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CONFORMAL DEFORMATION TO PRESCRIBED SCALAR CURVATURE ON COMPLETE NONCOMPACT RIEMANNIAN MANIFOLDS WITH NONPOSITIVE CURVATURE

Dedicated to Professor Hideki Ozeki on his 60th birthday

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Abstract. We consider the problem of deforming the metric on a complete negatively curved manifold conformally to another complete metric whose scalar curvature is positive in an unbounded domain. We also consider the case of the Euclidean space.

1. Introduction. Let (M, g) be a Riemannian manifold with or without boundary ($n=\dim M\geqq 2$), and f a smooth function on M. In this paper, we consider the problem of deforming the given metric g conformally to another metric

$$\hat{g}=\begin{cases} e^u g & \text{if} \quad n=2 \\ u^{4/(n-2)}g \quad (u>0) & \text{if} \quad n\geqq 3 \end{cases}$$

with the prescrived scalar curvature f. It is well-known that this problem is equivalent to solving the following elliptic differential equation:

$$(*2) \qquad -\Delta_g u+S_g=fe^u \qquad \text{if} \quad n=2\,,$$

$$(*\mathrm{n}) \qquad \begin{cases} -4\dfrac{n-1}{n-2}\Delta_g u+S_g u=fu^{(n+2)/(n-2)} \\ u>0 \end{cases} \qquad \text{if} \quad n\geqq 3\,,$$

where Δ_g is the Laplacian with respect to g, namely, $\Delta_g=\operatorname{trace}\nabla_g^2$, and S_g is the scalar curvature of g. This problem and related ones have been extensively investigated, mainly in the case (M, g) is a compact manifold. As for the case (M, g) is the Euclidean space $(\boldsymbol{R}^n, g_0)$, since Ni [13] was published, many authors have refined and generalized his results, and applied the method to other equations (see, for example, [7], [11], and their references).

We study first the case of $(\boldsymbol{R}^n, g_0)$, and show the following sufficient conditions for the existence of infinitely many metrics each of which is pointwise conformal and uniformly equivalent to g_0, and each of whose scalar curvature is the prescrived

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function f.

THEOREM I. *Let Σ be a submanifold of $\boldsymbol{R}^n$ $(n \geqq 3)$ with $m = \dim \Sigma \leqq n-3$, and f a bounded smooth function on $\boldsymbol{R}^n$. Suppose Σ and f satisfy the following conditions*:

(R.1) *Σ is the graph of some C^1-map from $\boldsymbol{R}^m$ to $\boldsymbol{R}^{n-m}$ whose gradient is bounded*;

(R.2)
$$|f| \leqq C/r_\Sigma^l \quad \text{on} \quad \boldsymbol{R}^n \setminus B_R(\Sigma)$$

for positive constants C, $l>2$ and R, where $r_\Sigma(x) := \inf_{y \in \Sigma} |x-y|$, and $B_R(\Sigma)$ is the R-neighborhood of Σ.

Then, for any small enough positive number β, the equation

$$(*\text{n}') \qquad \begin{cases} -4\dfrac{n-1}{n-2}\Delta u = f u^{(n+2)/(n-2)} \\ u > 0 \end{cases}$$

possesses a bounded smooth solution u which is also bounded away from zero, and which has the following property:

$$(\text{R.3}) \qquad |u-\beta| \leqq \begin{cases} C'/r_\Sigma^{l-2} & \text{if} \quad l < n-m \\ C'/r_\Sigma^{n-m-2-\varepsilon} & \text{if} \quad n-m \leqq l < n-m+2 \\ C'/r_\Sigma^{n-m-2} & \text{if} \quad n-m+2 \leqq l \end{cases}$$

for a positive constant C', where a positive number ε can be chosen arbitrarily small.

The same assertion holds when Σ is the union of a finite family of submanifolds of $\boldsymbol{R}^n$ each of which satisfies the condition (R.1) (cf. Remark 2.2). Ni [13] proved the same assertion as above in the case where Σ is an affine subspace of $\boldsymbol{R}^n$ (see [ibid., Theorem 1.4], and also [12] and [11]). He constructed a supersolution and a subsolution of the equation $(*\text{n}')$ which are symmetric with respect to Σ by solving certain ordinary differential equations. However, it seems to be difficult to apply the method to our case. Actually, we try to construct a supersolution and a subsolution directly, based on the condition on Σ.

Moreover, our method is applicable to other situations. In fact, it yields the following results.

THEOREM II. *Let Σ be a subset of $\boldsymbol{R}^n$ $(n \geqq 3)$ and f a bounded smooth function on $\boldsymbol{R}^n$. Suppose Σ and f satisfy the following conditions*:

$$(\text{R.1}') \qquad \sup_{x \in \boldsymbol{R}^n} \int_{B_\delta(\Sigma)} \frac{dy}{(|x-y|^2+1)^{\alpha/2}} < +\infty$$

for positive numbers δ and $\alpha \leqq n-2$;

(R.2) *as in Theorem* I *with*

$$l=\begin{cases}\alpha+2 & \text{if } \alpha<n-2\\ n+2 & \text{if } \alpha=n-2\,.\end{cases}$$

Then the equation (*n′) *possesses infinitely many bounded smooth solutions each of which is also bounded away from zero.*

THEOREM III. *Let* (M, g) *be a complete, noncompact, simply connected Riemannian manifold* ($n=\dim M\geqq 2$) *satisfying*

$$-A^2\leqq \text{the sectional curvatures}\leqq -B^2 \tag{1.1}$$

for some positive constants A and B such that

$$\left(\frac{A}{B}\right)^2\leqq\frac{(n-1)^2}{n(n-2)}\,.$$

Let Σ *be a subset of* M, *and* f *a bounded smooth function on* M. *Suppose* Σ *and* f *satisfy the following conditions*:

$$\sup_{x\in M}\int_{B_\delta(\Sigma)}\frac{dy}{[\cosh\{Bd_g(x,y)\}]^\alpha}<+\infty \tag{H.1}$$

for a positive number δ, *where*

$$\alpha=\frac{n-1+\{(n-1)^2-n(n-2)(A/B)^2\}^{1/2}}{2}\,(=1 \ \text{if } n=2)\,;$$

$$-a^2\leqq f\leqq\begin{cases}-b^2 & \text{on } M\setminus B_R(\Sigma)\\ \varepsilon & \text{on } M\end{cases} \tag{H.2}$$

for positive constants a, b, R and a certain positive constant ε *depending only on A, B, a, b, R and* Σ, *where* $\rho_\Sigma(x):=\inf_{y\in\Sigma} d_g(x,y)$.

Then the equation (*n) *possesses a bounded smooth solution* (*which is also bounded away from zero if* $n\geqq 3$).

Aviles and McOwen [2] proved the same assertion as above in the case Σ is a point or $B_R(\Sigma)$ is compact (see [ibid., Theorems 1 and 4]). Indeed, when Σ is compact, the condition (H.1) is obviously satisfied. On the other hand, even if Σ is noncompact, we can construct examples of Σ satisfying the condition (H.1). For instance, when Σ is the union of a certain family of totally geodesic submanifolds of M, the condition (H.1) is satisfied (cf. Section 5). Furthermore, if Σ is invariant under the action of a certain nontrivial subgroup Γ of Isom(M), then our supersolution and subsolution are also Γ-invariant. Hence, we can regard them as those on M/Γ which is not simply connected (cf. Section7).

Sections 2–3 (resp. 4–7) are devoted to the case where (M, g) is the Euclidean space $(\boldsymbol{R}^n, g_0)$ (resp. the case where (M, g) has negative curvature).

In Sections 2 and 3, we give the proofs of Theorems I and II, examples of Σ, and some remarks on generalization to other equations (cf. [14] etc.). A proof of Theorem III and examples of Σ are given in Sections 4 and 5. We observe in Section 6 that the condition (H.1) is sharp in a sense, when M is the hyperbolic plane $\boldsymbol{H}^2=\boldsymbol{H}^2(-1)$ of constant curvature -1. In Section 7, we discuss the case where $M=\boldsymbol{H}^2/\Gamma$ is not simply connected.

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2. Proofs of Theorems I and II. We recall first the following:

THE METHOD OF SUPERSOLUTIONS AND SUBSOLUTIONS. *Let (M, g) be a complete Riemannian mainfold, and f a smooth function on M. If there exist a weak supersolution u_+ and a weak subsolution u_- of the equation* (*n) *such that $u_+, u_- \in H^p_{2,\mathrm{loc}}(M)$ $(p>n)$ and $u_- \leqq u_+$, then the equation* (*n) *possesses a smooth solution u satisfying $u_- \leqq u \leqq u_+$.*

This is well-known and we omit the proof (see, for instance, [8], [10], [13] and [3]). First, we give a proof of Theorem II for convenience.

PROOF OF THEOREM II. For any $y\in\boldsymbol{R}^n$, let $r_y := r_{\{y\}}$ and $u_y := 1/(r_y^2+1)^{\alpha/2}$. By direct computation, we see that

$$-\Delta u_y=\alpha(r_y^2+1)^{-\alpha/2-2}\{(n-2-\alpha)r_y^2+n\}\ .$$

Set

$$u_\Sigma := \int_{B_\delta(\Sigma)} u_y dy - \inf \int_{B_\delta(\Sigma)} u_y dy$$

and $\bar{u}_\Sigma := \sup_{\boldsymbol{R}^n} u_\Sigma$ which is finite by the assumption (R.1′). Moreover, it follows easily that u_Σ is a smooth function on $\boldsymbol{R}^n$, and Δu_Σ satisfies

$$-\Delta u_\Sigma \geqq \begin{cases} \alpha(n-2-\alpha)\displaystyle\int_{B_\delta(\Sigma)} (r_y^2+1)^{-\alpha/2-1} dy & \text{if} \quad \alpha<n-2 \\ \alpha n \displaystyle\int_{B_\delta(\Sigma)} (r_y^2+1)^{-\alpha/2-2} dy & \text{if} \quad \alpha=n-2\ . \end{cases}$$

Now, for any $x\in\boldsymbol{R}^n$ and every $r_0>r_\Sigma(x)$, there is an $x_0\in\Sigma$ such that $r_\Sigma(x)\leqq|x-x_0|<r_0$. Since

$$r_y(x)=|x-y|\leqq|x-x_0|+|x_0-y|<r_0+\delta$$

for any $y\in B_\delta(x_0)\subset B_\delta(\Sigma)$, it follows that

$$
\begin{aligned}
-\Delta u_\Sigma &\geqq \alpha(n-2-\alpha)\int_{B_\delta(x_0)}(r_y^2+1)^{-\alpha/2-1}dy \\
&\geqq \alpha(n-2-\alpha)\operatorname{vol}(B_\delta)\{(r_0+\delta)^2+1\}^{-\alpha/2-1} \\
&=\alpha(n-2-\alpha)\operatorname{vol}(B_\delta)\{(r_0+\delta)^2+1\}^{-l/2} \qquad \text{if} \quad \alpha<n-2\,,
\end{aligned}
$$

and

$$
\begin{aligned}
-\Delta u_\Sigma &\geqq \alpha n\int_{B_\delta(x_0)}(r_y^2+1)^{-\alpha/2-2}dy \\
&\geqq \alpha n\cdot\operatorname{vol}(B_\delta)\{(r_0+\delta)^2+1\}^{-\alpha/2-2} \\
&=\alpha n\cdot\operatorname{vol}(B_\delta)\{(r_0+\delta)^2+1\}^{-l/2} \qquad \text{if} \quad \alpha=n-2\,.
\end{aligned}
$$

Since we can take r_0 arbitrarily near $r_\Sigma(x)$, we get

$$-\Delta u_\Sigma\geqq C_1/\{(r_\Sigma+\delta)^2+1\}^{l/2}\,,$$

where

$$C_1:=\begin{cases}\alpha(n-2-\alpha)\operatorname{vol}(B_\delta) & \text{if} \quad \alpha<n-2 \\ \alpha n\cdot\operatorname{vol}(B_\delta) & \text{if} \quad \alpha=n-2\,.\end{cases}$$

Now, we may assume $R\geqq 1$. It follows from simple computations that

$$(r_\Sigma+\delta)^2+1\leqq\{2+(2\delta^2+1)/R^2\}r_\Sigma^2 \qquad \text{on} \quad \boldsymbol{R}^n\setminus B_R(\Sigma)\,.$$

Hence we get

$$-\Delta u_\Sigma\geqq\begin{cases}C_2/r_\Sigma^l & \text{on} \quad \boldsymbol{R}^n\setminus B_R(\Sigma) \\ C_3 & \text{in} \quad B_R(\Sigma)\,,\end{cases}$$

where

$$C_2:=C_1/\{2+(2\delta^2+1)/R^2\}^{l/2}\,, \qquad C_3:=C_1/\{(R+\delta)^2+1\}^{l/2}\,.$$

Let us now set $u_\pm:=\beta(1\pm\eta_\pm u_\Sigma)$ with positive numbers β, η_+ and η_-, where η_+ is chosen arbitrarily, η_- is chosen so as to satisfy $\eta_-<1/\bar{u}_\Sigma$, while β is chosen so as to satisfy

$$\beta\leqq\beta_0:=\left[\frac{n-2}{4(n-1)}\max\left\{\frac{(1+\eta_+\bar{u}_\Sigma)^{(n+2)/(n-2)}C}{\eta_+C_2},\ \frac{(1+\eta_+\bar{u}_\Sigma)^{(n+2)/(n-2)}\sup f}{\eta_+C_3},\ \frac{C}{\eta_-C_2},\ \frac{-\inf f}{\eta_-C_3}\right\}\right]^{-(n-2)/4}.$$

Then we get

$$u_+^{-(n+2)/(n-2)}\left(-4\frac{n-1}{n-2}\Delta u_+\right)$$

$$=\beta^{-(n+2)/(n-2)}(1+\eta_+u_\Sigma)^{-(n+2)/(n-2)}\left(-4\frac{n-1}{n-2}\beta\eta_+\Delta u_\Sigma\right)$$

$$\geqq\beta^{-4/(n-2)}(1+\eta_+\bar{u}_\Sigma)^{-(n+2)/(n-2)}\cdot 4\frac{n-1}{n-2}\eta_+(-\Delta u_\Sigma)$$

$$\geqq\begin{cases}\beta_0^{-4/(n-2)}(1+\eta_+\bar{u}_\Sigma)^{-(n+2)/(n-2)}\cdot 4\dfrac{n-1}{n-2}\eta_+C_2/r_\Sigma^l\geqq C/r_\Sigma^l\geqq f & \text{on} \quad \boldsymbol{R}^n\setminus B_R(\Sigma)\\ \beta_0^{-4/(n-2)}(1+\eta_+\bar{u}_\Sigma)^{-(n+2)/(n-2)}\cdot 4\dfrac{n-1}{n-2}\eta_+C_3\geqq \sup f\geqq f & \text{in} \quad B_R(\Sigma)\,,\end{cases}$$

that is

$$-4\frac{n-1}{n-2}\Delta u_+\geqq f u_+^{(n+2)/(n-2)} \qquad \text{on} \quad \boldsymbol{R}^n\,.$$

On the other hand, we get

$$-u_-^{-(n+2)/(n-2)}\left(-4\frac{n-1}{n-2}\Delta u_-\right)$$

$$=\beta^{-(n+2)/(n-2)}(1-\eta_-u_\Sigma)^{-(n+2)/(n-2)}\left(-4\frac{n-1}{n-2}\beta\eta_-\Delta u_\Sigma\right)$$

$$\geqq\beta^{-4/(n-2)}\cdot 4\frac{n-1}{n-2}\eta_-(-\Delta u_\Sigma)$$

$$\geqq\begin{cases}\beta_0^{-4/(n-2)}\cdot 4\dfrac{n-1}{n-2}\eta_-C_2/r_\Sigma^l\geqq C/r_\Sigma^l\geqq -f & \text{on} \quad \boldsymbol{R}^n\setminus B_R(\Sigma)\\ \beta_0^{-4/(n-2)}\cdot 4\dfrac{n-1}{n-2}\eta_-C_3\geqq -\inf f\geqq -f & \text{in} \quad B_R(\Sigma)\,,\end{cases}$$

that is

$$-4\frac{n-1}{n-2}\Delta u_-\leqq f u_-^{(n+2)/(n-2)} \qquad \text{on} \quad \boldsymbol{R}^n\,.$$

Hence u_+ and u_- are respectively a supersolution and a subsolution of the equation (*n′). Since $u_-\leqq u_+$, by the method of supersolutions and subsolutions, the equation (*n′) possesses a bounded smooth solution u satisfying $u_-\leqq u\leqq u_+$. It is clear that

$$|u-\beta|\leqq\beta\cdot\max\{\eta_+,\eta_-\}u_\Sigma \qquad \text{on} \quad \boldsymbol{R}^n\,,$$

from which, for any positive number $\beta \leqq \beta_0$, the equation (*n′) possesses a solution satisfying $\lim_{i\to\infty} u(x_i) = \beta$ for any minimizing sequence $\{x_i\}_{i=1}^{\infty}$ of u_Σ. Namely the equation (*n′) possesses infinitely many solutions. q.e.d.

PROOF OF THEOREM I. Since the assertion in the case $m=0$ is known as we referred to in Section 1, we assume $m \geqq 1$ in what follows.

By the condition (R.1), we may assume that the submanifold Σ is given as

$$\Sigma = \{(x_1, h(x_1)) \in \boldsymbol{R}^m \times \boldsymbol{R}^{n-m}\},$$

where $h: \boldsymbol{R}^m \to \boldsymbol{R}^{n-m}$ is a C^1-map with $|\partial h/\partial x_1| \leqq C_\Sigma$ for a positive constant C_Σ.

Let r_y and u_y be as in the proof of Theorem II. Set

$$u_\Sigma := \int_\Sigma u_y ds(y) = \int_\Sigma (r_y^2+1)^{-\alpha/2} ds(y),$$

where $ds(y)$ is the volume element of Σ, and

$$\alpha := \begin{cases} l+m-2 & \text{if} \quad l<n-m \\ n-2-\varepsilon \quad \text{for some} \quad \varepsilon>0 & \text{if} \quad n-m \leqq l < n-m+2 \\ n-2 & \text{if} \quad n-m+2 \leqq l. \end{cases}$$

For any $x=(x_1, x_2) \in \boldsymbol{R}^m \times \boldsymbol{R}^{n-m} = \boldsymbol{R}^n$, let $s_\Sigma(x) := |x_2 - h(x_1)|$. By the assumption (R.1),

$$\Sigma \subset \{y=(y_1, y_2) \mid |y_2 - h(x_1)| \leqq C_\Sigma |y_1 - x_1|\}$$

for any $x_1 \in \boldsymbol{R}^m$, from which it follows that

$$\frac{s_\Sigma}{\sqrt{C_\Sigma^2+1}} \leqq r_\Sigma \leqq s_\Sigma \qquad \text{on} \quad \boldsymbol{R}^n.$$

On the other hand, since $ds(x) = \{\det(g_{ij})\}^{1/2} dx_1$ and

$$g_{ij} = \delta_{ij} + \sum_{k=1}^{n-m} \frac{\partial h^k}{\partial x_1^i} \cdot \frac{\partial h^k}{\partial x_1^j} \qquad \text{for} \quad 1 \leqq i, j \leqq m$$

and $|\partial h/\partial x_1| \leqq C_\Sigma$, we get

$$dx_1 \leqq ds(x) \leqq V dx_1$$

for some positive constant $V \geqq 1$ depending only on C_Σ.

Now, if we denote $s_1 := s_\Sigma/(C_\Sigma^2+1)^{1/2}$ and $r := |x_1 - y_1|$ for convenience, then obviously, for any $y \in \Sigma$,

$$u_y = (r_y^2+1)^{-\alpha/2} \leqq \min\{(s_1^2+1)^{-\alpha/2}, (r^2+1)^{-\alpha/2}\} \qquad \text{on} \quad \boldsymbol{R}^n,$$

from which it follows that

$$\begin{aligned}\int_\Sigma u_y ds(y) &\leqq \int_\Sigma u_y V dy_1\\ &\leqq V\left\{\int_{r\leqq s_1}(s_1^2+1)^{-\alpha/2}dy_1+\int_{r\geqq s_1}(r^2+1)^{-\alpha/2}dy_1\right\}\\ &\leqq V\omega\left(s_1^{-\alpha}\int_0^{s_1} r^{m-1}dr+\int_{s_1}^{\infty} r^{-\alpha+m-1}dr\right)\\ &= V\omega\left(\frac{1}{m}+\frac{1}{\alpha-m}\right)s_1^{m-\alpha},\end{aligned}$$

where $\omega := \mathrm{vol}\{S^{m-1}(1)\}$. On the other hand, since $u_y \leqq 1$,

$$\begin{aligned}\int_\Sigma u_y ds(y) &\leqq V\left\{\int_{r\leqq 1} dy_1+\int_{r\geqq 1}(r^2+1)^{-\alpha/2}dy_1\right\}\\ &\leqq V\omega\left(\frac{1}{m}+\frac{1}{\alpha-m}\right).\end{aligned}$$

Hence we get

$$u_\Sigma \leqq C_0 \min\{1, s_1^{m-\alpha}\},$$

where

$$C_0 := V\omega\left(\frac{1}{m}+\frac{1}{\alpha-m}\right).$$

From this estimate and the caluculation in the proof of Theorem II, it follows easily that u_Σ is a smooth function on $\boldsymbol{R}^n$, and Δu_Σ satisfies

$$-\Delta u_\Sigma \geqq \begin{cases} \alpha(n-2-\alpha)\displaystyle\int_\Sigma (r_y^2+1)^{-\alpha/2-1}ds(y) & \text{if } \alpha<n-2\\ \alpha n\displaystyle\int_\Sigma (r_y^2+1)^{-\alpha/2-2}ds(y) & \text{if } \alpha=n-2.\end{cases}$$

Now, for any $x\in\boldsymbol{R}^n$ and every $y\in\Sigma$, we have

$$\begin{aligned}r_y(x)^2 &= |x-y|^2 = |x_1-y_1|^2+|x_2-y_2|^2 \leqq |x_1-y_1|^2+(s_\Sigma(x)+C_\Sigma|x_1-y_1|)^2\\ &\leqq |x_1-y_1|^2+2s_\Sigma(x)^2+2C_\Sigma^2|x_1-y_1|^2 = (2C_\Sigma^2+1)|x_1-y_1|^2+2s_\Sigma(x)^2.\end{aligned}$$

Hence

$$\int_\Sigma (r_y^2+1)^{-\alpha/2-1}ds(y) \geqq \omega\int_0^\infty \{(2C_\Sigma^2+1)r^2+2s_\Sigma^2+1\}^{-\alpha/2-1}r^{m-1}dr$$

$$=\omega s_\Sigma^{m-\alpha-2}\int_0^\infty \{(2C_\Sigma^2+1)t^2+2+s_\Sigma^{-2}\}^{-\alpha/2-1}t^{m-1}dt\,.$$

If we denote $s_2:=(2+s_\Sigma^{-2})^{1/2}$ for convenience, then obviously

$$(2C_\Sigma^2+1)t^2+s_2^2\leqq\begin{cases}2(C_\Sigma^2+1)s_2^2 & \text{for any}\quad t\leqq s_2\\ 2(C_\Sigma^2+1)t^2 & \text{for any}\quad t\geqq s_2\,,\end{cases}$$

from which it follows that

$$\begin{aligned}&\int_\Sigma (r_y^2+1)^{-\alpha/2-1}ds(y)\\ &\geqq\omega s_\Sigma^{m-\alpha-2}\left[\int_0^{s_2}\{2(C_\Sigma^2+1)s_2^2\}^{-\alpha/2-1}t^{m-1}dt+\int_{s_2}^\infty\{2(C_\Sigma^2+1)t^2\}^{-\alpha/2-1}t^{m-1}dt\right]\\ &=\omega s_\Sigma^{m-\alpha-2}\left[\{2(C_\Sigma^2+1)\}^{-\alpha/2-1}s_2^{-\alpha-2}\frac{1}{m}s_2^m+\{2(C_\Sigma^2+1)\}^{-\alpha/2-1}\frac{1}{\alpha-m+2}s_2^{m-2-\alpha}\right]\\ &=\omega\{2(C_\Sigma^2+1)\}^{-\alpha/2-1}\left(\frac{1}{m}+\frac{1}{\alpha-m+2}\right)(s_2s_\Sigma)^{m-2-\alpha}\\ &\geqq\omega\{2(C_\Sigma^2+1)\}^{-\alpha/2-1}\left(\frac{1}{m}+\frac{1}{\alpha-m+2}\right)\{2(C_\Sigma^2+1)r_\Sigma^2+1\}^{(m-2-\alpha)/2}\,.\end{aligned}$$

Similarly

$$\begin{aligned}&\int_\Sigma (r_y^2+1)^{-\alpha/2-2}ds(y)\\ &\geqq\omega\{2(C_\Sigma^2+1)\}^{-\alpha/2-2}\left(\frac{1}{m}+\frac{1}{\alpha-m+4}\right)\{2(C_\Sigma^2+1)r_\Sigma^2+1\}^{(m-4-\alpha)/2}\,.\end{aligned}$$

Hence we get

$$-\Delta u_\Sigma\geqq C_1/\{2(C_\Sigma^2+1)r_\Sigma^2+1\}^{l/2}\,,$$

where

$$C_1:=\begin{cases}\alpha(n-2-\alpha)\omega\{2(C_\Sigma^2+1)\}^{-\alpha/2-1}\left(\dfrac{1}{m}+\dfrac{1}{\alpha-m+2}\right) & \text{if}\quad l<n-m+2\\ \alpha n\omega\{2(C_\Sigma^2+1)\}^{-\alpha/2-2}\left(\dfrac{1}{m}+\dfrac{1}{\alpha-m+4}\right) & \text{if}\quad n-m+2\leqq l\,.\end{cases}$$

Now, we may assume $R\geqq 1$. It follows easily that

$$2(C_\Sigma^2+1)r_\Sigma^2+1\leqq\{2(C_\Sigma^2+1)+1/R^2\}r_\Sigma^2\qquad\text{on}\quad \boldsymbol{R}^n\setminus B_R(\Sigma)\,.$$

Hence we get

$$-\Delta u_\Sigma \geqq \begin{cases} C_2/r_\Sigma^l & \text{on} \quad \boldsymbol{R}^n \setminus B_R(\Sigma) \\ C_3 & \text{in} \quad B_R(\Sigma)\,, \end{cases}$$

where

$$C_2 := C_1/\{2(C_\Sigma^2+1)+1/R^2\}^{l/2}\,, \qquad C_3 := C_1/\{2(C_\Sigma^2+1)R^2+1\}^{l/2}\,.$$

Using this estimate, we can prove, by the same method as in the proof of Theorem II, that there exist positive numbers η_+, η_- and β_0 such that, for any positive number $\beta \leqq \beta_0$, the equation (*n′) possesses a smooth bounded solution u satisfying

$$|u-\beta| \leqq \beta \cdot \max\{\eta_+, \eta_-\} u_\Sigma \qquad \text{on} \quad \boldsymbol{R}^n\,.$$

Moreover, since

$$0 < u_\Sigma \leqq C_0 s_1^{m-\alpha} = C_0\left(\frac{s_\Sigma}{\sqrt{C_\Sigma^2+1}}\right)^{m-\alpha}$$

$$\leqq C_0\left(\frac{r_\Sigma}{\sqrt{C_\Sigma^2+1}}\right)^{m-\alpha} = C_0(C_\Sigma^2+1)^{(\alpha-m)/2}/r_\Sigma^{\alpha-m} \qquad \text{on} \quad \boldsymbol{R}^n\,,$$

we get the estimate (R.3) with

$$C' := \beta \cdot \max\{\eta_+, \eta_-\} C_0 (C_\Sigma^2+1)^{(\alpha-m)/2}\,.$$

q.e.d.

REMARK 2.1. From our proof, it is not hard to see that we can replace the condition (R.1) by the following condition:

(R.1″) $\quad \langle\!\langle G(x), q\rangle\!\rangle := \min\{|\pi_q(v)| \mid v \in G(x), |v|=1\} > \varepsilon_\Sigma \qquad \text{on} \quad \Sigma \setminus \Sigma_0\,,$

for some $q \in G(m, n-m)$, a positive number ε_Σ, and a compact subset Σ_0 of Σ, where $G\colon \Sigma \to G(m, n-m)$ is the Gauss map of Σ.

REMARK 2.2. We can replace Σ in Theorem I by the union of a finite family $\{\Sigma_i\}_{i=1}^k$ of submanifolds of $\boldsymbol{R}^n$ with $m_i = \dim \Sigma_i \leqq n-3$ such that each Σ_i satisfies the condition (R.1) with $h = h_i$. Indeed, for any $1 \leqq i \leqq k$, set

$$u_{\Sigma_i} := \int_{\Sigma_i} u_y\, ds(y) = \int_{\Sigma_i} (r_y^2+1)^{-\alpha_i/2} ds(y)\,,$$

where

$$\alpha_i := \begin{cases} l+m_i-2 & & \text{if} \quad l < n-m_i \\ n-2-\varepsilon & \text{for some} \quad \varepsilon > 0 & \text{if} \quad n-m_i \leqq l < n-m_i+2 \\ n-2 & & \text{if} \quad n-m_i+2 \leqq l\,. \end{cases}$$

Then, by the proof of Theorem I,

$$-\Delta u_{\Sigma_i} \geqq \begin{cases} C_{2i}/r_{\Sigma_i}^l & \text{on} \quad \boldsymbol{R}^n \setminus B_R(\Sigma_i) \\ C_{3i} & \text{in} \quad B_R(\Sigma_i)\,, \end{cases}$$

$$u_{\Sigma_i} \leqq C_{4i}/r_{\Sigma_i}^{\alpha_i - m_i} \qquad \text{on} \quad \boldsymbol{R}^n\,,$$

where

$$C_{4i} := C_{0i}(C_\Sigma^2 + 1)^{(\alpha_i - m_i)/2}\,.$$

Set

$$u_\Sigma := \sum_{i=1}^k u_{\Sigma_i}\,.$$

Now, since $r_\Sigma = \min_i r_{\Sigma_i}$, it is clear that

$$\begin{aligned} & r_\Sigma \leqq r_{\Sigma_i} && \text{for any} \quad 1 \leqq i \leqq k\,, \\ & r_\Sigma(x) = r_{\Sigma_i}(x) && \text{for some} \quad i \text{ depending on} \quad x \in \boldsymbol{R}^n\,, \end{aligned}$$

from which it follows that

$$-\Delta u_\Sigma = \sum_{i=1}^k (-\Delta u_{\Sigma_i}) \geqq \begin{cases} \min_i C_{2i}/r_\Sigma^l & \text{on} \quad \boldsymbol{R}^n \setminus B_R(\Sigma) = \bigcap_i \{\boldsymbol{R}^n \setminus B_R(\Sigma_i)\} \\ \min_i C_{3i} & \text{in} \quad B_R(\Sigma) = \bigcup_i B_R(\Sigma_i)\,, \end{cases}$$

$$u_\Sigma \leqq \sum_{i=1}^k C_{4i} \Big/ r_{\Sigma_i}^{\alpha_i - m_i} \leqq \left(\sum_{i=1}^k C_{4i} \right) \Big/ r_\Sigma^{\min_i \{\alpha_i - m_i\}} \qquad \text{on} \quad \boldsymbol{R}^n \setminus B_R(\Sigma)\,,$$

where

$$\min_i \{\alpha_i - m_i\} = \begin{cases} l - 2 & \text{if} \quad l < n - m \\ n - m - 2 - \varepsilon & \text{if} \quad n - m \leqq l < n - m + 2 \\ n - m - 2 & \text{if} \quad n - m + 2 \leqq l \end{cases}$$

and $m := \max_i m_i$. Using these estimates, we can prove our assertion by the same method as in the proof of Theorem I.

3. Examples and generalization. In this section, we first give examples for submanifolds Σ of $\boldsymbol{R}^n$ such that the assertion of Theorem I holds. Secondly, we discuss certain equations in more general forms.

EXAMPLE 3.1. Let $\tilde{h} := \boldsymbol{R} \to \boldsymbol{R}$ be a C^1-function with $d\tilde{h}/dt$ bounded above or below, and

$$\Sigma := \{(t, \tilde{h}(t), 0, \dots, 0) \in \boldsymbol{R}^n \,|\, t \in \boldsymbol{R}\} \qquad (n \geqq 4)\,.$$

Even if $|d\tilde{h}/dt|$ is unbounded, e.g., $\tilde{h}(t)=e^t$ or $\tilde{h}(t)$ is a polynomial of odd degree, we see in this case that Σ satisfies the condition (R.1) by a suitable coordinate change (c.f. Remark 2.1).

EXAMPLE 3.2. Let Σ be as in Example 3.1. When $\tilde{h}(t)$ is a polynomial of even degree, it is clear that $d\tilde{h}/dt$ is unbounded above and below. However, we can easily show that Σ satisfies the condition (R.1′) in Theorem II with $\alpha > m$. Moreover, if we set

$$\Sigma_i := \{(t, \tilde{h}_i(t), 0, \dots, 0)\} \qquad \text{for} \quad i=1, 2\,,$$

$$\tilde{h}_1(t) := \begin{cases} \tilde{h}(t) & \text{for} \quad t \geqq 0 \\ \tilde{h}'(0)t + \tilde{h}(0) & \text{for} \quad t \leqq 0\,, \end{cases}$$

$$\tilde{h}_2(t) := \begin{cases} \tilde{h}'(0)t + \tilde{h}(0) & \text{for} \quad t \geqq 0 \\ \tilde{h}(t) & \text{for} \quad t \leqq 0\,, \end{cases}$$

then obviously both Σ_1 and Σ_2 satisfy the assumption in Example 3.1. Hence they satisfy the condition (R.1). Since $\Sigma \subset \Sigma_1 \cup \Sigma_2$, we see that the assertion of Theorem I holds for Σ with the property which is somewhat weaker than the property (R.3) (cf. Remark 2.2).

In the remainder of this section, we mention some generalization of the method used so far to the following equation:

$$(**) \qquad -\Delta u(x) = f(x)F(x, u(x)) \qquad \text{on} \quad \boldsymbol{R}^n\,,$$

where $f(x)$ is a bounded locally Hölder continuous function on $\boldsymbol{R}^n$, and $F(x, t)$ is a nonnegative locally Hölder continuous function on $\boldsymbol{R}^n \times (a, b)$ $(-\infty \leqq a < b \leqq +\infty)$ with one of the following properties:

(F.1) $\quad a > -\infty$ and $\dfrac{F(x, t)}{t-a} \to 0$ as $t \to a+0$ uniformly in x;

(F.2) $\quad a = -\infty$ and $\dfrac{F(x, t)}{-t} \to 0$ as $t \to -\infty$ uniformly in x;

(F.3) $\quad b < +\infty$ and $\dfrac{F(x, t)}{b-t} \to 0$ as $t \to b-0$ uniformly in x;

(F.4) $\quad b = +\infty$ and $\dfrac{F(x, t)}{t} \to 0$ as $t \to +\infty$ uniformly in x;

(F.5) $\quad a = -\infty$, $f(x) \leqq 0$ and $F(x, t)$ is bounded on $(a, c]$ for any $c \in (a, b)$;

(F.6) $\quad b = +\infty$, $f(x) \geqq 0$ and $F(x, t)$ is bounded on $[c, b)$ for any $c \in (a, b)$;

(F.7) $\quad a = -\infty$, $b = +\infty$ and $F(x, t)$ is bounded.

In this situation, we can apply the same method as in the proofs of Theorems I and II to showing the following existence results which respectively include the assertions of Theorems I and II.

THEOREM 3.3. *Let Σ be a submanifold of $\boldsymbol{R}^n$ $(n \geqq 3)$ with $m = \dim \Sigma \leqq n-3$, and f and F as above. Suppose Σ and f satisfy the conditions* (R.1) *and* (R.2). *Then, for any $\beta \in I$, the equation* (**) *possesses a C^2-solution u which is bounded away from both a and b, and which has the property* (R.3), *where*

$$I = \begin{cases} (a, b_0) & \text{for some} \quad b_0 \in (a, b] \quad \text{when (F.1) or (F.2) holds} \\ (a_0, b) & \text{for some} \quad a_0 \in [a, b) \quad \text{when (F.3) or (F.4) holds} \\ (a, b) & \qquad\qquad\qquad\qquad \text{when (F.5), (F.6) or (F.7) holds}\,. \end{cases}$$

THEOREM 3.4. *Let Σ be a subset of $\boldsymbol{R}^n$ $(n \geqq 3)$, and f and F as above. Suppose Σ and f satisfy the conditions* (R.1′) *with $\alpha \leqq n-2$ and* (R.2) *with the same l as in Theorem* II. *Then the equation* (**) *possesses infinitely many C^2-solutions each of which is bounded away from both a and b.*

REMARKS 3.5. (1) The equation $-\Delta u = f u^p$ $(p > 1)$ satisfies (F.1) with $a = 0$ and $b = +\infty$.

(2) The equation $-\Delta u = f u^p$ $(p < 1)$ satisfies (F.4) with $a = 0$.

(3) The equation $-\Delta u = f e^u$ satisfies (F.2) with $b = +\infty$. In addition, if $f \leqq 0$, then this equation satisfies (F.5).

4. Proof of Theorem III.

We recall first the following standard theorem:

COMPARISON THEOREM. *Let (M, g) be a complete, noncompact, simply connected Riemannian manifold $(n = \dim M \geqq 2)$ satisfying*

$$(1.1) \qquad -A^2 \leqq \text{the sectional curvatures} \leqq -B^2$$

for some positive constants A and B, and let Σ be a totally geodesic submanifold of M with $m = \dim \Sigma \leqq n-1$. Then the distance $\rho_\Sigma := d_g(\cdot, \Sigma)$ to Σ satisfies the following estimates on $M \setminus \Sigma$:

$$(4.1) \qquad |\nabla_g \rho_\Sigma| \equiv 1\,,$$

$$(4.2) \qquad |\nabla_g^2 \rho_\Sigma| \leqq nA \coth A\rho_\Sigma\,,$$

$$(4.3) \qquad \Delta_g \rho_\Sigma \geqq B\{(n-m-1)(\coth B\rho_\Sigma) + m(\tanh B\rho_\Sigma)\}\,.$$

The equality holds in (4.3) *if and only if (M, g) is the hyperbolic space $\boldsymbol{H}^n(-B^2)$ of constant curvature $-B^2$.*

This is well-known and we omit the proof (see, for instance, [5] and [6]).

PROOF OF THEOREM III. For any $y \in M$, let $\rho_y := \rho_{\{y\}}$, and $u_y := 1/(\cosh B\rho_y)^\alpha$. By direct computation, we see that

$$-\Delta_g u_y = B^2(\cosh B\rho_y)^{-\alpha-2}$$
$$\cdot\left\{-\alpha^2(\cosh B\rho_y)^2+\frac{\alpha}{B}(\cosh B\rho_y)(\sinh B\rho_y)\Delta_g\rho_y+\alpha(\alpha+1)\right\}.$$

Now by (4.3) with $\Sigma=\{y\}$ (hence $m=\dim\{y\}=0$), we have

$$\Delta_g\rho_y \geqq B(n-1)\coth B\rho_y ,$$

from which

$$-\Delta_g u_y \geqq B^2(\cosh B\rho_y)^{-\alpha-2}\{-\alpha(\alpha-n+1)(\cosh B\rho_y)^2+\alpha(\alpha+1)\}.$$

Set

$$u_\Sigma := \int_{B_\delta(\Sigma)} u_y\,dy ,$$

and $\bar{u}_\Sigma := \sup_M u_\Sigma$ which is finite by the assumption (H.1). By (4.1) and (4.2), we can easily get $u_\Sigma \in C^2(M)$, and $\Delta_g u_\Sigma$ satisfies

$$(4.4)\qquad -\Delta_g u_\Sigma \geqq \int_{B_\delta(\Sigma)} B^2(\cosh B\rho_y)^{-\alpha-2}\{-\alpha(\alpha-n+1)(\cosh B\rho_y)^2+\alpha(\alpha+1)\}\,dy .$$

THE CASE $n=2$. In this case, since $\alpha=1$,

$$-\Delta_g u_\Sigma \geqq \int_{B_\delta(\Sigma)} 2B^2(\cosh B\rho_y)^{-3}\,dy > 0 .$$

Now, for any $x\in M$ and every $\rho_0>\rho_\Sigma(x)$, there is an $x_0\in\Sigma$ such that $\rho_\Sigma(x)\leqq d_g(x, x_0)<\rho_0$. Since

$$\rho_y(x)=d_g(x, y)\leqq d_g(x, x_0)+d_g(x_0, y)<\rho_0+\delta$$

for any $y\in B_\delta(x_0)\subset B_\delta(\Sigma)$, it follows that

$$-\Delta_g u_\Sigma \geqq \int_{B_\delta(x_0)} 2B^2(\cosh B\rho_y)^{-3}\,dy \geqq 2B^2\operatorname{vol}_B(B_\delta)\,[\cosh\{B(\rho_0+\delta)\}]^{-3} ,$$

where vol_B is the volume with respect to the metric of $H^n(-B^2)$. Since we can take ρ_0 arbitrarily near $\rho_\Sigma(x)$, we get

$$-\Delta_g u_\Sigma \geqq 2B^2\operatorname{vol}_B(B_\delta)\,[\cosh\{B(\rho_\Sigma+\delta)\}]^{-3} .$$

It is clear that

$$-\Delta_g u_\Sigma \geqq 2B^2\operatorname{vol}_B(B_\delta)\,[\cosh\{B(R+\delta)\}]^{-3} =: C_1>0 \qquad \text{in} \quad B_R(\Sigma) .$$

Set $u_+ := \beta u_\Sigma + \log(2A^2/b^2)$, where β is chosen so as to satisfy $\beta>2A^2/C_1$. If we take $\varepsilon := (\beta C_1-2A^2)b^2/2A^2\exp(\beta\bar{u}_\Sigma)>0$, then we get

$$-\Delta_g u_+ + S_g \geqq -\beta\Delta_g u_\Sigma - 2A^2 \geqq \beta C_1 - 2A^2 = \varepsilon \exp\left(\beta \bar{u}_\Sigma + \log\frac{2A^2}{b^2}\right)$$
$$\geqq \varepsilon e^{u_+} \geqq f e^{u_+} \qquad \text{in} \quad B_R(\Sigma)\,,$$

and

$$-\Delta_g u_+ + S_g > -2A^2 = -b^2 \exp\left(\log\frac{2A^2}{b^2}\right) > -b^2 e^{u_+} \geqq f e^{u_+} \qquad \text{on} \quad M\,.$$

On the other hand, if we set $u_- := \log(2B^2/a^2)$, then we have

$$-\Delta_g u_- + S_g \leqq -2B^2 = -a^2 \exp\left(\log\frac{2B^2}{a^2}\right) = -a^2 e^{u_-} \leqq f e^{u_-} \qquad \text{on} \quad M\,.$$

Hence u_+ and u_- are respectively a supersolution and a subsolution of the equation (*2). Since $u_- \leqq u_+$, by the method of supersolutions and subsolutions, the equation (*2) possesses a bounded solution.

THE CASE $n \geqq 3$. We have first by (4.4),

$$-4\frac{n-1}{n-2}\Delta_g u_\Sigma + S_g u_\Sigma \geqq -4\frac{n-1}{n-2}\Delta_g u_\Sigma - A^2 n(n-1) u_\Sigma$$
$$\geqq \int_{B_\delta(\Sigma)} \left[4\frac{n-1}{n-2} B^2 (\cosh B\rho_y)^{-\alpha-2} \{-\alpha(\alpha-n+1)(\cosh B\rho_y)^2 + \alpha(\alpha+1)\} - A^2 n(n-1)(\cosh B\rho_y)^{-\alpha} \right] dy$$
$$= \int_{B_\delta(\Sigma)} 4\frac{n-1}{n-2} B^2 (\cosh B\rho_y)^{-\alpha-2} \left[-\left\{\alpha^2 - (n-1)\alpha + \frac{n(n-2)}{4}\left(\frac{A}{B}\right)^2\right\} \cdot (\cosh B\rho_y)^2 + \alpha(\alpha+1) \right] dy$$
$$= \int_{B_\delta(\Sigma)} 4\frac{n-1}{n-2} B^2 \alpha(\alpha+1)(\cosh B\rho_y)^{-\alpha-2}\, dy > 0\,.$$

Now, by the same observation as in the case $n=2$, we get

$$-4\frac{n-1}{n-2}\Delta_g u_\Sigma + S_g u_\Sigma \geqq 4\frac{n-1}{n-2} B^2 \alpha(\alpha+1)\,\mathrm{vol}_B(B_\delta)[\cosh\{B(\rho_\Sigma+\delta)\}]^{-\alpha-2}\,.$$

It is clear that

$$-4\frac{n-1}{n-2}\Delta_g u_\Sigma + S_g u_\Sigma \geqq 4\frac{n-1}{n-2} B^2 \alpha(\alpha+1)\,\mathrm{vol}_B(B_\delta)[\cosh\{B(R+\delta)\}]^{-\alpha-2}$$
$$=: C_2 > 0 \qquad \text{in} \quad B_R(\Sigma)\,.$$

Set $u_+ := \beta u_\Sigma + \{A^2n(n-1)/b^2\}^{(n-2)/4} > 0$, where β is chosen so as to satisfy $\beta > \{A^2n(n-1)/b^2\}^{(n-2)/4} \cdot A^2n(n-1)/C_2$. If we take

$$\varepsilon := \left[\beta \bar{u}_\Sigma + \left\{\frac{A^2n(n-1)}{b^2}\right\}^{(n-2)/4}\right]^{-(n+2)/(n-2)} \cdot \left[\beta C_2 - A^2n(n-1)\left\{\frac{A^2n(n-1)}{b^2}\right\}^{(n-2)/4}\right] > 0\,,$$

then we get

$$\begin{aligned}
-4\frac{n-1}{n-2}\Delta_g u_+ + S_g u_+ &= \beta\left(-4\frac{n-1}{n-2}\Delta_g u_\Sigma + S_g u_\Sigma\right) + S_g\left\{\frac{A^2n(n-1)}{b^2}\right\}^{(n-2)/4} \\
&\geqq \beta C_2 - A^2n(n-1)\left\{\frac{A^2n(n-1)}{b^2}\right\}^{(n-2)/4} \\
&= \varepsilon\left[\beta \bar{u}_\Sigma + \left\{\frac{A^2n(n-1)}{b^2}\right\}^{(n-2)/4}\right]^{(n+2)/(n-2)} \\
&\geqq \varepsilon u_+^{(n+2)/(n-2)} \geqq f u_+^{(n+2)/(n-2)} \qquad \text{in} \quad B_R(\Sigma)\,,
\end{aligned}$$

and

$$\begin{aligned}
-4\frac{n-1}{n-2}\Delta_g u_+ + S_g u_+ &> -A^2n(n-1)\left\{\frac{A^2n(n-1)}{b^2}\right\}^{(n-2)/4} \\
&= -b^2\left[\left\{\frac{A^2n(n-1)}{b^2}\right\}^{(n-2)/4}\right]^{(n+2)/(n-2)} \\
&> -b^2 u_+^{(n+2)/(n-2)} \geqq f u_+^{(n+2)/(n-2)} \qquad \text{on} \quad M\,.
\end{aligned}$$

On the other hand, if we set $u_- := \{B^2n(n-1)/a^2\}^{(n-2)/4} > 0$, then we have

$$\begin{aligned}
-4\frac{n-1}{n-2}\Delta_g u_- + S_g u_- &\leqq -B^2n(n-1)\left\{\frac{B^2n(n-1)}{a^2}\right\}^{(n-2)/4} \\
&= -a^2\left[\left\{\frac{B^2n(n-1)}{a^2}\right\}^{(n-2)/4}\right]^{(n+2)/(n-2)} \\
&= -a^2 u_-^{(n+2)/(n-2)} \leqq f u_-^{(n+2)/(n-2)} \qquad \text{on} \quad M\,.
\end{aligned}$$

Hence u_+ and u_- are respectively a supersolution and a subsolution of the equation (*n). Since $0 < u_- \leqq u_+$, by the method of supersolutions and subsolutions, the equation (*n) possesses a solution u which is bounded between two positive constants. q.e.d.

5. Examples for the negative case. In this section, we give examples for subsets Σ of M such that the assertion of Theorem III holds. To begin with, we shall prove the following:

THEOREM 5.1. *Let* (M, g), Σ *and* f *be as in Theorem* III. *Suppose* Σ *and* f *satisfy the following conditions*:

(H.1′) Σ *is the union of a family* $\{\Sigma_i\}_{i\in I}$ *of totally geodesic submanifolds of* M *with*

$$m_i = \dim \Sigma_i \leqq \begin{cases} 1 & \text{if } n=2 \\ \dfrac{n+1}{2} & \text{if } n\geqq 3 \text{ and } \left(\dfrac{A}{B}\right)^2 < \dfrac{(n-1)^2}{n(n-2)} \\ \dfrac{n}{2} & \text{if } n\geqq 3 \text{ and } \left(\dfrac{A}{B}\right)^2 = \dfrac{(n-1)^2}{n(n-2)}, \end{cases}$$

such that the condition

$$\sup_{x\in M} \sum_{i\in I} \frac{1}{[\cosh\{Bd_g(x, \Sigma_i)\}]^\alpha} < +\infty \tag{5.1}$$

holds with the same α *as in Theorem* III;

(H.2) *as in Theorem* III.

Then the equation (∗n) *possesses a bounded smooth solution* (*which is also bounded away from zero if* $n\geqq 3$).

Although we can derive this theorem (except when $m_i = \dim \Sigma_i = (n+1)/2$ if $n\geqq 3$ and $(A/B)^2 \geqq \{(n-1)^2/n(n-2)\}\cdot\{1-1/(2n^2-4n+1)^2\}$) as a corollary of Theorem III, we will prove it directly.

PROOF OF THEOREM 5.1. For any $i\in I$, let $\rho_i := \rho_{\Sigma_i}$, and $u_i := 1/(\cosh B\rho_i)^\alpha$. By direct computation, we see that

$$-\Delta_g u_i = B^2(\cosh B\rho_i)^{-\alpha-2}$$
$$\cdot\left\{-\alpha^2(\cosh B\rho_i)^2 + \frac{\alpha}{B}(\cosh B\rho_i)(\sinh B\rho_i)\Delta_g\rho_i + \alpha(\alpha+1)\right\}.$$

Now by (4.3) with $\Sigma = \Sigma_i$, we have

$$\Delta_g\rho_i \geqq B\{(n-m_i-1)(\coth B\rho_i) + m_i(\tanh B\rho_i)\},$$

from which

$$-\Delta_g u_i \geqq B^2(\cosh B\rho_i)^{-\alpha-2}\{-\alpha(\alpha-n+1)(\cosh B\rho_i)^2 + \alpha(\alpha-m_i+1)\}.$$

Set $u_\Sigma := \sum_{i\in I} u_i$, and $\bar{u}_\Sigma := \sup_M u_\Sigma$ which is finite by the assumption (5.1). By using (4.1) and (4.2), we can easily get $u_\Sigma \in C^2(M)$, and $\Delta_g u_\Sigma$ satisfies

$$-\Delta_g u_\Sigma \geqq \sum_{i\in I} B^2(\cosh B\rho_i)^{-\alpha-2}\{-\alpha(\alpha-n+1)(\cosh B\rho_i)^2 + \alpha(\alpha-m_i+1)\}.$$

In the case $n=2$,

$$-\Delta_g u_\Sigma \geqq \sum_{i\in I}(2-m_i)B^2(\cosh B\rho_i)^{-3}>0\,.$$

Since, for any $x\in B_R(\Sigma)$, there exists an $i\in I$ such that $x\in B_R(\Sigma_i)$, it follows that

$$-\Delta_g u_\Sigma \geqq (2-m_i)B^2(\cosh BR)^{-3}\geqq B^2(\cosh BR)^{-3}>0 \qquad \text{in} \quad B_R(\Sigma)\,.$$

In the case $n\geqq 3$,

$$\begin{aligned}-4\frac{n-1}{n-2}\Delta_g u_\Sigma + S_g u_\Sigma &\geqq \sum_{i\in I} 4\frac{n-1}{n-2}B^2(\cosh B\rho_i)^{-\alpha-2}\\ &\quad\cdot\left[-\left\{\alpha^2-(n-1)\alpha+\frac{n(n-2)}{4}\left(\frac{A}{B}\right)^2\right\}(\cosh B\rho_i)^2+\alpha(\alpha-m_i+1)\right]\\ &= \sum_{i\in I} 4\frac{n-1}{n-2}B^2\alpha(\alpha-m_i+1)(\cosh B\rho_i)^{-\alpha-2}>0\,.\end{aligned}$$

Since, for any $x\in B_R(\Sigma)$, there exists an $i\in I$ such that $x\in B_R(\Sigma_i)$, it follows that

$$\begin{aligned}-4\frac{n-1}{n-2}\Delta_g u_\Sigma + S_g u_\Sigma &\geqq 4\frac{n-1}{n-2}B^2\alpha(\alpha-m_i+1)(\cosh BR)^{-\alpha-2}\\ &\geqq 4\frac{n-1}{n-2}B^2\alpha C_3(\cosh BR)^{-\alpha-2}>0 \qquad \text{in} \quad B_R(\Sigma)\,,\end{aligned}$$

where

$$C_3 := \begin{cases} \dfrac{\{(n-1)^2-n(n-2)(A/B)^2\}^{1/2}}{2} & \text{if} \quad \left(\dfrac{A}{B}\right)^2<\dfrac{(n-1)^2}{n(n-2)} \\ \dfrac{1}{2} & \text{if} \quad \left(\dfrac{A}{B}\right)^2=\dfrac{(n-1)^2}{n(n-2)}\,. \end{cases}$$

Now we can prove our assertion by the same method as in the proof of Theorem III. q.e.d.

In Theorem 5.1, if $\sharp I$ is finite, then the condition (5.1) is obviously satisfied, and hence the assertion holds. Even if $\sharp I$ is infinite, we can construct examples satisfying the condition (5.1), which is illustrated in the following:

PROPOSITION 5.2. *In Theorem* 5.1, *the condition* (5.1) *is satisfied provided that* $I=N$, *and that there exists a sequence* $\{D_i\}_{i\in N}$ *of domains of* M *with the following properties*:
(1) $D_i\subset D_j$ *for all* $i<j$,
(2) $d:=\inf_{i\in N} d_g(\partial D_i, \partial D_{i+1})>0$,
(3) Σ_i *is contained in* $\bar{D}_i\setminus D_{i-1}$ *for any* $i\in N$, *where* $D_0:=\varnothing$.

PROOF. For any $i, j \in \boldsymbol{N}$ and $x \in \bar{D}_j \setminus D_{j-1}$,

$$\rho_i(x) = d_g(x, \Sigma_i) \geqq \begin{cases} (j-i-1)d & \text{if } i<j \\ 0 & \text{if } i=j \\ (i-j-1)d & \text{if } i>j . \end{cases}$$

Hence

$$\begin{aligned}\sum_{i \in N} \frac{1}{(\cosh B\rho_i)^\alpha} &\leqq \sum_{i<j} \frac{1}{(\cosh B\rho_i)^\alpha} + \frac{1}{(\cosh B\rho_j)^\alpha} + \sum_{i>j} \frac{1}{(\cosh B\rho_i)^\alpha} \\ &\leqq \sum_{i<j} \frac{1}{[\cosh\{(j-i-1)Bd\}]^\alpha} + \frac{1}{(\cosh 0)^\alpha} + \sum_{i>j} \frac{1}{[\cosh\{(i-j-1)Bd\}]^\alpha} \\ &< 1 + 2\sum_{i \in N} \frac{1}{[\cosh\{(i-1)Bd\}]^\alpha} \\ &< 1 + \frac{2^{\alpha+1}}{1-e^{-Bd\alpha}} < +\infty .\end{aligned}$$

Now the condition (5.1) is satisfied. q.e.d.

When $M = \boldsymbol{H}^2$ or $\boldsymbol{H}^3$, we can construct examples having the properties above with $\partial D_i = \Sigma_i$ for any $i \in \boldsymbol{N}$.

In Section 7, the idea of Proposition 5.2 will be applied to the case where M is not simply connected.

Now we replace the assumption on $m_i = \dim \Sigma_i$ in Theorem 5.1 by another.

THEOREM 5.3. *Let (M, g), Σ and f be as in Theorem* III *(without the assumption on A/B). Suppose Σ is a totally geodesic submanifold of M with $m = \dim \Sigma \leqq n-1$, and f satisfies the condition* (H.2) *with $R < R_0/B$, where R_0 is a positive constant* (*or* $+\infty$) *depending only on A/B, m, and n. Then the equation* (*n) *possesses a bounded smooth solution* (*which is also bounded away from zero if $n \geqq 3$*).

PROOF. It is enough to prove the case $n \geqq 3$. Set $u_\Sigma := 1/(\cosh B\rho_\Sigma)^\alpha$, where a positive number α will be chosen later. By direct computation, we see that

$$-4\frac{n-1}{n-2}\Delta_g u_\Sigma + S_g u_\Sigma \geqq 4\frac{n-1}{n-2}B^2(\cosh B\rho_\Sigma)^{-\alpha-2}\{-F(\alpha)(\cosh B\rho_\Sigma)^2 + \alpha(\alpha-m+1)\} ,$$

where

$$F(\alpha) := \alpha^2 - (n-1)\alpha + \frac{n(n-2)}{4}\left(\frac{A}{B}\right)^2 .$$

Set

$$R_0 := \sup\left\{\mathrm{Cosh}^{-1}\sqrt{\frac{\alpha(\alpha-m+1)}{F(\alpha)}}\,\middle|\, F(\alpha)>0,\ \alpha>\frac{n(n-2)}{4(n-m)}\left(\frac{A}{B}\right)^2\right\}>0\,.$$

Now since $R<R_0/B$, there exists some α such that

$$BR<R_1 := \mathrm{Cosh}^{-1}\sqrt{\frac{\alpha(\alpha-m+1)}{F(\alpha)}}\leqq R_0\,.$$

For any $x\in B_R(\Sigma)$,

$$[\cosh\{B\rho_\Sigma(x)\}]^2\leqq(\cosh BR)^2=\left(\frac{\cosh BR}{\cosh R_1}\right)^2\cdot\frac{\alpha(\alpha-m+1)}{F(\alpha)}\,,$$

and hence

$$\begin{aligned}&-4\frac{n-1}{n-2}\Delta_g u_\Sigma+S_g u_\Sigma\\ &\geqq 4\frac{n-1}{n-2}B^2(\cosh BR)^{-\alpha-2}\left\{1-\left(\frac{\cosh BR}{\cosh R_1}\right)^2\right\}\alpha(\alpha-m+1)>0 \qquad \text{in}\quad B_R(\Sigma)\,.\end{aligned}$$

On the other hand, we have

$$-4\frac{n-1}{n-2}\Delta_g u_\Sigma+S_g u_\Sigma>-4\frac{n-1}{n-2}B^2F(\alpha)\qquad \text{on}\quad M\,.$$

Hence we can prove our assertion by the same method as in the proof of Theorem III. q.e.d.

When $m\leqq(n+1)/2$, we can easily see

$$R_0=+\infty\qquad \text{if}\quad \left(\frac{A}{B}\right)^2\leqq\frac{(n-1)^2}{n(n-2)}\,,$$

and

$$R_0\to+\infty\qquad \text{as}\quad \left(\frac{A}{B}\right)^2\to\frac{(n-1)^2}{n(n-2)}+0\,.$$

6. The case $M=H^2$. In this section, we consider the case $n=2$. Under the condition (1.1), if $n=2$, then by the Ahlfors-Schwarz Lemma (cf. [1]), (M,g) is conformally and uniformly equivalent to the hyperbolic plane $H^2=H^2(-1)$. Hence we restrict our attention to the case $M=H^2$.

Now we provide a certain necessary condition for the same assertion as in Theorem III to hold. We begin with the following:

LEMMA 6.1. *If there exists a bounded solution of the equation* (*2) *on* H^2, *then for* *any* $1<\alpha\leqq 2\sqrt{2}$ *and* $x\in \boldsymbol{H}^2$,

$$\int_{\boldsymbol{H}^2}\frac{f(y)}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}dy<0\,. \tag{6.1}$$

PROOF. Under the assumption above, [2, Theorem 2] showed that if we regard $\boldsymbol{H}^2$ as the Poincaré disk D, then $\int_D f dv_0<0$, where dv_0 is the volume element with respect to the flat metric. By the same method, we can show that $\int_D f(1-r^2)^{\alpha-2}dv_0<0$ for any $1<\alpha\leqq 2\sqrt{2}$, where r is the distance to the origin with respect to the flat metric. Now if we regard $x\in\boldsymbol{H}^2$ as the origin of D, and use the hyperbolic distance d_g, then we get the condition (6.1). q.e.d.

THEOREM 6.2. *Let Σ be a subset of $\boldsymbol{H}^2$, and f a bounded function $\boldsymbol{H}^2$. Suppose Σ satisfies $B_\delta(\Sigma')\subset\Sigma$ for a positive number δ and a measurable subset Σ' of $\boldsymbol{H}^2$,*

$$\sup_{x\in\boldsymbol{H}^2}\int_{\Sigma'}\frac{dy}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}=\frac{4\pi}{2^{\alpha}(\alpha-1)} \tag{6.2}$$

with some $1<\alpha\leqq 2\sqrt{2}$, and $f\geqq\varepsilon$ on Σ' for a positive number ε. Then the equation $(*2)$ possesses no bounded smooth solution.

PROOF. Clearly, there is a positive number a such that $f\geqq -a^2$ on $\boldsymbol{H}^2$. If the equation $(*2)$ possesses a bounded solution u, then, from Lemma 6.1,

$$\begin{aligned}
0&>\int_{\boldsymbol{H}^2}\frac{f(y)}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}dy\\
&\geqq\int_{\Sigma'}\frac{\varepsilon}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}dy-\int_{\boldsymbol{H}^2\setminus\Sigma'}\frac{a^2}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}dy\\
&=(\varepsilon+a^2)\int_{\Sigma'}\frac{dy}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}-a^2\int_{\boldsymbol{H}^2}\frac{dy}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}\\
&=(\varepsilon+a^2)\int_{\Sigma'}\frac{dy}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}-a^2\frac{4\pi}{2^{\alpha}(\alpha-1)}
\end{aligned}$$

for any $x\in\boldsymbol{H}^2$. Hence

$$\int_{\Sigma'}\frac{dy}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}<\frac{a^2}{\varepsilon+a^2}\cdot\frac{4\pi}{2^{\alpha}(\alpha-1)}\,.$$

Finally we have

$$\sup_{x\in\boldsymbol{H}^2}\int_{\Sigma'}\frac{dy}{[\cosh\{d_g(x,y)\}+1]^{\alpha}}\leqq\frac{a^2}{\varepsilon+a^2}\cdot\frac{4\pi}{2^{\alpha}(\alpha-1)}<\frac{4\pi}{2^{\alpha}(\alpha-1)}\,.$$

This contradicts the assumption (6.2). q.e.d.

Observe $4\pi/2^{\alpha}(\alpha-1)\to+\infty$ as $\alpha\to 1+0$. Moreover,

$$\int_{\Sigma'}\frac{dy}{\cosh\{d_g(x, y)\}+1}<+\infty\,,$$

if and only if

$$\int_{\Sigma'}\frac{dy}{\cosh\{d_g(x, y)\}}<+\infty\,.$$

In this sense, the sufficient condition (H.1) in Theorem III is sharp.

EXAMPLE 6.3. Let Σ be a subset of H^2. Suppose there exists a horocyclic region $U\subset\Sigma$. Then, by replacing U, we may assume $B_\delta(U)\subset\Sigma$ with a positive number δ. Now, for any positive number R, there exists an $x_R\in U$ satisfing $B_R(x_R)\subset U$. Hence

$$\int_U\frac{dy}{[\cosh\{d_g(x_R, y)\}+1]^\alpha}>\int_{B_R(x_R)}\frac{dy}{[\cosh\{d_g(x_R, y)\}+1]^\alpha}\,,$$

from which

$$\begin{aligned}\sup_{x\in H^2}\int_U\frac{dy}{[\cosh\{d_g(x, y)\}+1]^\alpha}&\geq\lim_{R\to+\infty}\int_{B_R(x_R)}\frac{dy}{[\cosh\{d_g(x_R, y)\}+1]^\alpha}\\&=\int_{H^2}\frac{dy}{[\cosh\{d_g(\cdot\,, y)\}+1]^\alpha}=\frac{4\pi}{2^\alpha(\alpha-1)}\,.\end{aligned}$$

Namely, the condition (6.2) is satisfied for $\Sigma'=U$, hence the same assertion as in Theorem III does not hold for Σ.

On the other hand, in the following case, the same conclusion as in Theorem III holds.

EXAMPLE 6.4. Suppose Σ is a horocycle of H^2, and f satisfies the condition (H.2). Then the equation (*2) possesses a bounded smooth solution. Indeed, let Σ' be a horocycle which is the component of $\partial B_R(\Sigma)$ contained in the smaller component of $H^2\setminus\Sigma$. Denote the Busemann function with respect to a point on Σ' and the end point of Σ' by ρ. Then we get $B_R(\Sigma)=\{x\in H^2\,|\,0<\rho(x)<2R\}$, $|\nabla_g\rho|\equiv 1$, and $\Delta_g\rho\equiv 1$. Let $u_\Sigma:=1/\cosh\rho$. By direct computation, we see that

$$\begin{aligned}-\Delta_g u_\Sigma&=(\cosh\rho)^{-3}\{-(\cosh\rho)^2+(\cosh\rho)(\sinh\rho)\Delta_g\rho+2\}\\&=(\cosh\rho)^{-3}\{-(\cosh\rho)^2+(\cosh\rho)(\sinh\rho)+2\}\\&=(\cosh\rho)^{-3}(3-e^{-2\rho})/2\,.\end{aligned}$$

Since $0<\rho(x)<2R$ for any $x\in B_R(\Sigma)$,

$$-\Delta_g u_\Sigma>(\cosh\rho)^{-3}>(\cosh 2R)^{-3}>0\qquad\text{in}\quad B_R(\Sigma)\,.$$

On the other hand, we have,

$$-\Delta_g u_\Sigma \geqq \inf\{(\cosh \rho)^{-3}(3-e^{-2\rho})/2 \,|\, 3-e^{-2\rho}<0\}$$

$$> -4\sup\left\{e^\rho \,\middle|\, \rho < -\frac{1}{2}\log 3\right\} = -\frac{4}{\sqrt{3}} \qquad \text{on} \quad \boldsymbol{H}^2 .$$

Hence we can prove our assertion by the same method as in the proof of Theorem III.

REMARK 6.5. We can give an example similar to that above when M is the hyperbolic space $\boldsymbol{H}^3$ and Σ is a horosphere.

Next, as a generalization of [2, Theorem 3] on the behavior of a solution of the equation (*2), we get the following:

THEOREM 6.6. *Let Σ be a subset of $\boldsymbol{H}^2$, and f a bounded smooth function on $\boldsymbol{H}^2$. Suppose Σ is the union of a finite family $\{\Sigma_i\}_{i\in I}$ of complete geodesics of $\boldsymbol{H}^2$, and f satisfies the following condition*:

(H.2′) $f \leqq 0$, *and*

$$|f+2b^2| \leqq C \sum_{i\in I} e^{-\alpha\rho_{\Sigma_i}}$$

for positive constants b, C and $\alpha<1$.

*Then the equation (*2) possesses a bounded smooth solution u which has the following property*:

$$\text{(H.3)} \qquad |u+2\log b| \leqq C' \sum_{i\in I} e^{-\alpha\rho_{\Sigma_i}}$$

for a positive constant C'.

PROOF. Let ρ_i, u_i and u_Σ be as in the proof of Theorem 5.1. Then

$$-\Delta_g u_\Sigma = \sum_{i\in I} (\cosh \rho_i)^{-\alpha-2}\{\alpha(1-\alpha)(\cosh \rho_i)^2 + \alpha^2\}$$

$$> \alpha(1-\alpha)\sum_{i\in I} (\cosh \rho_i)^{-\alpha} > \alpha(1-\alpha)\sum_{i\in I} e^{-\alpha\rho_i} .$$

Set $u_\pm := \pm\beta u_\Sigma - 2\log b$, where $\beta := b^{-2}C/\alpha(1-\alpha)$. Now we get

$$-\Delta_g u_+ + S_g - fe^{u_+} = -\beta\Delta_g u_\Sigma - 2 - fb^{-2}e^{\beta u_\Sigma} \geqq -\beta\Delta_g u_\Sigma - 2 - fb^{-2}$$

$$= -\beta\Delta_g u_\Sigma - b^{-2}(2b^2+f) > \{\beta\alpha(1-\alpha) - b^{-2}C\}\sum_{i\in I} e^{-\alpha\rho_i} = 0 .$$

On the other hand, we get

$$-\Delta_g u_- + S_g - f e^{u_-} = \beta\Delta_g u_\Sigma - 2 - f b^{-2} e^{-\beta u_\Sigma} \leqq \beta\Delta_g u_\Sigma - 2 - f b^{-2}$$
$$= \beta\Delta_g u_\Sigma - b^{-2}(2b^2 + f) < \{-\beta\alpha(1-\alpha) + b^{-2}C\} \sum_{i\in I} e^{-\alpha\rho_i} = 0 .$$

Hence u_+ and u_- are respectively a supersolution and a subsolution of the equation $(*2)$. Since $u_- \leqq u_+$, by the method of supersolutions and subsolutions, the equation $(*2)$ possesses a bounded solution u satisfying $u_- \leqq u \leqq u_+$. Namely, u satisfies the estimate (H.3). q.e.d.

Under the assumption of Theorem 6.6, we do not have much information on u at $\Sigma(\infty)$, but we see

$$u \to -2\log b \qquad \text{as} \quad x \to \boldsymbol{H}^2(\infty) \setminus \Sigma(\infty) .$$

7. The case $M = \boldsymbol{H}^2/\Gamma$. In this section, we consider the case where $M = \boldsymbol{H}^2/\Gamma$ is not simply connected. First, from Theorem 5.1 and Example 6.4, we immediately get the following:

COROLLARY 7.1. *Let Σ be a subset of $M = \boldsymbol{H}^2/\Gamma$, and f a bounded smooth function on M. Suppose $\Gamma \cong \boldsymbol{Z}$, Σ is compact, and f satisfies the condition* (H.2). *Then the equation* $(*2)$ *possesses a bounded smooth solution.*

PROOF. There are two cases. When Γ is a hyperbolic subgroup of Isom($\boldsymbol{H}^2$), let Σ be the lift of the minimal closed geodesic of M, and when Γ is a parabolic subgroup of Isom($\boldsymbol{H}^2$), let Σ be a suitable horocycle on M. In both cases, since $u_\Sigma = 1/\cosh\{d_g(\cdot, \Sigma)\}$ on $\tilde{M} = \boldsymbol{H}^2$ is Γ-invariant, we can regard u_Σ as a function on M. Hence, by the method of supersolutions and subsolutions, we get a bounded solution of the equation $(*2)$ also on M. q.e.d.

Now we consider the case where Γ is purely hyperbolic.

DEFINITION 7.2. Let $(M = \boldsymbol{H}^2/\Gamma, g)$ be a complete, noncompact, oriented surface which is finitely connected with h handles and e ends. Set

$$d_\Gamma := \sup_{\{T_i\}} \left[\min_{i \neq j} \{d_{\tilde{g}}(T_i, T_j)\} \right] ,$$

where $\{T_i\}_{i=1}^N$ runs through families of complete geodesics of $\tilde{M} = \boldsymbol{H}^2$ which bounds a fundamental domain of Γ, $N := 2(2h + e - 1)$, and $\tilde{g}$ is the standard metric on $\boldsymbol{H}^2$.

It is easy to verify that $d_\Gamma > 0$ if and only if Γ is purely hyperbolic.

THEOREM 7.3. *Let (M, g) be as in Definition 7.2, Σ a subset of M, and f a bounded smooth function on M. Suppose $d_\Gamma > \log(N-1)$, Σ is the union of a finite family $\{\Sigma_i\}_{i\in I}$ of complete geodesics of M, and f satisfies the condition* (H.2). *Then the equation* $(*2)$ *possesses a bounded smooth solution.*

PROOF. Let $\{T_i\}_{i=1}^N$ be a family of complete geodesics of $\tilde{M}=\boldsymbol{H}^2$ which bounds a fundamental domain Ω of Γ, and $d:=\min_{i\neq j} d_{\tilde{g}}(T_i, T_j)>\log(N-1)$, that is $(N-1)e^{-d}<1$. Without loss of generality, we may assume that some lift $\tilde{\Sigma}_i$ of Σ_i is contained in $\bar{\Omega}$ for any $i\in I$. Now, since

$$\#\{\gamma\in\Gamma \mid d_{\tilde{g}}(\gamma\tilde{x}, \bar{\Omega})\leqq ld\}\leqq 1+\sum_{j=1}^{l} N(N-1)^{j-1}$$

for any $\tilde{x}\in\boldsymbol{H}^2$ and every $l\in N$, it is clear that

$$\sum_{\gamma\in\Gamma}\frac{1}{\cosh\{d_{\tilde{g}}(\gamma\tilde{x}, \tilde{\Sigma}_i)\}}\leqq 1+\sum_{j=1}^{\infty}\frac{N(N-1)^{j-1}}{\cosh\{(j-1)d\}}\leqq 1+2N\sum_{j=1}^{\infty}\{(N-1)e^{-d}\}^{j-1}$$
$$=1+\frac{2N}{1-(N-1)e^{-d}}<+\infty$$

for any $i\in I$. Hence we can define

$$u_\Sigma(\tilde{x}):=\sum_{i\in I}\sum_{\gamma\in\Gamma}\frac{1}{\cosh\{d_{\tilde{g}}(\gamma\tilde{x}, \tilde{\Sigma}_i)\}}$$

on $\boldsymbol{H}^2$. Since u_Σ is Γ-invariant, we can regard u_Σ as a function on $M=\boldsymbol{H}^2/\Gamma$. Hence by the method of supersolutions and subsolutions, we get a bounded solution of the equation $(*2)$ on M. q.e.d.

The hyperbolic case of Corollary 7.1 is obtained also as a corollary to Theorem 7.3. Indeed, in this case, we have $N=2$ since $h=0$ and $e=2$. Hence $\log(N-1)=0$.

EXAMPLE 7.4. Let D $(=\boldsymbol{H}^2)$ be the Poincaré disk, and $\{T_i\}_{i=1}^8$ a family of circular arcs in D which are orthogonal to ∂D $(=\boldsymbol{H}^2(\infty))$ (those are geodesics of $(\boldsymbol{H}^2, \tilde{g})$) as in

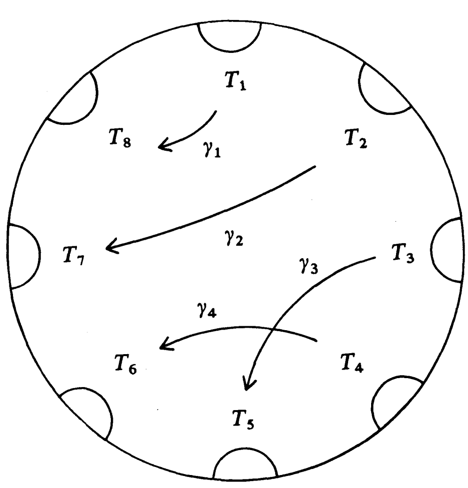

FIGURE.

Figure. We can easily take $\{T_i\}_{i=1}^8$ satisfying $d = \min_{i \neq j} d_{\tilde{g}}(T_i, T_j) > \log 7$. (In fact, we can take it for arbitrarily large d.) Let γ_1 be the hyperbolic isometry (with respect to $\tilde{g}$) such that $\gamma_1(T_1) = T_8$, and that the axis is the geodesic orthogonal to T_1 and T_8. Define γ_2 similarly by T_2 and T_7, γ_3 by T_3 and T_5, and γ_4 by T_4 and T_6. Suppose that Γ is the purely hyperbolic subgroup of $\mathrm{Isom}(\boldsymbol{H}^2, \tilde{g})$ generated by γ_1, γ_2, γ_3 and γ_4 (see, for instance, [4]). Then clearly $M = \boldsymbol{H}^2/\Gamma$ has one handle and three ends. Hence $N - 1 = 7$, and M satisfies the assumption of Theorem 7.3.

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