non-negative numbers not all zero. Let $\rho(a_1,\dots,a_L)$ denote the class of conformally invariant metrics $\rho(z)|dz|$ on $\mathscr{R}$ with ρ measurable, non-negative and such that for γ_j rectifiable in $\mathscr{H}_j$, $\int_{\gamma_j}\rho|dz|$ exists and satisfies

$$\int_{\gamma_j}\rho|dz|\geq a_j, \qquad j=1,\dots,L.$$

Find the greatest lower bound $M(a_1,\dots,a_L)$ of

$$\iint_{\mathscr{R}}\rho^2 dA_z$$

for $\rho|dz|\in\rho(a_1,\dots,a_L)$.

Problem $\mathscr{P}(a_1,\dots,a_L)$. Let a_j, $j=1,\dots,L$, be non-negative numbers not all zero. For an admissible family of domains D_j of module M_j, $j=1,\dots,L$, find the least upper bound of $\sum_{j=1}^{L}a_j^2M_j$.

2. We will now state our fundamental theorem in the case of a finite Riemann surface. Many extensions are possible, for example, allowing punctures in the surface, homotopy classes of paths running from one border component to another or circle domains with assigned centres in $\mathscr{R}$ (using reduced modules rather than modules). All of these are readily reduced to the present case or proved in a completely analogous manner as indicated in [3]. Indeed it is possible to reduce the proof to the case of a closed surface but this proves not to be advantageous.

In what follows we will tacitly exclude the cases where $\mathscr{R}$ is a disc, a (proper) doubly-connected domain or a closed surface of genus 1. In the first case the theorem is vacuously true. In the second the theorem is trival. In the third everything is true apart from some uniqueness statements which occurs because the torus is exceptional for the Basic Structure Theorem. In this case the result of the theorem is long and well known.

THEOREM F. *Let $\mathscr{R}$ be a finite Riemann surface, $\mathscr{H}_j$, $j=1,\dots,L$, a free family of homotopy classes on $\mathscr{R}$. Then the solution of problem $P(a_1,\dots,a_L)$ is given by the (essentially) unique extremal metric $|Q(z)|^{1/2}|dz|$ where $Q(z)dz^2$ is a regular positive quadratic differential on $\mathscr{R}$, all of whose structure domains are ring domains. Enumerating these appropriately as $D_j(a_1,\dots,a_L)$, $j=1,\dots,L$, they form an admissible family for the $\mathscr{H}_j$. If $D_j(a_1,\dots,a_L)$ is not degenerate its level curves all have length a_j in the metric $|Q(z)|^{1/2}|dz|$. If it is degenerate there is a geodesic in this metric belonging to $\mathscr{H}_j$ composed of trajectory arcs joining zeros of $Q(z)dz^2$ plus end points and of length $\geq a_j$ in this metric.*

If $D_j(a_1, \ldots, a_L)$ *has module* $M_j(a_1, \ldots, a_L)$

$$M(a_1, \ldots, a_L) = \sum_{j=1}^{L} a_j^2 M_j(a_1, \ldots, a_L)\,.$$

The solution of problem $\mathscr{P}(a_1, \ldots, a_L)$ *is given by the domains* $D_j(a_1, \ldots, a_L)$, $j=1, \ldots, L$, *the least upper bound* $M(a_1, \ldots, a_L)$ *being a maximum attained uniquely for these domains.*

It is convenient to separate out the uniqueness aspects of this theorem.

THEOREM U. *Let* $\mathscr{R}$ *be a finite Riemann surface,* $\mathscr{H}_j$, $j=1, \ldots, L$, *a free family of homotopy classes on* $\mathscr{R}$. Let $Q(z)dz^2$ *be a regular positive quadratic differential on* $\mathscr{R}$ *all of whose structure domains are ring domains such that suitably enumerated and allowing for degenerate domains they form an admissible family* D_j^*, $j=1, \ldots, L$, *for the* $\mathscr{H}_j$. *Let the module of* D_j^* *be* M_j^*. *Suppose that for a non-degenerate domain* D_j^* *all trajectories in* D_j^* *have length* a_j *while for a degenerate* D_j^* *there is a geodesic in the Q-metric belonging to* $\mathscr{H}_j$ *composed of trajectories of* $Q(z)dz^2$ *joining zeros plus their end points of length* a_j^*. *Then for* a_j, *non-negative numbers with* $a_j \le a_j^*$ *in the case of degenerate domains* $|Q(z)|^{1/2}|dz|$ *provides the (essentially) unique solution of the problem* $P(a_1, \ldots, a_L)$ *while the domains* D_j^*, $j=1, \ldots, L$, *provide the unique solution of the problem* $\mathscr{P}(a_1, \ldots, a_L)$.

This result is proved in [3] without it being explicitly formulated. However the proof there depends slightly on the viewpoint of that paper so we will give a brief sketch of the proof. It uses only standard techniques of the method of the extremal metric.

The metric $|Q(z)|^{1/2}|dz|$ is seen to be in $\rho(a_1, \ldots, a_L)$ by [5; Lemma 4.6] and

$$\iint_{\mathscr{R}} |Q(z)| dA_z = \sum_{j=1}^{L} a_j^2 M_j^*\,.$$

If $\rho(z)|dz| \in \rho(a_1, \ldots, a_L)$,

$$\iint_{D_j^*} \rho^2(z) dA_z \ge a_j^2 M_j^*\,.$$

Thus $|Q(z)|^{1/2}|dz|$ is an extremal metric and by a standard result essentially unique.

Now let D_j be an admissible family of domains with modules M_j for the $\mathscr{H}_j$, $j=1, \ldots, L$. As above

$$\iint_{D_j} |Q(z)| dA_z \ge a_j^2 M_j\,.$$

Equality can occur only if each level curve λ of D_j (if non-degenerate) has length a_j in the Q-metric. Thus λ must be a geodesic. In any case there will be a geodesic γ composed of trajectories in the Q-metric of length $\ge a_j$. If λ and γ meet they must coincide by uniqueness of geodesics. Otherwise there is a doubly-connected domain Δ on $\mathscr{R}$ bounded by λ and γ. On λ, $e^{i\varphi}Q(z)dz^2 \geq 0$ for real φ; on γ, $Q(z)dz^2 \geq 0$. If $q(z)dz^2$ is the basic quadratic differential for Δ, the quotient of $Q(z)dz^2$ by $q(z)dz^2$ (appropriately understood) is regular on Δ, non-negative and bounded on γ, of constant argument and bounded on λ. Thus it must be a positive constant, λ is a trajectory of $Q(z)dz^2$ and each non-degenerate D_j lies in a structure domain of $Q(z)dz^2$. For equality above the D_j must coincide with the structure domains D_j^*, $j=1, \ldots, L$.

The existence part of Theorem F is proved by induction, first for schlichtartig surfaces ($g=0$) by induction on the connectivity. Theorem F holds for a triply-connected domain ($n=3$, $g=0$). This can be proved in a variety of ways. All the material is in [1] although it is not formulated in terms of quadratic differentials. It can also be proved as in [4; Lemma 3.7] although an auxiliary consideration showing that all values of a_1, a_2, a_3 are possible would be required. Probably the simplest and most direct method is to proceed as in [2; §6]. This proof can be made more elementary by using the continuity method rather than quasiconformal mappings and Teichmüller space.

We can always assume that the free family of homotopy classes is maximal by taking some of the a_j to be zero. (This maximal number is readily seen to be $2n-3$ when $g=0$.) Then suppose that $n>3$ and Theorem F has been proved for connectivity less than n. Among the classes $\mathscr{H}_j$ there will be one such that when it is represented by a Jordan curve and $\mathscr{R}$ is severed along this curve each component has connectivity ≥ 3. We will choose the notation so that this is $\mathscr{H}_1$. Consider the problem $\mathscr{P}(a_1, \ldots, a_L)$. By a normal families argument there is always an admissible family of domains D_j of modules M_j, $j=1, \ldots, L$, for which $\sum_{j=1}^{L} a_j^2 M_j$ attains its maximal value. Suppose that $a_1>0$ and $M_1>0$. We sever $\mathscr{R}$ along a level curve c of D_1 to obtain surfaces $\mathscr{R}^{(1)}$, $\mathscr{R}^{(2)}$ each of connectivity $<n$. The given free family of homotopy classes induces such a family on each of them, say $\mathscr{H}_j^{(1)}$, $j=1, \ldots, L^{(1)}$, and $\mathscr{H}_j^{(2)}$, $j=1, \ldots, L^{(2)}$ $(L^{(1)}+L^{(2)}=L+1)$. We take $\mathscr{H}_1^{(1)}$, $\mathscr{H}_1^{(2)}$ to be the classes induced by $\mathscr{H}_1$. The original admissible family of domains D_j gives admissible families for these, $D_j^{(1)}$, $j=1, \ldots, L^{(1)}$, $D_j^{(2)}$, $j=1, \ldots, L^{(2)}$, of modules $M_j^{(1)}$, $j=1, \ldots, L^{(1)}$, $M_j^{(2)}$, $j=1, \ldots, L^{(2)}$. Here $D_1^{(1)}$ and $D_1^{(2)}$ are the domains into which D_1 is severed by c so that $M_1^{(1)}+M_1^{(2)}=M_1$. We renumber the original a_j appropriately to be $a_j^{(1)}$, $j=1, \ldots, L^{(1)}$, $a_j^{(2)}$, $j=1, \ldots, L^{(2)}$, $(a_1^{(1)}=a_1^{(2)}=a_1)$. By induction Theorem F is valid for $\mathscr{R}^{(1)}$ and $\mathscr{R}^{(2)}$ thus there are positive quadratic differentials $Q^{(1)}(z)dz^2$, $Q^{(2)}(z)dz^2$ on these all of whose structure domains are ring domains $D_j^{(1)}(a_1^{(1)}, \ldots, a_{L^{(1)}}^{(1)})$ of modules $M_j^{(1)}(a_1^{(1)}, \ldots, a_{L^{(1)}}^{(1)})$, $j=1, \ldots, L^{(1)}$, and $D_j^{(2)}(a_1^{(2)}, \ldots, a_{L^{(2)}}^{(2)})$ of modules $M_j^{(2)}(a_1^{(2)}, \ldots, a_{L^{(2)}}^{(2)})$, $j=1, \ldots, L^{(2)}$. In particular the union of $D_1^{(1)}(a_1^{(1)}, \ldots, a_{L^{(1)}}^{(1)})$ and $D_1^{(2)}(a_1^{(2)}, \ldots, a_{L^{(2)}}^{(2)})$ is a doubly-connected domain $\hat{D}$ of module $\hat{M}$ with $\hat{M} \geq M_1^{(1)}(a_1^{(1)}, \ldots, a_{L^{(1)}}^{(1)})+M_1^{(2)}(a_1^{(2)}, \ldots, a_{L^{(2)}}^{(2)})$ (by Grötzsch's Lemma, with the proper understanding in case of degeneracy).

Now on the one hand

$$a_1^2\hat{M}+\sum_{j=2}^{L^{(1)}}(a_j^{(1)})^2 M_j^{(1)}(a_1^{(1)}, \ldots, a_{L^{(1)}}^{(1)})+\sum_{j=2}^{L^{(2)}}(a_j^{(2)})^2 M_j^{(2)}(a_1^{(2)}, \ldots, a_{L^{(2)}}^{(2)}) \leq \sum_{j=1}^{L} a_j^2 M_j .$$

On the other hand by Theorem F for $\mathscr{R}^{(1)}$, $\mathscr{R}^{(2)}$

$$a_1^2 M_1^{(1)} + a_1^2 M_1^{(2)} + \sum_{j=2}^{L} a_j^2 M_j \leq \sum_{j=1}^{L^{(1)}} (a_j^{(1)})^2 M_j^{(1)}(a_1^{(1)}, \ldots, a_{L^{(1)}}^{(1)})$$
$$+ \sum_{j=1}^{L^{(2)}} (a_j^{(2)})^2 M_j^{(2)}(a_1^{(2)}, \ldots, a_{L^{(2)}}^{(2)}) .$$

Thus the two sides must be equal. In particular $D_1^{(k)}$ and $D_1^{(k)}(a_1^{(k)}, \ldots, a_{L^{(k)}}^{(k)})$, $k=1, 2$ coincide. If $q^{(k)}(z)dz^2$ is the basic quadratic differential for $D_1^{(k)}$, $Q^{(k)}(z)dz^2$ induces the quadratic differential $a_1^2 q^{(k)}(z)dz^2$ on $D_1^{(k)}$, $k=1, 2$. If $q(z)dz^2$ is the basic quadratic differential for D_1 because of the above equality $a_1^2 q^{(k)}(z)dz^2$ coincides with $a_1^2 q(z)dz^2$ on $D_1^{(k)}$, $k=1$, 2, and thus the quadratic differentials $Q^{(1)}(z)dz^2$, $Q^{(2)}(z)dz^2$ are induced by a single positive quadratic differential on $\mathscr{R}$. It is verified at once to satisfy the conditions of Theorem F.

In particular the previous situation obtains for the problems $P(1, 0, \ldots, 0)$, $\mathscr{P}(1, 0, \ldots, 0)$. Let $\mathscr{Q}(z)dz^2$ be the quadratic differential assigned by Theorem F. In the metric $|\mathscr{Q}(z)|^{1/2}|dz|$ the lengths of elements in $\mathscr{H}_j$, $j=2, \ldots, L$, are bounded from zero. Let a_j, $j=2, \ldots, L$, be non-negative values not all zero. For X sufficiently large the quadratic differential $X^2\mathscr{Q}(z)dz^2$ will provide the solution of the problems $P(X, a_2, \ldots, a_L)$, $\mathscr{P}(X, a_2, \ldots, a_L)$ as in Theorem F. Let A be the greatest lower bound of (non-negative) numbers T such that for $X > T$ the domain for $\mathscr{H}_1$ in the problem $\mathscr{P}(X, a_2, \ldots, a_L)$ is non-degenerate.

A normal families argument shows that there will be a sequence of values $X_n \downarrow A$ and corresponding positive quadratic differentials $Q_n(z)dz^2$ for the problems $P(X_n, a_2, \ldots, a_L)$, $\mathscr{P}(X_n, a_2, \ldots, a_L)$ assigned by Theorem F converging uniformly on $\mathscr{R}$ to a positive quadratic differential $Q(z)dz^2$. Further the structure domains $D_{n,j}$ of module $M_{n,j}$ will converge to domains D_j of module M_j, $j=1, \ldots, L$, whose level sets (if D_j is non-degenerate) are trajectories of $Q(z)dz^2$. Moreover the elements of $\mathscr{H}_1$ all have length $\geq A$ and those of $\mathscr{H}_j$ all have length $\geq a_j$, $j=2, \ldots, L$, in the Q-metric.

Suppose that we had $A=0$. Let $\hat{D}_j$ of module $\hat{M}_j$ be an admissible family of domains for the family $\mathscr{H}_j$, $j=1, \ldots, L$, giving a maximal value in $\mathscr{P}(0, a_2, \ldots, a_L)$. On the one hand

$$\sum_{j=2}^{L} a_j^2 M_j \leq \sum_{j=2}^{L} a_j^2 \hat{M}_j ,$$

while

$$X_n^2 M_{n,1} + \sum_{j=2}^{L} a_j^2 M_{n,j} \geq \sum_{j=2}^{L} a_j^2 \hat{M}_j .$$

Passing to the limit

$$\sum_{j=2}^{L} a_j^2 M_j \geq \sum_{j=2}^{L} a_j^2 \hat{M}_j .$$

Thus D_1 must be degenerate and the D_j, $j=2, \ldots, L$, must be the structure domains for $Q(z)dz^2$. Moreover there is a positive lower bound for the lengths of the elements in $\mathscr{H}_1$ in the metric $|Q(z)|^{1/2}|dz|$ so that this would be the extremal metric for the problem $P(X, a_2, \ldots, a_L)$ for X sufficiently close to 0. Thus $A>0$.

There will be a sequence of values $Y_m \uparrow A$ such that for them there are admissible families $\hat{D}_{m,j}$ of modules $\hat{M}_{m,j}$, $j=1, \ldots, L$, maximizing for the problem $\mathscr{P}(Y_m, a_2, \ldots, a_L)$ with $\hat{D}_{m,1}$ degenerate and these can further be chosen to converge to an admissible family of domains $\hat{D}_j$ of modules $\hat{M}_j$, $j=1, \ldots, L$, with D_1 degenerate. On the one hand

$$Y_m^2 M_1 + \sum_{j=2}^{L} a_j^2 M_j \leq \sum_{j=2}^{L} a_j^2 \hat{M}_{m,j} ,$$

while

$$X_n^2 M_{n,1} + \sum_{j=2}^{L} a_j^2 M_{n,j} \geq \sum_{j=2}^{L} a_j^2 \hat{M}_{m,j} .$$

Passing to the limit $n, m \to \infty$ we have

$$A^2 M_1 + \sum_{j=2}^{L} a_j^2 M_j \leq \sum_{j=2}^{L} a_j^2 \hat{M}_j$$

and

$$A^2 M_1 + \sum_{j=2}^{L} a_j^2 M_j \geq \sum_{j=2}^{L} a_j^2 \hat{M}_j$$

so that we have equality. Thus the domains D_j if not degenerate must be the structure domains of $Q(z)dz^2$ and D_1 must be degenerate. There will be a geodesic in the Q-metric in $\mathscr{H}_1$ composed of trajectories plus end points and having length A. Thus $Q(z)dz^2$ provides the solution of problem $P(T, a_2, \ldots, a_L)$ for $T \leq A$ as in Theorem F and the domains D_j, $j=2, \ldots, L$, provide the solution of problem $\mathscr{P}(T, a_2, \ldots, a_L)$. This completes the proof of Theorem F for schlichtartig surfaces.

For surfaces of positive genus Theorem F is proved by induction on the genus assuming it is known for all surfaces of lower genus and arbitrary connectivity. Let $\mathscr{R}$ be a finite Riemann surface of genus g (>0) and connectivity n, $\mathscr{H}_j$, $j=1, \ldots, L$, a free family of homotopy classes on $\mathscr{R}$ which we may again assume to be a maximal family. There will be a class which we choose to be $\mathscr{H}_L$ such that when $\mathscr{R}$ is severed along a Jordan curve representing it we obtain a surface of genus one less. We consider the problem $\mathscr{P}(a_1, \ldots, a_L)$ and take an admissible family of domains D_j of modules M_j, $j=1, \ldots, L$, maximizing $\sum_{j=1}^{L} a_j^2 M_j$. We suppose that we have $a_L>0$, $M_L>0$ and sever $\mathscr{R}$ along a level curve c of D_L. In this way we obtain a finite Riemann surface $\tilde{\mathscr{R}}$

of genus $g-1$ and connectivity $n+2$. On it the family $\mathscr{H}_j$ determines a free family $\tilde{\mathscr{H}}_j, j=1, \ldots, L+1$, which $\tilde{\mathscr{H}}_j$ determined by $\mathscr{H}_j, j=1, \ldots, L-1$, and $\tilde{\mathscr{H}}_L, \tilde{\mathscr{H}}_{L+1}$ determined by $\mathscr{H}_L$. The original admissible family of domains is replaced by $\tilde{D}_j=D_j, j=1, \ldots, L-1$, and the two domains $\tilde{D}_L, \tilde{D}_{L+1}$ into which D_L is divided by c. We take the values $\tilde{a}_j, j=1, \ldots, L+1$ where $\tilde{a}_j=a_j, j=1, \ldots, L-1, \tilde{a}_L=\tilde{a}_{L+1}=a_L$. $\tilde{D}_L, \tilde{D}_{L+1}$ have modules $\tilde{M}_L, \tilde{M}_{L+1}$ with $\tilde{M}_L+\tilde{M}_{L+1}=M_L$. By induction Theorem F is valid for $\tilde{\mathscr{R}}$ and the remainder of the proof follows exactly on the lines of the preceding one.

3. Several authors have formulated other problems in this same context. As before $\mathscr{R}$ is a finite Riemann surface, $\mathscr{H}_j, j=1, \ldots, L$, a free family of homotopy classes on $\mathscr{R}$.

Strebel [8] enunciated the following rather artificial problem, designed purely to produce a positive quadratic differential all of whose structure domains are ring domains such that all such domains in an admissible family are non-degenerate. Strebel's method was taken directly from [3].

Problem $S(m_1, \ldots, m_L)$. Let $m_j, j=1, \ldots, L$, be positive numbers. For an admissible family of domains D_j of modules $M_j, j=1, \ldots, L$, find the least upper bound of $\min_{j=1, \ldots, L} M_j/m_j$.

Renelt [7] gave a somewhat more natural problem which produces the same sort of admissible family. It does not, however, have the intimate connection with quadratic differentials which makes possible the above proof of Theorem F.

Problem $R(b_1, \ldots, b_L)$. Let $b_j, j=1, \ldots, L$, be positive numbers. For an admissible family of domains D_j of modules $M_j, j=1, \ldots, L$, find the greatest lower bound of $\sum_{j=1}^{L} b_j^2 M_j^{-1}$.

Renelt used variation of the Dirichlet integral. He actually dealt only with plane domains.

Both of these problems are very easily solved by the use of Theorem F. For Strebel's problem this was done in [6]. (Unfortunately in the course of publication of that paper part of the text was transposed. Lines 7 through 16 on page 68 should be moved to between lines 15 and 16 on page 69.)

Renelt's problem can be treated just as easily. The expression for $\iint_{\mathscr{R}} |Q(z)|\, dA_z$ can be written in the form $\sum_{j=1}^{L} a_j \beta_j(a_1, \ldots, a_L)$ where

$$\beta_j(a_1, \ldots, a_L)=a_j M_j(a_1, \ldots, a_L)\,.$$

Some authors refer to these quantities as the "heights" of the domains $D_j(a_1, \ldots, a_L)$. Widths might seem to be a more appropriate term.

The following lemma is easily proved (see for example a proof in [9; p. 106]).

LEMMA. *Let $\mathscr{R}$ be a finite Riemann surface, $\mathscr{H}_j$, $j=1, \ldots, L$, a free family of homotopy classes on $\mathscr{R}$, $Q(z)dz^2$ a quadratic differential providing the solution of problem $\mathscr{P}(a_1, \ldots, a_L)$ with $D_j(a_1, \ldots, a_L)$ non-degenerate, $j=1, \ldots, L$. Let $b_j=\beta_j(a_1, \ldots, a_L)$,*

$j=1,\dots,L$. Then for an admissible family of domains D_j of modules M_j for $\mathscr{H}_j$, $j=1,\dots,L$.

$$\sum_{j=1}^{L} b_j^2 M_j^{-1} \geq \sum_{j=1}^{L} b_j^2 M_j^{-1}(a_1,\dots,a_L)$$

with equality only if the D_j coincide with $D_j(a_1,\dots,a_L)$, $j=1,\dots,L$.

Now consider the mapping from the (a_j) L-dimesnional Euclidean space to the (b_j) space by $(a_j)\to(\beta_j(a_1,\dots,a_L))$. Since for $a>0$

$$\beta_j(aa_1,\dots,aa_L)=a^2\beta_j(a_1,\dots,a_L)\,,\qquad j=1,\dots,L\,,$$

we normalize taking $\sum_{j=1}^{L} a_j=1$. (This is slightly nicer than the normalization in [6] since it gives immediately an $(L-1)$-simplex Σ.) As in [6] the mapping is continuous. We project the image of Σ radially back onto Σ and this carries each r-face of Σ into itself. Proceeding stepwise and using the lemma we see the latter mapping is (1, 1) on the open faces and thus the (oriented) boundary of each face is mapped onto itself with degree 1. Hence for every set of positive b_j we can find a_j and $\alpha>0$ so that

$$b_j=\alpha\beta_j(a_1,\dots,a_L)\,,\qquad j=1,\dots,L\,.$$

This completes the solution of problem $R(b_1,\dots,b_L)$. The derivations in the opposite direction are much more troublesome.

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