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CLASSIFICATION OF NON-RIGID FAMILIES OF ABELIAN VARIETIES

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Abstract. We will give a complete classification of non-rigid families of abelian varieties by means of the endomorphism algebra of the variation of Hodge structure. As a consequence, we can obtain several conditions of rigidity for abelian schemes. For example, we show that an abelian scheme which has no isotrivial factor is rigid if the relative dimension is less than 8. Moreover, examples of non-rigid abelian schemes are obtained as Kuga fiber spaces associated to symplectic representations classified by Satake.

Introduction. Let Y be an algebraic curve defined over an algebraically closed field k of characteristic zero, and let $\Sigma \subset Y$ be a finite set of points. Faltings [F] has shown a theorem of Arakelov-type for abelian varieties, that is, there are only finitely many families of principally polarized abelian varieties of relative dimension g on Y, with good reduction outside Σ, and satisfying the condition $(*)$ in [F].

His proof consists of two ingredients. First he showed that the moduli space of families of principally polarized abelian varieties on Y with good reduction outside Σ is a scheme of finite type over k (a boundedness result). Next he proved that a family of abelian varieties cannot be deformed (i.e., a family is rigid) if and only if the condition $(*)$ is satisfied.

The condition $(*)$ says essentially that all endomorphisms of the local system of the first (co-)homology groups of fibers come from endomorphism of the abelian varieties, and Deligne [D] has shown that the condition is satisfied by a family of abelian varieties which has no isotrivial factors and the relative dimension ≤ 3.

On the other hand, following Deligne's suggestion, Faltings [F] gave an example of non-rigid families of abelian varieties with relative dimension 8 which has no isotrivial factors. So it is interesting to ask, for example, whether there exists a non-rigid family of abelian varieties of relative dimension d, $4 \leq d \leq 7$, which has no isotrivial factors.

In this paper, we will give a complete classification of non-rigid families of abelian varieties by means of the endomorphism algebra of the variation of Hodge structure of the first homology (or cohomology) groups of the fibers.

Let S be a connected smooth quasi-projective variety over $\mathbb{C}$, and $f: X \to S$ an

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abelian scheme over S. Consider the local system $\boldsymbol{W}_{\mathbb{Z}}:=\boldsymbol{R}_1 f_* \mathbb{Z}_X$ of free $\mathbb{Z}$-modules, which come from the first homology groups of fibers. Then $\boldsymbol{W}_{\mathbb{Z}}$ underlies a (polarized) variation of Hodge structure (VHS) of weight -1, and of type $(-1, 0)$, $(0, -1)$. On the other hand, if a polarized VHS of weight -1, and of type $(-1, 0)$, $(0, -1)$ on S is given, one has the corresponding abelian scheme on S. The algebra $\boldsymbol{E}:= H^0(S, \mathcal{E}nd(\boldsymbol{W}_{\mathbb{Q}}))$, consisting of all flat global endomorphisms of $\boldsymbol{W}_{\mathbb{Q}}:=\boldsymbol{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, has a natural pure Hodge structure of weight 0. Let Q denote a symplectic bilinear form on $\boldsymbol{W}_{\mathbb{Z}}$ induced by a polarization of the abelian scheme. Let us denote by $\boldsymbol{E}^Q$ the subalgebra of $\boldsymbol{E}$ which consists of all skew endomorphisms with respect to Q. Then the abelian scheme satisfies the condition $(*)$, if $\boldsymbol{E}^Q \otimes \mathbb{C} = (\boldsymbol{E}^Q \otimes \mathbb{C})^{(0,0)}$, i.e., all skew endomorphisms of $\boldsymbol{W}_{\mathbb{Q}}$ are of type $(0, 0)$. More precisely, the Zariski tangent space of the moduli space of abelian schemes over S with a fixed polarization type is isomorphic to the space $(\boldsymbol{E}^Q \otimes_{\mathbb{Q}} \mathbb{C})^{-1,1}$. Therefore, in order to classify non-rigid abelian scheme over a fixed base space S, we only classify polarized VHS's of weight -1 and of type $(-1, 0)$, $(0, -1)$ such that $\dim(\boldsymbol{E}^Q \otimes \mathbb{C})^{-1,1} > 0$.

We have a primary decomposition of $\boldsymbol{W}_{\mathbb{Q}}$ (cf. §3), and each primary component is a $\mathbb{Q}$-subVHS over S, hence we can reduce the problem to the primary $\mathbb{Q}$-VHS. (We can see that if the generic fiber of the corresponding abelian scheme is simple then $\boldsymbol{W}_{\mathbb{Q}}$ is primary.)

Let us assume that $\boldsymbol{W}_{\mathbb{Q}} = \boldsymbol{R}_1 f_* \mathbb{Q}_X$ is a primary $\mathbb{Q}$-VHS over S, (and of weight -1, and of type $(0, -1)$, $(-1, 0)$). Denote by $\boldsymbol{V}$ an irreducible $\mathbb{Q}$-local subsystem of $\boldsymbol{W}_{\mathbb{Q}}$, and set

$$D = \mathrm{End}(\boldsymbol{V}), \quad F = \mathrm{Cent}\, D, \quad \boldsymbol{U} = \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{W}_{\mathbb{Q}}).$$

By Schur's lemma, D is a division algebra over $\mathbb{Q}$, and the polarization Q on $\boldsymbol{W}_{\mathbb{Q}}$ induces an involultion ι on D. The center F of D is stable under ι, and let F^+ denote the subfield of F fixed by ι. From the positivity condition of Q, one can deduce that ι is a positive involution on F, hence F is either (i) a totally real number field and $F = F^+$, or (ii) a CM field and $[F : F^+] = 2$.

Moreover we have a tensor product decomposition of $\boldsymbol{W}_{\mathbb{Q}}$

$$\boldsymbol{W}_{\mathbb{Q}} \cong \boldsymbol{U} \otimes_D \boldsymbol{V}$$

(see (3.11)).

The main theorem of this paper, which will give a classification of non-isotrivial, non-rigid, abelian schemes, can be stated as follows.

(0.1) Theorem (cf. Theorem (8.1)). *Let $f: X \to S$ be an abelian scheme such that the corresponding $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}} = \boldsymbol{R}_1 f_* \mathbb{Q}_X$ is primary (e.g., the generic fiber X_η of f is simple). Let $\boldsymbol{W}_{\mathbb{Q}} = \boldsymbol{U} \otimes_D \boldsymbol{V}$ be the tensor product decomposition of $\boldsymbol{W}_{\mathbb{Q}}$ as above. Set* $\mathrm{rank}_D\, \boldsymbol{U} = m$, $\mathrm{rank}_D\, \boldsymbol{V} = n$, *and* $t = [F^+ : \mathbb{Q}]$. *Assume that $f: X \to S$ is non-isotrivial and non-rigid.*

(i) *If the center F of D is totally real (i.e., $F = F^+$), then D is a quaternion algebra* *over $F=F^+$ such that*

$$D\otimes_{\mathbb{Q}}\mathbb{R}\cong\underbrace{\mathbb{H}\times\cdots\times\mathbb{H}}_{t'}\times\underbrace{M_2(\mathbb{R})\times\cdots\times M_2(\mathbb{R})}_{t-t'},$$

where $\mathbb{H}$ denotes the Hamilton quaternion algebra.

Hence if one denotes by $r(f)$ the relative dimension of $f: X\to S$, one has

$$r(f)=\frac{1}{2}\cdot\mathrm{rank}_{\mathbb{Q}}(U\otimes_D V)=2tmn. \tag{0.2}$$

Here, one must have $t'>0$ and $t-t'>0$, hence in particular $t=[F:\mathbb{Q}]\geq 2$, and one of the following cases occurs.

Case (R2, 1) $\quad n\geq 1$ *and* $m\geq 2$,

Case (R2, -1) $\quad n\geq 2$ *and* $m\geq 1$.

In particular, the relative dimension $r(f)$ is even, and ≥ 8.

(ii) *If the center F of D is a* CM *field (i.e., $[F: F^+]=2$), then D is a central simple division algebra over F such that $[D: F]=r^2$ and*

$$D\otimes_{\mathbb{Q}}\mathbb{R}\cong\underbrace{M_r(\mathbb{C})\times\cdots\times M_r(\mathbb{C})}_{t}.$$

In this case, one has

$$r(f)=\frac{1}{2}\cdot 2tnmr^2=t(nr)(mr), \tag{0.3}$$

and $t=[F^+:\mathbb{Q}]=(1/2)[F:\mathbb{Q}]\geq 2$, $nr\geq 2$, $mr\geq 2$. In particular, $r(f)\geq 8$.

From this one can obtain the following:

(0.4) Corollary (cf. Corollary (8.4)). *Let $f: X\to S$ be an abelian scheme which has no isotrivial factors. If the relative dimension $r(f)$ of f is less than* 8, *the abelian scheme is rigid.*

(0.5) Corollary (cf. Corollary (8.5)). *Let $f: X\to S$ be an abelian scheme whose generic fiber X_η is simple. Assume that f has no-isotrivial factor and the relative dimension of f is a prime integer. Then $f: X\to S$ is rigid.*

On the other hand, as a by-product of the proof of Theorem (0.1), we can obtain the following result, which we call the *monodromy theorem*.

(0.6) Theorem (cf. Theorem (8.6)). *Let $f: X\to S$ be an abelian scheme such that the corresponding $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}}=\boldsymbol{R}_1 f_*\mathbb{Q}_X$ is primary (e.g., the generic fiber of f is simple). Assume that S is non-compact and a local monodromy around a point in the boundary has infinite order. Then $f: X\to S$ is rigid.*

The organization of this paper is as follows. In § 1 and § 2, we shall review some fundamental facts on VHS, abelian schemes, and deformation theory of VHS, or abelian schemes. In § 3, we shall study the structure of the $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}}=\boldsymbol{R}_1 f_* \mathbb{Q}_X$ and its endomorphism algebra $\boldsymbol{E}$. We will introduce a tensor product decomposition of a primary $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}}$ following Satake [S1], [S2]. In § 4, the decomposition of the polarization Q on $\boldsymbol{W}_{\mathbb{Q}}$ will be introduced, which is also due to Satake. In § 5, we shall investigate the scalar extension of a primary $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}}$ and a polarization Q. In § 6, we shall introduce the notion of $\mathbb{Q}$-symplectic representation of a $\mathbb{Q}$-Hermitian pair $(G_{\mathbb{Q}}, H_0)$ due to Satake [S1], [S2], and show our fundamental result, i.e., Theorem 6.17. This theorem says that to each $\mathbb{Q}$-primary VHS $\boldsymbol{W}_{\mathbb{Q}}$ on S we can associate two $\mathbb{Q}$-Hermitian pairs $(G_{\mathbb{Q}}, H_0)$, $(G'_{\mathbb{Q}}, H'_0)$ and their $\mathbb{Q}$-symplectic representations. One can show that $\mathbb{Q}$-primary VHS $\boldsymbol{W}_{\mathbb{Q}}$ is non-rigid if and only if the $\mathbb{R}$-valued point $G'_{\mathbb{R}}$ of $G'_{\mathbb{Q}}$ is non-compact. (See Theorem (6.21) and Corollary (6.23).) In § 7, we shall review the classification of $\mathbb{Q}$-primary sympletic representations of $\mathbb{Q}$-Hermitian pair $(G_{\mathbb{Q}}, H_0)$ due to Satake [S1], [S2]. Then in § 8, we shall obtain our main results. In § 9, we shall give an examples of non-rigid abelian schemes. Such examples are constructed by Kuga fiber spaces of abelian varieties associated to $\mathbb{Q}$-symplectic representations. These examples show that Theorem (0.1) (=Theorem (8.1)) is best possible, or complete.

Here are some remarks on works related to our results. Naturally, Faltings [F] and Peters [P] are starting points of this paper. Besides these works, Noguchi [N], which studied the structure of the space $\mathrm{Hol}(S, \Gamma\backslash\mathscr{D})$ of the holomorphic mapping from a Zariski open set S of a compact complex manifold to the arithmetic quotient of a Hermitian symmetric space, is another motivation for this work. (There are also other previous works due to Kuga-Ihara [K-I], and Sunada [Su1], [Su2] when $\Gamma\backslash\mathscr{D}$ and S is compact.) Actually, he showed that $\mathrm{Hol}(S, \Gamma\backslash\mathscr{D})$ is a quasi-projective variety whose irreducible components are also arithmetic quotients of Hermitian symmetric spaces. We can deduce the boundedness of $\mathrm{VH}^{-1}_{g,\delta}(S)$ from his result. (Note that this follows from Faltings' original theorem in [F] or a result due to Deligne in [D2].) Moreover Noguchi [N] obtained some interesting results on the rigidity of holomorphic mapping in $\mathrm{Hol}(S, \Gamma\backslash\mathscr{D})$. One can regard our Theorem (6.17) as a refinement of his results.

It is obvious that the work on $\mathbb{Q}$-symplectic representations and Kuga fiber spaces of Satake [S1], [S2] is essential for our work. In fact, he considered the rigidity of $\mathbb{Q}$-symplectic representation in § 4 and § 6, Ch. IV of [S1]. After getting Theorem (6.17), the classification of non-rigid families is reduced to his classification of $\mathbb{Q}$-symplectic representations. Some of these non-rigid $\mathbb{Q}$-symplectic representations and corresponding Kuga fiber spaces (i.e., Kuga fiber spaces of type (R2, ± 1)) were first studied by Shimura [Sh3], in which he has already remarked that such Kuga fiber spaces have non-holomorphic real-analytic endomorphisms. Similar classification of non-rigid families of K3 surfaces were carried out by Saito-Zucker in [S-Z].

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1. VHS and abelian schemes.

(1.1) VHS. Let S be a connected smooth qausi-projective variety defined over $\mathbb{C}$.

(1.2) DEFINITION. A *polarized $\mathbb{Z}$-variation of Hodge structure* ($\mathbb{Z}$-VHS for short) on S of weight -1 (resp. weight 1) and of type $(0, -1)$, $(-1, 0)$ (resp. of type $(1, 0)$, $(0, 1)$) consists of

(i) a local system of free $\mathbb{Z}$-modules $W_{\mathbb{Z}}$ on S,

(ii) a decreasing filtration

$$0=\mathscr{F}^1\subset\mathscr{F}^0\subset\mathscr{F}^{-1}=W_{\mathcal{O}_S}\quad(\text{resp. }0=\mathscr{F}^2\subset\mathscr{F}^1\subset\mathscr{F}^0=W_{\mathcal{O}_S}),$$

of $W_{\mathcal{O}_S}:=W_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathcal{O}_S$ such that

$$\mathscr{F}^0\oplus\overline{\mathscr{F}^0}\cong W_{\mathcal{O}_S}\quad(\text{resp. }\mathscr{F}^1\oplus\overline{\mathscr{F}^1}\cong W_{\mathcal{O}_S}).$$

(For each point $s\in S$, the Weil operator C_s on $W_{\mathbb{R},s}=W_{\mathbb{Z},s}\otimes_{\mathbb{Z}}\mathbb{R}$ is defined by the above Hodge decomposition so that

$$C_s u=\sqrt{-1}u\qquad\text{for}\quad u\in\mathscr{F}_s^0\ (\text{resp. }\mathscr{F}_s^1).)$$

(iii) A flat $\mathbb{Z}$-valued symplectic non-degenerate bilinear from Q on $W_{\mathbb{Z}}$ such that the form $Q_s(x, C_s y)$ on $W_{\mathbb{R},s}$ is symmetric and positive definite, which we write symbolically

$$Q_s\cdot C_s\gg 0. \tag{1.3}$$

Let $(W_{\mathbb{Z}}, \{F^p\}, Q)$ be a polarized VHS over S, and s a geometric point S. Then by choosing a basis of $W_{\mathbb{Z},s}$, we can transform the symplectic bilinear form Q_s into the following standard form

$$J(\delta)=\begin{pmatrix} 0 & \begin{matrix}\delta_1 & & \\ & \ddots & \\ & & \delta_g\end{matrix} \\ \begin{matrix}-\delta_1 & & \\ & \ddots & \\ & & -\delta_g\end{matrix} & 0 \end{pmatrix}, \tag{1.4}$$

where $g=(1/2)\,\mathrm{rank}\, W_{\mathbb{Z}}$ and $\delta=\{\delta_1, \cdots, \delta_g\}$ is a sequence of positive integers such that

$$\delta_1 \,|\, \delta_2 \,|\, \cdots \,|\, \delta_g . \tag{1.5}$$

The sequence δ of integers does not depend on the choice of points $s \in S$. We say that such a polarization Q is of type δ, and we set

$$d=\prod_{i=1}^{g} \delta_i , \tag{1.6}$$

which is *the degree of the polarization.*

We denote by $\mathrm{VH}^{-1}_{g,\delta}(S)$ the set of isomorphism classes of $\mathbb{Z}$-VHS $(W_{\mathbb{Z}}, \{F^p\}, Q)$ over S of weight -1 and of type $(-1, 0)+(0, -1)$ with a local system $W_{\mathbb{Z}}$ of free $\mathbb{Z}$-modules of rank $2g$ and a polarization Q of type δ.

Then this set $\mathrm{VH}^{-1}_{g,\delta}(S)$ has a natural analytic structure (see e.g., [S-Z, §3]). Moreover, $\mathrm{VH}^{-1}_{g,\delta}(S)$ turns out to be a quasi-projective variety with only quotient singularities (cf. [F], [N] or [S-Z]).

(1.7) Abelian schemes. Let S be as above. An abelian scheme over S is a smooth proper group scheme $f: X \to S$ of finite type with connected geometric fibers. By definition, each geometric fiber is a proper group variety over $\mathbb{C}$, hence is an abelian variety. Since S is smooth, by a theorem of Grothendieck [R, Théorème XI, 1.4], X is projective over S. Therefore the dual abelian scheme $f^\vee : X^\vee = \mathrm{Pic}^0(X/S) \to S$ exists (see e.g., [M-F, Cor. 6.8]). A polarization is an S-homomorphism $\lambda: X \to X^\vee$ such that for any geometric point $s \in S$, the induced homomorphism $\lambda_s: X_s \to X_s^\vee$ is of the form $\lambda_s = \phi_{\mathscr{L}_s}$ for some ample invertible sheaf $\mathscr{L}_s$ on X_s, where $\phi_{\mathscr{L}_s}$ is given by the formula

$$X_s \ni a \mapsto \phi_{\mathscr{L}_s}(a) := t_a^* \mathscr{L}_s \otimes \mathscr{L}_s^{-1} \in X_s^\vee . \tag{1.8}$$

Let $\lambda: X/S \to X^\vee/S$ be a polarization as above. Then λ is a surjective homomorphism and $\ker \lambda$ is a finite group scheme whose geometric fibers are isomorphic to $(\mathbb{Z}/\delta_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\delta_g\mathbb{Z})^{\oplus 2}$ where $g = \dim X - \dim S$, and $\delta=\{\delta_1, \delta_2, \cdots \delta_g\}$ is a set of positive integers such that $\delta_1 \,|\, \delta_2 \,|\, \cdots \,|\, \delta_g$. We say that such a polarization is of type δ.

We denote by $A_{g,\delta}(S)$ the set of isomorphism classes of abelian schemes over S of relative dimension g with polarizations of type δ.

(1.9) Equivalence between $A_{g,\delta}(S)$ and $\mathrm{VH}^{-1}_{g,\delta}(S)$. Let $(X/S, \lambda)$ be an abelian scheme over S with a polarization λ of type δ. Denote by $\boldsymbol{R}_1 f_* \mathbb{Z}_X$ (resp. $\boldsymbol{R}^1 f_* \mathbb{Z}_X$) the local system of the first homology (resp. cohomology) groups of the fibers of f. Let us denote by $\mathscr{L}ie(X/S)$ the locally free sheaf on S^{an} which is a pull-back by the zero section of f of the sheaf of the Lie algebras of the fibers. The relative exponential map induces an exact sequence of the sheaf

$$0 \to \boldsymbol{R}_1 f_* \mathbb{Z}_X \to \mathscr{L}ie(X/S) \to \mathscr{O}_S^{\mathrm{an}}(X) \to 0 .$$

Setting

$$\mathscr{F}^0 = \ker\{R_1 f_* \mathbb{Z}_X \otimes_{\mathbb{Z}_S} \mathcal{O}_S^{\mathrm{an}} \to \mathscr{L}ie(X/S)\},$$

we have the Hodge filtration of $R_1 f_* \mathbb{Z}_X \otimes_{\mathbb{Z}_S} \mathcal{O}_S^{\mathrm{an}}$ of weight -1 and of type $(0, -1)$, $(-1, 0)$ (see [D, 4.4.2]). Moreover, one easily sees that

$$\mathscr{L}ie(X/S) \cong \mathrm{Gr}_{\mathscr{F}}^{-1} \cong R^1 f_* \mathcal{O}_X^{\mathrm{an}}.$$

On the other hand, taking higher direct images of the usual exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X^{\mathrm{an}} \to (\mathcal{O}_X^{\mathrm{an}})^\times \to 1$, we get the exact sequence

$$0 \to R^1 f_* \mathbb{Z}_X \to R^1 f_* \mathcal{O}_X^{\mathrm{an}} \to \mathcal{O}_S^{\mathrm{an}}(X^\vee) \to 0,$$

which defines a VHS on $R^1 f_* \mathbb{Z}_X$ of weight 1 and of type $(1, 0)$, $(0, 1)$.

The polarization $\lambda\colon (X/S) \to (X^\vee/S)$ induces a surjective sheaf homomorphism

$$\lambda\colon \mathcal{O}_S(X)^{\mathrm{an}} \to \mathcal{O}_S^{\mathrm{an}}(X^\vee),$$

which can be lifted to an isomorphism between locally free $\mathcal{O}_S^{\mathrm{an}}$-modules

$$\hat{\lambda}\colon \mathscr{L}ie(X/s) \to \mathscr{L}ie(X^\vee/S).$$

This $\hat{\lambda}$ induces an injective homomorphism

$$R_1 f_* \mathbb{Z}_X \to R^1 f_* \mathbb{Z}_X = \mathscr{H}om(R_1 f_* \mathbb{Z}_X, \mathbb{Z}_S),$$

and hence corresponds to a flat bilinear form Q on $R_1 f_* \mathbb{Z}_X$. It is easy to see that Q satisfies the condition (iii) of (1.2) and if the polarization λ is of type δ then the corresponding bilinear form Q is also of type δ. Therefore, we have the natural morphism

$$(1.10) \qquad \Phi_S\colon A_{g,\delta}(S) \to \mathrm{VH}_{g,\delta}^{-1}(S), \quad (X/S, \lambda) \mapsto (R_1 f_* \mathbb{Z}_X, \{F^p\}, Q).$$

Deligne [D] showed the following:

(1.11) PROPOSITION (cf. [D, 4.3.3]). *The morphism Φ_S induces an isomorphism between $A_{g,\delta}(S)$ and $\mathrm{VH}_{g,\delta}^{-1}$.*

2. Deformation of abelian schemes and VHS. Let S be a connected smooth quasi-projective variety over $\mathbb{C}$, $f\colon X \to S$ an abelian scheme over S of relative dimension g, λ its polarization of type δ, and $(W_{\mathbb{Z}} := R_1 f_* \mathbb{Z}, \{F^p\}, Q)$ the corresponding polarized VHS of weight -1, and of type $(0, -1)$, $(-1, 0)$.

It can be proved that the moduli space $A_{g,\delta}(S) \cong \mathrm{VH}_{g,\delta}^{-1}(S)$ defined in §1 has a natural structure of a quasi-projective variety with at most quotient singularities (see [F], [No], and [S-Z]). Due to Faltings [F] and Peters [P], the local analytic structure of $\mathrm{VH}_{g,\delta}^{-1}(S)$ at the point $[W_{\mathbb{Z}}]$ can be described as follows.

Let $E = \mathrm{End}(W_{\mathbb{Q}}) = H^0(S, \mathscr{E}nd(W_{\mathbb{Q}}))$ denote the algebra of the global flat endomorphisms of $W_{\mathbb{Q}} := R_1 f_* \mathbb{Q}$, and $E^Q = \mathrm{End}^Q(W_{\mathbb{Q}})$ the subalgebra of E consisting of the elements skew with respect to the polarization Q. Then by [D], E underlies a pure Hodge structure of weight 0 (see Theorem (3.1) below), i.e., one has a decomposition

$$E \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_p E^{-p,p},$$

such that $\overline{E^{-p,p}} \cong E^{p,-p}$.

(2.1) THEOREM. *Let* $[W_{\mathbb{Z}}] \in \mathrm{VH}^{-1}_{g,\delta}(S)$ *be as above. Then the Zariski tangent space of the local semi-universal deformation space of* $[W_{\mathbb{Z}}]$ *is isomorphic to*

$$(2.2) \qquad (E^Q \otimes_{\mathbb{Q}} \mathbb{C})^{-1,1} = (\mathrm{End}^Q(W_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C})^{-1,1}.$$

The local analytic structure of $\mathrm{VH}^{-1}_{g,\delta}(S)$ *at* $[W_{\mathbb{Z}}]$ *is isomorphic to*

$$(2.3) \qquad (E^Q \otimes_{\mathbb{Q}} \mathbb{C})^{-1,1}/G,$$

where G is a finite group induced by the automorphism group of the given polarized VHS $[W_{\mathbb{Z}}]$.

PROOF. The first assertion is due to Faltings [F] in the case S is a curve, and the result was extended to the case of arbitrary quasi-projective bases by Peters [P]. The rest of the proof is similar to the proof of Theorem (3.5.2) in [S-Z].

(2.4) COROLLARY. *An abelian scheme* $f: X \to S$ *with a polarization* λ *is rigid, that is, has no non-trivial deformation with the base scheme S and the polarization fixed, if and only if*

$$(2.5) \qquad (\mathrm{End}^Q(\boldsymbol{R}_1 f_* \mathbb{Z}_X) \otimes_{\mathbb{Z}} \mathbb{C})^{-1,1} \cong \{0\}.$$

(2.6) REMARK. Since the Hodge types of $\boldsymbol{R}_1 f_* \mathbb{Z}_X \otimes_{\mathbb{Z}_S} \mathbb{C}$ are only $(-1, 0)$ and $(0, -1)$, the Hodge types of $\mathrm{End}^Q(\boldsymbol{R}_1 f_* \mathbb{Z}_X) \otimes_{\mathbb{Z}} \mathbb{C}$ are $(-1, 1)$, $(0, 0)$, and $(1, -1)$. Therefore the condition (2.5) is equivalent to Faltings' condition $(*)$ in [F], i.e.,

$$\mathrm{End}^Q(\boldsymbol{R}_1 f_* \mathbb{Z}_X) \cong (\mathrm{End}^Q(\boldsymbol{R}_1 f_* \mathbb{Z}_X))^{0,0}.$$

3. The endomorphism algebra of $\boldsymbol{R}_1 f_* \mathbb{Z}_X$. Let us keep the notation in §2. In this section, we will study the structure of the endomorphism algebra $E_{\mathbb{Z}} := \mathrm{End}(\boldsymbol{R}_1 f_* \mathbb{Z}_X)$ or $E = \mathrm{End}(\boldsymbol{R}_1 f_* \mathbb{Q}_X)$ for an abelian scheme $f: X \to S$. Let us recall the following fundamental results:

(3.1) THEOREM (cf. [D, Théorème (4.2.6) and Corollarie (4.2.8)]). *Let* $s \in S$ *be a geometric point. Then we have the following.*

(i) *The action of the fundamental group* $\pi_1(S, s)$ *on the fiber* $(\boldsymbol{R}_1 f_* \mathbb{Q}_X)_s$ *is semi-simple.*

(ii) *The endomorphism algebra* $E = \mathrm{End}(\boldsymbol{R}_1 f_* \mathbb{Q}_X)$ *is a semi-simple algebra, and admits a natural Hodge structure of weight* 0.

(iii) *The center of* E *is of type* $(0, 0)$.

(3.2) DEFINITION. A $\mathbb{Q}$-local system $\boldsymbol{T}_{\mathbb{Q}}$ on D is said to be *primary* if $\boldsymbol{T}_{\mathbb{Q}}$ is a sum of irreducible local systems which are mutually isomorphic to each other.

Set $\boldsymbol{W}_{\mathbb{Q}} := \boldsymbol{R}_1 f_* \mathbb{Q}_X$. From (i) of (3.1), we can write $\boldsymbol{W}_{\mathbb{Q}}$ as

$$\boldsymbol{W}_{\mathbb{Q}} = (\boldsymbol{W}_1)^{\oplus n_1} \oplus (\boldsymbol{W}_2)^{\oplus n_2} \oplus \cdots \oplus (\boldsymbol{W}_t)^{\oplus n_t}, \tag{3.3}$$

where $\boldsymbol{W}_i$'s are irreducible $\mathbb{Q}$-local systems such that $\boldsymbol{W}_i \not\simeq \boldsymbol{W}_j$ for $i \neq j$. In the decomposition (3.3), each local system $(\boldsymbol{W}_i)^{\oplus n_i}$ is primary, and is called a *primary component* of $\boldsymbol{W}_{\mathbb{Q}}$. It is easy to show the following:

(3.4) LEMMA. *For a polarized* $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}}$, *each primary component forms a polarized* $\mathbb{Q}$-*sub*VHS, *and hence the primary decomposition* (3.3) *is orthogonal decomposition with respect to the given polarization* Q.

From the decomposition (3.3), we can write $\boldsymbol{E}$ as

$$\boldsymbol{E} = \bigoplus_{i=1}^{t} \mathrm{End}(\boldsymbol{W}_i^{\oplus n_i}) \cong \bigoplus_{i=1}^{t} M_{n_i}(D_i), \tag{3.5}$$

where we have set $D_i = \mathrm{End}(\boldsymbol{W}_i)$, which are division algebras over $\mathbb{Q}$ by Schur's lemma. By Lemma (3.4), the Hodge decomposition of $\boldsymbol{E} \otimes_{\mathbb{Q}} \mathbb{C}$ is compatible with the decomposition (3.5). Hence, in order to classify non-rigid $\mathbb{Q}$-VHS's over S, it suffices to classify primary ones.

(3.6) REMARK. Let η be the generic point of S. Then the generic fiber X_η is an abelian variety over the field of rational functions $K = \mathbb{C}(S)$. We have an isomorphism $\mathrm{End}_S(X) \cong \mathrm{End}_K(X_\eta)$, because $f: X \to S$ is an abelian scheme. Moreover we have an isomorphism

$$\mathrm{End}_S(X) \cong \mathrm{End}(\boldsymbol{R}_1 f_* \mathbb{Z}_X)^{0,0}.$$

Assume that X_η is simple over K. Then the center Z of $\mathrm{End}(X_\eta) \otimes \mathbb{Q}$ is a field, and so is the center of $\mathrm{End}(\boldsymbol{R}_1 f_* \mathbb{Q}_X)$, because of (iii) of (3.1). In view of Lemma (3.4) and (3.5), $\boldsymbol{R}_1 f_* \mathbb{Q}_X$ must be a primary $\mathbb{Q}$-VHS in this case.

(3.7) Tensor product decomposition of primary $\boldsymbol{W}_{\mathbb{Q}}$. From now on, we assume that $\boldsymbol{W}_{\mathbb{Q}} = \boldsymbol{R}_1 f_* \mathbb{Q}_X$ is a primary $\mathbb{Q}$-VHS over S, (and of weight -1, and of type $(0, -1)$, $(-1, 0)$). In this subsection, we recall the tensor product decomposition of $\boldsymbol{W}_{\mathbb{Q}}$ following [S1, Ch. IV]. Denote by $\boldsymbol{V}$ a non-trivial irreducible $\mathbb{Q}$-local subsystem of $\boldsymbol{W}_{\mathbb{Q}}$, and set

$$D = \mathrm{End}(\boldsymbol{V}), \quad F = \mathrm{Cent}\, D, \quad U = \mathrm{Hom}(\boldsymbol{V}, \boldsymbol{W}_{\mathbb{Q}}). \tag{3.8}$$

By Schur's lemma, D is a division algebra over $\mathbb{Q}$, and F is a finite extension field of $\mathbb{Q}$. The local system $\boldsymbol{V}$ has a natural structure of a left D-module, and the $\mathbb{Q}$-vector space U has a natural structure of a right D-module. We put:

(3.9) $$[F:\mathbb{Q}]=d\,,\quad [D:F]=r^2\,,$$

(3.10) $$\dim_D U=m\,,\quad \mathrm{rank}_D V=n\,.$$

Denote by $\bar{D}$ the division algebra opposite to D. Then U can be regarded as a left $\bar{D}$-module.

We have the following assertions, which follow from [S1, Lemma 1.1, Ch. IV].

(3.11) PROPOSITION. *For a primary* $\mathbb{Q}$-VHS $W_{\mathbb{Q}}$ *as above, one has the following isomorphisms*:

(3.12) $$W_{\mathbb{Q}}\cong U\otimes_D V\,,$$

(3.13) $$E=\mathrm{End}(W_{\mathbb{Q}})=\mathrm{End}_{\bar{D}}(U)\cong M_m(D)\,.$$

Here $U\otimes_D V$ *denotes the tensor product of* U *and* V *over the division algebra* D.

(3.14) Involutions on E. Since the polarization Q induces an isomorphism $W_{\mathbb{Q}}\cong W_{\mathbb{Q}}^{\vee}:=\mathcal{H}om(W_{\mathbb{Q}},\mathbb{Q}_S)$, we have an involution on E, which plays an important roll in the classification.

Fixing a geometric points $s\in S$, we have an isomorphism

$$E\simeq \mathrm{End}_{\pi_1(S,s)}(W_{\mathbb{Q},s})(\subset \mathrm{End}(W_{\mathbb{Q},s}))\,.$$

Then we can define an involution ι_s on $\mathrm{End}(W_{\mathbb{Q},s})$ by a^{ι_s} as the adjoint of $a\in\mathrm{End}(W_{\mathbb{Q},s})$ with respect to Q_s, namely,

$$Q_s(a\cdot x, y)=Q_s(x, a^{\iota_s}\cdot y)\,,$$

for all $x, y\in W_{\mathbb{Q},s}$ and $a\in\mathrm{End}(W_{\mathbb{Q},s})$. Since Q_s is invariant under the action of $\pi_1(S,s)$, the subalgebra $E\subset\mathrm{End}(W_{\mathbb{Q},s})$ is stable under ι_s. Moreover, it is easy to check that ι_s is compatible with the Hodge decomposition on $E\otimes_{\mathbb{Q}}\mathbb{C}$, and if we restrict ι_s to $\mathrm{End}(W_{\mathbb{Q},s})^{0,0}\cong\mathrm{End}(A_s)\otimes_{\mathbb{Z}}\mathbb{Q}$, then it coincides with the *Rosati* involution on the abelian variety X_s induced by the polarization λ_s. On $E=\mathrm{End}_{\pi_1(S,s)}(W_{\mathbb{Q},s})$, the involution ι_s does not depend on the choice of the point $s\in S$, so we denote it by ι.

From the self-duality $W_{\mathbb{Q}}\cong W_{\mathbb{Q}}^{\vee}$, we can deduce that the irreducible local system V is also self-dual, that is, $V\cong V^{\vee}$. Therefore there exists an involution ι_0 on $D=\mathrm{End}(V)$. The center F of E (F is also equal to $\mathrm{Cent}(D)$) is stable under both involutions ι and ι_0, and one has

(3.15) $$\iota_{0|F}=\iota_{|F}\,.$$

In general, an involution on an algebra is said to be *of the first kind* if it fixes all elements in the center of the algebra, and *of the second kind* otherwise.

(3.16) PROPOSITION. *The center* F *of the endomorphism algebra of a primary* $\mathbb{Q}$-VHS $W_{\mathbb{Q}}$ *is a finite extension field of* $\mathbb{Q}$ *with a positive involution* $\iota=\iota_0$, *so it is one of the following*:

(i) *F is a totally real number field and* $\iota=\mathrm{id}$, *or*

(ii) *F is a purely imaginary quadratic extension of a totally real number field and* $\iota =$ *the complex conjugation.*

Proof. Let C_s denote the Weil operator on $W_{\mathbb{R},s}$. Then we have the positivity condition

(3.17) $$Q_s(x, C_s x) > 0 \qquad \text{for all} \quad x \in W_{\mathbb{R},s} .$$

For $a \in \mathrm{End}(W_{\mathbb{Q},s})$ and $x \in W_{\mathbb{Q},s}$ such that $ax \neq 0$, we have

$$0 < Q_s(a \cdot x, C_s a \cdot x) = Q_s(x, C_s(C_s(a^\iota) \cdot a) \cdot x) ,$$

where $C_s(a^\iota) = C_s^{-1} a^\iota C_s$. Hence we have

(3.18) $$\mathrm{Tr}(C_s(a^\iota) \cdot a) > 0 .$$

By (iii) of Theorem (3.1), F has a Hodge type $(0, 0)$, so it commutes with the Weil operator C_s. Hence from (3.18), for $a \in F - \{0\}$, one has

$$\mathrm{Tr}_{F/\mathbb{Q}}(a^\iota a) > 0 .$$

Since F is a finite extension field of $\mathbb{Q}$ with a positive involution ι, according to Albert, we obtain the classification.

4. Decomposition of polarization.

(4.1) (D, ε)-Hermitian form. Let k be a field of characteristic zero and D a division algebra over k. Denoting by F the center of D, we set

$$[F : k] = d , \qquad [D : F] = r^2 .$$

Consider a finite dimensional k-vector space T with a structure of a *right* D-module, and set $n = \mathrm{rank}_D T$.

Let ι_0 be an involution on D, i.e., an anti-automorphism of order ≤ 2, and let $\varepsilon = \pm 1$. A (D, ε)*-Hermitian form* h on T with respect to ι_0 is, by definition, a k-bilinear mapping $h : T \times T \to D$ satisfying the following conditions:

(4.2) $$h(v, v'\alpha) = h(v, v')\alpha ,$$

(4.3) $$h(v', v) = \varepsilon h(v, v')^{\iota_0} \qquad \text{for all} \quad v, v' \in T , \ \alpha \in D .$$

A (D, ε)-Hermitian form h is said to be *non-degenerate* if a intersection matrix $L = (h(e_i, e_j))$ for a D-basis (e_i) of T is invertible. For a non-degenerate (D, ε)-Hermitian form h on T with respect to ι_0, we define the *unitary group* and the *special unitary group* for h by

(4.4) $$U(T, h) = \{ g \in GL(T/D) \mid h(gv, gv') = h(v, v')(v, v' \in T) \} ,$$

(4.5) $$SU(T, h) = U(T, h) \cap SL(T/D) .$$

Note that these are F-algebraic groups.

If T' is a left D-module, we can define a $(\bar{D}, \varepsilon)$-Hermitian form h' on T' with respect to $\overline{\iota_0}$, by regarding T' as a right $\bar{D}$-module.

(4.6) Recall that a primary $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}}$ has a tensor product decomposition $\boldsymbol{W}_{\mathbb{Q}} = U \otimes_D \boldsymbol{V}$ as in (3.12). The following theorem shows that the polarization Q is also decomposed according to this decomposition.

As in (3.14), one obtains an involution ι on $\boldsymbol{E}$ induced from the polarization Q and an involution ι_0 on D such that $(\iota_0)_{|F} = \iota_{|F}$.

(4.7) THEOREM. *In the notation in* (3.7), *there exist a flat non-degenerate* $(\bar{D}, \varepsilon)$-*Hermitian form* $\boldsymbol{h}$ *on* $\boldsymbol{V}$ *with respect to* $\overline{\iota_0}$, *and a non-degenerate* $(D, -\varepsilon)$-*Hermitian form* h' *on* U *with respect to* ι_0 *such that the polarization* Q *on* $\boldsymbol{W}_{\mathbb{Q}}$ *can be written as*

$$Q = \mathrm{Tr}_{D/\mathbb{Q}}(h' \otimes_D \boldsymbol{h}) . \tag{4.8}$$

Here the sign ε *is uniquely determined by* Q *if* ι_0 *is of the first kind, but arbitrary if* ι_0 *is of the second kind.*

The proof is similar to that of Lemma 2.2 and Theorem 2.3 in [S1, Ch. IV].

5. Scalar extension. In §3 and §4, we obtained the tensor product decomposition of a primary $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}}$

$$\boldsymbol{W}_{\mathbb{Q}} = U \otimes_D \boldsymbol{V} ,$$

as in (3.12) and the decomposition of the polarization $Q = \mathrm{Tr}_{D/\mathbb{Q}}(h' \otimes_D \boldsymbol{h})$. In this section, we will study the structure of $\mathbb{R}$-VHS $\boldsymbol{W}_{\mathbb{R}} := \boldsymbol{W}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ which is obtained by scalar extension.

The center F of $D = \mathrm{End}(\boldsymbol{V})$ is a finite extension field of $\mathbb{Q}$ with a positive involution ι_0 (see (3.16)), so set $F^+ = \{z \in F \mid z^{\iota_0} = z\}$. Then, from (3.16), F^+ is a totally real number field, and either

(R) $F = F^+$, so F is a total real fields, or

(C) F is a CM field, i.e., a purely imaginary quadratic extension of F^+.

Setting $t = [F^+ : \mathbb{Q}]$, let $\{\tau_i : F^+ \hookrightarrow \mathbb{R},\ 1 \le i \le t\}$ be the set of t distinct embeddings of F^+ into $\mathbb{R}$. Regarding $\boldsymbol{W}_{\mathbb{Q}}$ as a local system of F^+-vector spaces, we can decompose $\boldsymbol{W}_{\mathbb{R}} := \boldsymbol{W}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ as

$$\boldsymbol{W}_{\mathbb{R}} = \bigoplus_{i=1}^{t} \boldsymbol{W}^{(i)} , \tag{5.1}$$

where we have set

$$\boldsymbol{W}^{(i)} = \boldsymbol{W}_{\mathbb{Q}} \otimes_{F^+, \tau_i} \mathbb{R} . \tag{5.2}$$

Since F^+ is of Hodge type (0, 0), this decomposition is compatible with the Hodge decomposition of each fiber. Denote by $Q^{(i)}$ the bilinear form on $\boldsymbol{W}^{(i)}$ induced by Q.

Then we have the following:

(5.3) LEMMA. *The local systems $W^{(i)}$ are $\mathbb{R}$-subVHS's of $W_{\mathbb{R}}$ with a polarization $Q^{(i)}$, and the decomposition* (5.1) *is an orthogonal sum with respect to $Q_{\mathbb{R}}$.*

From this lemma, we have the decomposition of the Weil operator

$$C_s = \bigoplus_{i=1}^{t} C_s^{(i)} \quad \text{for each} \quad s \in S \tag{5.4}$$

and the polarization

$$Q_{\mathbb{R}} = \bigoplus_{i=1}^{t} Q^{(i)}, \tag{5.5}$$

according to (5.1).

For each embedding $\tau_i \colon F^+ \hookrightarrow \mathbb{R}$, we put

$$F^{(i)} = F \otimes_{F^+, \tau_i} \mathbb{R}, \tag{5.6}$$

$$D^{\tau_i} = D \otimes_{F^+, \tau_i} \mathbb{R}, \tag{5.7}$$

$$V^{\tau_i} = V \otimes_{F^+, \tau_i} \mathbb{R}, \tag{5.8}$$

$$U^{\tau_i} = U \otimes_{F^+, \tau_i} \mathbb{R}. \tag{5.9}$$

The algebra D^{τ_i} becomes a central simple algebra over $F^{(i)}$. Hence there exists a division algebra $D^{(i)}$ over $F^{(i)}$ such that

$$D^{\tau_i} \cong M_s(D^{(i)}).$$

Fixing an above isomorphism, we denote by $\varepsilon_{\nu\mu}^{i}$ the corresponding matrix unit in D^{τ_i}. We moreover set

$$V^{(i)} := \varepsilon_{11}^{i} V^{\tau_i}, \quad U^{(i)} = U^{\tau_i} \varepsilon_{11}^{i}. \tag{5.10}$$

Then $V^{(i)}$ is a local system of left $D^{(i)}$-modules, and $U^{(i)}$ is a right $D^{(i)}$-module, and we have an isomorphism (cf. [S1, p. 189]),

$$W^{(i)} = U^{(i)} \otimes_{D^{(i)}} V^{(i)}. \tag{5.11}$$

Note that $F^{(i)}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, corresponding to the case (R) or (C), so $D^{(i)}$ is isomorphic to $\mathbb{R}$, $\mathbb{H}$, or $\mathbb{C}$.

(5.12) LEMMA. *Let $W_{\mathbb{Q}}$ be a primary $\mathbb{Q}$-VHS over S of weight $(0, 0)$ and of type $(-1, 0)+(0, -1)$. Let $W_{\mathbb{Q}} = U \otimes_D V$ be the tensor product decomposition in* (3.12). *For each embedding $\tau_i \colon F^+ \hookrightarrow \mathbb{R}$, let $W^{(i)}$, $F^{(i)}$, D^{τ_i}, $V^{(i)}$, $U^{(i)}$ be as above. There exists an isomorphism*

$$W^{(i)} \cong U^{(i)} \otimes_{D^{(i)}} V^{(i)}, \tag{5.13}$$

such that for every geometric point $s \in S$, the Weil operator $C_s^{(i)}$ can be written as

$$C_s^{(i)} = I'^{(i)} \otimes I_s^{(i)}, \tag{5.14}$$

where $I'^{(i)}$ and I_s are $\mathbb{R}$-linear automorphisms of $U^{(i)}$ and $V_s^{(i)}$, respectively.

Moreover, one of the following cases occurs:

$$I'^{(i)} = 1_{U^{(i)}} \quad \text{and} \quad (I_s^{(i)})^2 = -1_{V_s^{(i)}}, \tag{5.15}$$

and

$$(I'^{(i)})^2 = -1_{U^{(i)}} \quad \text{and} \quad I_s^{(i)} = 1_{V_s^{(i)}}. \tag{5.16}$$

Proof. Regarding $\boldsymbol{E}$ as an F^+-vector space, we set

$$\boldsymbol{E}^{(i)} = \boldsymbol{E} \otimes_{F^+, \tau_i} \mathbb{R}.$$

Since from (3.13) we have an isomorphism $\boldsymbol{E} \cong \mathrm{End}_{\bar{D}}(U)$, we get an isomorphism

$$\boldsymbol{E}^{(i)} \cong \mathrm{End}_{\overline{D^{(i)}}}(U^{(i)}).$$

Since $\boldsymbol{E}^{(i)}$ has a natural Hodge structure of weight 0, there esists a corresponding Weil operator $C_s'^{(i)}$ on $\boldsymbol{E}^{(i)}$, which is induced by an $\mathbb{R}$-linear automorphism $I'^{(i)}$ on $U^{(i)}$. For each point $s \in S$, the natural map $\boldsymbol{E}^{(i)} \otimes \boldsymbol{W}_s^{(i)} \to \boldsymbol{W}_s^{(i)}$ is a morphism of Hodge structures. Hence, the Weil operator $C_s^{(i)}$ on $\boldsymbol{W}_s^{(i)}$ can be written as in (5.14). Since $\boldsymbol{W}_s^{(i)} \otimes_{\mathbb{R}} \mathbb{C}$ is of type $(-1, 0) + (0, -1)$, one of the cases (5.15) and (5.16) occurs. (See [D, (4.4.8)].)

(5.17) Remark. In the case (5.15), $\boldsymbol{E}^{(i)} \otimes_{\mathbb{R}} \mathbb{C}$ consists of elements of type (0, 0), while $I_s^{(i)}$ determines a complex structure on each fiber $V_s^{(i)}$. In the case (5.16), $\boldsymbol{E}^{(i)} \otimes_{\mathbb{R}} \mathbb{C}$ consists of elements of type $(-1, 1)$, $(0, 0)$, $(1, -1)$, but $V_s^{(i)} \otimes_{\mathbb{R}} \mathbb{C}$ consists of bihomogeneous elements.

Now let us study the scalar extension of the polarization Q.

(5.18) Lemma. *Keeping the notation in Proposition* (5.12), *let Q be a polarization of $\boldsymbol{W}_{\mathbb{Q}}$ with a decomposition $Q = \mathrm{Tr}_{D/\mathbb{Q}}(h' \otimes_D \boldsymbol{h})$ as in* (4.8), *ι_0 the involution on D defined in* (3.14), *and $\iota_0^{(i)}$ the induced involution on $D^{(i)}$.*

Then for each i, $1 \le i \le t$, $\boldsymbol{h}$ induces a $(\overline{D^{(i)}}, \varepsilon\eta_i)$-Hermitian form $\boldsymbol{h}^{(i)}$ on $V^{(i)}$ (with respect to $\overline{\iota_0^{(i)}}$), and a $(D, -\varepsilon)$-Hermitian form h' on U induces a $(D^{(i)}, -\varepsilon\eta_i)$-Hermitian form $h'^{(i)}$ on $U^{(i)}$ (with respect to $\iota_0^{(i)}$), where $\eta_i = \pm 1$, so that

$$Q^{(i)} = \mathrm{Tr}_{D^{(i)}/\mathbb{R}}(h'^{(i)} \otimes_{D^{(i)}} \boldsymbol{h}^{(i)}). \tag{5.19}$$

For the proof, see [S1, Ch. IV, §3].

(5.20) Proposition. *According to the cases* (5.15) *and* (5.16), *one can assume the following*:

$$\textit{Case } (5.15) \quad \varepsilon\eta_i = -1, \quad \text{and} \quad h'^{(i)} \gg 0 \quad \text{and} \quad \boldsymbol{h}_s^{(i)} I_s^{(i)} \gg 0. \tag{5.21}$$

(5.22) *Case* (5.16) $\varepsilon\eta_i=1$, *and* $h'^{(i)}I'^{(i)}\gg 0$ *and* $\boldsymbol{h}_s^{(i)}\gg 0$.

Here, for example, $\boldsymbol{h}_s^{(i)}I_s^{(i)}$ *denotes the bilinear form* $\boldsymbol{h}_s^{(i)}(x, I_s^{(i)}y)$, *and* "$h'^{(i)}\gg 0$" *means that the bilinear form* $h'^{(i)}$ *is symmetric* ($=D^{(i)}$*-Hermitian*) *and positive definite.*

PROOF. First, note that $D^{(i)}=\mathbb{R}$, $\mathbb{H}$ or $\mathbb{C}$ and $\iota_0^{(i)}$ is the standard involution on the division algebra $D^{(i)}$, (cf. Proposition (3.16) and §5). Assume first that we are in the case (5.15). Setting $x=u\otimes_{D^{(i)}}v$, $y=u'\otimes_{D^{(i)}}v'\in \boldsymbol{W}_s^{(i)}\cong U^{(i)}\otimes_{D^{(i)}}\boldsymbol{V}_s^{(i)}$, from (5.15) and (5.19), one has

$$Q^{(i)}(x, C_s^{(i)}y)=Q_s^{(i)}(u\otimes_{D^{(i)}}v, C_s^{(i)}(u'\otimes_{D^{(i)}}v'))=\mathrm{Tr}_{D^{(i)}/\mathbb{R}}(h'^{(i)}(u, u')\cdot \boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v')^{\iota_0^{(i)}})\,. \tag{5.23}$$

Since the bilinear form $Q_s^{(i)}(x, C_s^{(i)}y)$ is a symmetric form by Definition (1.3), one has

$$\mathrm{Tr}_{D^{(i)}/\mathbb{R}}(h'^{(i)}(u, u')\{\boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v')^{\iota_0^{(i)}}+\boldsymbol{h}_s^{(i)}(I_s^{(i)}v, v')^{\iota_0^{(i)}}\})=0\,.$$

Since $h'^{(i)}(u, u')$ takes arbitrary values in $D^{(i)}$, this implies that

$$\boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v')=-\boldsymbol{h}_s^{(i)}(I_s^{(i)}v, v')\,. \tag{5.24}$$

Now we show that $\varepsilon\eta_i=-1$. Assume the contrary. Then $h'^{(i)}(u, u')$ (resp. $\boldsymbol{h}_s^{(i)}(v, v')$) is a $(D^{(i)}, -1)$-Hermitian form (resp. a $(D^{(i)}, 1)$-Hermitian form), hence together with (5.24) one has

$$h'^{(i)}(u, u)^{\iota_0^{(i)}}=-h'^{(i)}(u, u')\,,\quad\text{and}\quad \boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v)^{\iota_0^{(i)}}=-\boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v)\,.$$

Hence both $h'^{(i)}(u, u)$ and $\boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v)$ are purely imaginary numbers in $D^{(i)}$. On the other hand, the positivity condition of $Q_s^{(i)}(x, C_s^{(i)}x)>0$ implies that

$$\mathrm{Tr}_{D^{(i)}/\mathbb{R}}(h'^{(i)}(u, u)\boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v)^{\iota_0^{(i)}})>0\qquad\text{for all}\quad u\in U^{(i)}-\{0\}\,,\ v\in \boldsymbol{V}_s^{(i)}-\{0\}\,. \tag{5.25}$$

Thus in the case $D^{(i)}=\mathbb{R}$, this is obviously impossible, and in the case $D^{(i)}=\mathbb{H}$, it is easy to find u and v for which the condition (5.25) does not hold. In the case $D^{(i)}=\mathbb{C}$, if we replace $h'^{(i)}$ and $\boldsymbol{h}^{(i)}$ by $\sqrt{-1}h'^{(i)}$ and $-\sqrt{-1}\boldsymbol{h}^{(i)}$, one can assume that $\varepsilon\eta_i=-1$. Thus one may assume that $\varepsilon\eta_i=-1$ in the case (5.15). In case (5.16), one may similarly assume that $\varepsilon\eta_i=1$.

Now both $h'^{(i)}(u, u')$ and $\boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v')$ (resp. $h'^{(i)}(u, I'^{(i)}u')$ and $\boldsymbol{h}_s^{(i)}(v, v')$) are $D^{(i)}$-Hermitian forms in the case (5.15) (resp. (5.16)). These also imply that both of $h'^{(i)}(u, u)$ and $\boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v)$ (resp. $h'^{(i)}(u, I'^{(i)}u)$ and $\boldsymbol{h}_s^{(i)}(v, v)$) are real numbers in the case (5.15) (resp. (5.16)). Hence (5.25) implies that

$$\text{Case (5.15)}\quad h'^{(i)}(u, u)\cdot \boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v)>0\quad\text{for all}\quad u\in U^{(i)}-\{0\}\,,\ v\in \boldsymbol{V}_s^{(i)}-\{0\}\,.$$

$$\text{(resp. Case (5.16)}\quad h'^{(i)}(u, I'^{(i)}u)\cdot \boldsymbol{h}_s^{(i)}(v, v)>0\quad\text{for all}\quad u\in U^{(i)}-\{0\},\ v\in \boldsymbol{V}_s^{(i)}-\{0\}.)$$

Thus $h'^{(i)}(u, u)$ and $\boldsymbol{h}_s^{(i)}(v, I_s^{(i)}v)$ (resp. $h'^{(i)}(u, I'^{(i)}u)$ and $\boldsymbol{h}_s^{(i)}(v, v)$) are both negative or positive real numbers. By a well-known theorem of algebraic number theory, one can find an element $\alpha\in(F^+)^\times$ such that $\tau_i(\alpha)\cdot h'^{(i)}(u, u)>0$ for all i in the case (5.15) and

$\tau_j(\alpha)\cdot h'^{(j)}(u, I'^{(j)}u)>0$ for all j in the case (5.16). Replacing h' and $\boldsymbol{h}$ by $\alpha\cdot h'$ and $\alpha^{-1}\cdot h$, one can get the assertion.

6. $\mathbb{Q}$-symplectic representations. Let $G_{\mathbb{Q}}$ be a $\mathbb{Q}$-algebraic group such that the group $G_{\mathbb{R}}$ of its $\mathbb{R}$-valued points is a Zariski connected semi-simple $\mathbb{R}$-group of Hermitian type. Let K be a maximal compact subgroup of $G_{\mathbb{R}}$ and $\mathscr{D}=G_{\mathbb{R}}/K$ the corresponding Hermitian bounded symmetric space. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ Lie algebras of $G_{\mathbb{R}}$ and K respectively, and by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Then the complex structure of $\mathscr{D}$ is induced by an element $H_0\in \mathrm{Cent}(\mathfrak{k})$ such that $(\mathrm{ad}_{\mathfrak{p}}(H_0))^2=-1_{\mathfrak{p}}$. We call such an element H_0 an H-element of $G_{\mathbb{R}}$. A pair $(G_{\mathbb{Q}}, H_0)$ consisting of the above $G_{\mathbb{Q}}$ and H_0 is called a *$\mathbb{Q}$-Hermitian pair.*

(6.1) DEFINITION. A *$\mathbb{Q}$-symplectic representation* of a $\mathbb{Q}$-Hermitian pair $(G_{\mathbb{Q}}, H_0)$ is a quadruple $(W_{\mathbb{Q}}, \rho_{\mathbb{Q}}, Q_{\mathbb{Q}}, I)$ consisting of

(i) a $\mathbb{Q}$-vector space $W_{\mathbb{Q}}$ of dimension $2g$,
(ii) a non-degenerate symplectic bilinear form $Q_{\mathbb{Q}}$ on $W_{\mathbb{Q}}$,
(iii) a faithful representation $\rho_{\mathbb{Q}}: G_{\mathbb{Q}}\to Sp(W_{\mathbb{Q}}, Q_{\mathbb{Q}})$ and
(iv) a complex structure $I\in\mathscr{D}(W_{\mathbb{R}}, Q_{\mathbb{R}})$ satisfying the condition

(6.2) $$[d\rho_{\mathbb{R}}(H_0)-(1/2)I, d\rho_{\mathbb{R}}(X)]=0 \qquad \text{for all} \quad X\in\mathfrak{g}_{\mathbb{R}},$$

where $\mathscr{D}(W_{\mathbb{R}}, Q_{\mathbb{R}})$ denotes

(6.3) $$\{I\in \mathrm{End}(W_{\mathbb{R}})\,|\,I^2=-1_{W_{\mathbb{R}}}, Q_{\mathbb{R}}(x, Iy) \text{ is a positive definite } \mathbb{R}\text{-symmetric form}\}.$$

Moreover, a $\mathbb{Q}$-symplectic representation $(W_{\mathbb{Q}}, \rho_{\mathbb{Q}}, Q_{\mathbb{Q}}, I)$ of a $\mathbb{Q}$-Hermitian pair $(G_{\mathbb{Q}}, H_0)$ is said to be *$\mathbb{Q}$-primary* if $(W_{\mathbb{Q}}, \rho_{\mathbb{Q}})$ is a sum of $G_{\mathbb{Q}}$-stable subspaces isomorphic to an irreducible $\mathbb{Q}$-representation $\rho_1: G_{\mathbb{Q}}\to GL(V/\mathbb{Q})$.

In this section, we will show that one can obtain a $\mathbb{Q}$-symplectic representation from a given primary $\mathbb{Q}$-VHS $W_{\mathbb{Q}}$ on S.

(6.4) Let us fix a geometric point $s\in S$. Then, from Theorem (4.7), the fiber $\boldsymbol{V}_s$ is a right $\bar{D}$-module with $(\bar{D}, \varepsilon)$-Hermitian form $\boldsymbol{h}_s$, and U a right D-module with $(D, -\varepsilon)$-Hermitian form h'. Denote by $SU(\boldsymbol{V}_s, \boldsymbol{h}_s)$ and $SU(U, h')$ the special unitary group corresponding to $(\boldsymbol{V}_s, \boldsymbol{h}_s)$ and (U, h'), respectively. Then these groups are F-algebraic groups. Consider the $\mathbb{Q}$-algebraic groups

(6.5) $$G_{\mathbb{Q}}=R_{F/\mathbb{Q}}(SU(\boldsymbol{V}_s, \boldsymbol{h}_s)), \quad G'_{\mathbb{Q}}=R_{F/\mathbb{Q}}(SU(U, h'))$$

obtained by the scalar restriction $R_{F/\mathbb{Q}}$ of Weil [W, 1.3]. Let

(6.6) $$\rho_1: G_{\mathbb{Q}}=R_{F/\mathbb{Q}}(SU(\boldsymbol{V}_s, \boldsymbol{h}_s))\to SU(\boldsymbol{V}_s, \boldsymbol{h}_s),$$

(6.7) $$\rho'_1: G'_{\mathbb{Q}}=R_{F/\mathbb{Q}}(SU(U, h'))\to SU(U, h'),$$

be the natural homomorphisms. Then, from Proposition (3.11) and Theorem (4.7), we have natural representations

$$\rho = 1_U \otimes \rho_1 : G_{\mathbb{Q}} = R_{F/\mathbb{Q}}(SU(V_s, h_s)) \to Sp(W_{\mathbb{Q},s}, Q_s) , \tag{6.8}$$

$$\rho' = \rho'_1 \otimes 1_{V_s} : G'_{\mathbb{Q}} = R_{F/\mathbb{Q}}(SU(U, h')) \to Sp(W_{\mathbb{Q},s}, Q) , \tag{6.9}$$

which commute with each other.

(6.10) Let B be a division algebra over $\mathbb{R}$, i.e., $B = \mathbb{R}$, $\mathbb{H}$, or $\mathbb{C}$, T a right B-vector space of dimension n, and h a non-degenerate (B, ε)-Hermitian form with respect to the standard involution ι_0 on B, where $\varepsilon = \pm 1$. In case $\varepsilon = 1$, we assume that h is positive definite, so that we have an orthonormal B-basis of T with respect to h and identify $SU(T, h)$ with

$$SU_n(B) = \{g \in SL_n(B) \mid {}^t g^{\iota_0} g = 1_n\} .$$

Note that the $\mathbb{R}$-group $SU_n(B)$ is always compact.

Next in the case $\varepsilon = -1$, we can choose a basis $\{e_i\}$ of T so that the intersection matrix $H = (h(e_i, e_j))$ can be written as follows:

(i) $B = \mathbb{R}$; $n = 2m$ is an even integer

$$H = J_m = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix} .$$

(ii) $B = \mathbb{H}$;

$$H = j 1_n .$$

(iii) $B = \mathbb{C}$; (p, q) is a pair of non-negative integers such that $p + q = n$.

$$H = -i 1_{pq} = \begin{pmatrix} -i 1_p & 0 \\ 0 & i 1_q \end{pmatrix} .$$

In the last case (iii), (p, q) is called the signature of h.

Then in each case, the group $SU(T, h)$ is isomorphic to the following groups.

(i′) $B = \mathbb{R}$; $n = 2m$ is even.

$$SU_n(\mathbb{R}, h) = Sp_{n/2}(\mathbb{R}) = \{g \in SL_n(\mathbb{R}) \mid {}^t g J_{n/2} g = J_{n/2}\} .$$

(ii′) $B = \mathbb{H}$

$$SU_n(\mathbb{H}, h) = SU_n(\mathbb{H})^- = \{g \in SL_n(\mathbb{H}) \mid {}^t g^{\iota}(j 1_n) g = j 1_n\} .$$

(iii′) $B = \mathbb{C}$; $p + q = n$.

$$SU_n(\mathbb{C}, h) = SU(p, q, \mathbb{C}) = \{g \in SL_n(\mathbb{C}) \mid {}^t \bar{g} 1_{pq} g = 1_{pq}\} .$$

In case $\varepsilon = -1$, the group $G_{\mathbb{R}} = SU(T, h)$ is a connected semi-simple $\mathbb{R}$-group unless $SU_1(\mathbb{H})^- \cong S^1$, and is non-compact of Hermitian type unless $SU(n, 0, \mathbb{C}) \cong SU(0, n, \mathbb{C})$ or

$SU_1(\mathbb{H})^-$. Let $\mathscr{D}(T, h)=G_{\mathbb{R}}/K$ denote the corresponding Hermitian symmetric bounded domain where K is a maximal compact subgroup of $G_{\mathbb{R}}$. Then we have an isomorphism

(6.11) $\mathscr{D}(T, h)=\{I\in \mathrm{End}_{\mathbb{R}}(T)\,|\,I^2=-1_T,\ h(x, Iy)$ is a positive definite B-Hermitian$\}$.

Corresponding to each case above, $\mathscr{D}(T, h)$ is isomorphic to one of the following bounded symmetric domains:

(i) $(\mathrm{II})_m=\{Z\in M_m(\mathbb{C})\,|\,{}^tZ=Z,\ 1_m-{}^t\bar{Z}Z\gg 0\}$,
(ii) $(\mathrm{II})_n=\{Z\in M_n(\mathbb{C})\,|\,{}^tZ=-Z,\ 1_n-{}^t\bar{Z}Z\gg 0\}$,
(iii) $(\mathrm{I})_{pq}=\{Z\in M(p, q, \mathbb{C})\,|\,1_q-{}^t\bar{Z}Z\gg 0\}$.

The relation between $SU(T, h)$ and $\mathscr{D}(T, h)$, and the $\mathbb{R}$-rank of $SU(T, h)$ are shown in the following table.

(6.12)

B	$G=SU(T, h)$	$\mathscr{D}=\mathscr{D}(T, h)$	$\dim_{\mathbb{C}}\mathscr{D}$	$\mathbb{R}$-rank
$\mathbb{R}$	$Sp_{n/2}(\mathbb{R})$	$(\mathrm{III})_{n/2}$	$(n/2)(n/2+1)/2$	$n/2$
$\mathbb{H}$	$SU_n(\mathbb{H})^-$	$(\mathrm{II})_n$	$n(n-1)/2$	$[n/2]$
$\mathbb{C}$	$SU(p, q, \mathbb{C})$	$(\mathrm{I})_{pq}$	$p\cdot q$	$\min(p, q)$

(6.13) REMARK. If h is a positive definite B-Hermitian form, the group $SU(T, h)\cong SU_n(B)$ is simple, and non-abelian unless $SU_1(\mathbb{R})$, $SU_2(\mathbb{R})$ and $SU_1(\mathbb{C})$. If h is a B-skew Hermitian form, then $G_{\mathbb{R}}=SU_n(B, h)$ is simple and non-abelian unless $SU_n(\mathbb{H})^-\cong S^1$, $SU_2(\mathbb{H})^-\cong SL_2(\mathbb{R})\times SU_2(\mathbb{C})$, and $SU(1, 0, \mathbb{C})\cong SU(0, 1, \mathbb{C})\cong S^1$.

Let $G_{\mathbb{R}}$ (resp. $G'_{\mathbb{R}}$) be the group of $\mathbb{R}$-valued points of $G_{\mathbb{Q}}$ (resp. $G'_{\mathbb{Q}}$). From Lemma (5.12) and (5.18), we have the following decomposition of $G_{\mathbb{R}}$ (resp. $G'_{\mathbb{R}}$):

(6.14) $$G_{\mathbb{Q}}=\prod_{i=1}^{t} SU(V_s^{(i)}, \boldsymbol{h}_s^{(i)})\,,$$

(6.15) $$G'_{\mathbb{R}}=\prod_{i=1}^{t} SU(U^{(i)}, h'^{,(i)})\,.$$

Moreover, from ρ one has a natural representation

(6.16) $$\rho^{(i)}\colon G_{\mathbb{R}}=R_{F/\mathbb{Q}}(SU(V_s, \boldsymbol{h}_s))_{\mathbb{R}}\to Sp(U^{(i)}\otimes_{D^{(i)}} V_s^{(i)}, h_s^{(i)})\cong Sp(W_s^{(i)}, Q_s)\,.$$

(One can also obtain a representation $\rho'^{(i)}$ of $G'_{\mathbb{R}}$.) Note that the isomorphism classes of $G_{\mathbb{Q}}$ and ρ_1 do not depend on the point $s\in S$.

The most fundamental result in this paper is the following:

(6.17) THEOREM. *Let the notation be as above.*

(i) *The $\mathbb{Q}$-algebraic groups $G_{\mathbb{Q}}$ and $G'_{\mathbb{Q}}$ are Zariski connected, and the groups $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$ of their $\mathbb{R}$-valued points are reductive $\mathbb{R}$-groups of Hermitian type.*

(ii) *If, moreover, $G_{\mathbb{R}}$ (resp. $G'_{\mathbb{R}}$) is non-compact, then $G_{\mathbb{R}}$ (resp. $G'_{\mathbb{R}}$) is a semi-simple $\mathbb{R}$-group of Hermitian type.*

(iii) *Assume that* $G_{\mathbb{R}}$ *(resp.* $G'_{\mathbb{R}}$*) is non-compact. For each point* $s \in S$*, there exists an H-element* $H_{0,s}$ *(resp.* H'_0*) of* $G_{\mathbb{R}}$ *(resp.* $G'_{\mathbb{R}}$*) such that* $(G_{\mathbb{Q}}, H_{0,s})$ *(resp.* $(G'_{\mathbb{Q}}, H'_0)$ *is a* $\mathbb{Q}$*-Hermitian pair and the data* $(\boldsymbol{W}_{\mathbb{Q},s}, \rho, Q_s, C_s)$ *(resp.* $(\boldsymbol{W}_{\mathbb{Q},s}, \rho', Q, C_s)$*) become a* $\mathbb{Q}$*-symplectic representation of* $(G_{\mathbb{Q}}, H_{0,s})$ *(resp.* $(G'_{\mathbb{Q}}, H'_0)$*).*

PROOF. (i) The Zariski connectedness of $G_{\mathbb{Q}}$ and $G'_{\mathbb{Q}}$ follows from the argument in [S1, Appendix, § 1]. In view of (6.10) and (6.11), we only have to show that $SU(\boldsymbol{V}_s^{(i)}, \boldsymbol{h}_s^{(i)})$ and $SU(U^{(i)}, h'^{,(i)})$ are reductive groups. From Proposition (5.20), $\boldsymbol{h}_s^{(i)}$ and $h'^{(i)}$ are $D^{(i)}$-skew-Hermitian or positive definite Hermitian. Hence this follows from Remark (6.13).

(ii) We only have to prove the assertion (ii) for $G_{\mathbb{R}}$. If $G_{\mathbb{R}}$ is non-compact, one of $SU(\boldsymbol{V}_s^{(i)}, \boldsymbol{h}_s^{(i)})$ is non-compact. Hence in particular $\boldsymbol{h}_s^{(i)}$ is a $D^{(i)}$-skew-Hermitian form and $SU(\boldsymbol{V}_s^{(i)}, \boldsymbol{h}_s^{(i)})$ is a sem-simple $\mathbb{R}$-group of Hermitian type. Then by Remark (6.13), the group $SU(\boldsymbol{V}_s^{(k)}, \boldsymbol{h}_s^{(k)})$ for $k \neq i$ is semi-simple of Hermitian type. Therefore, we obtain the assertion (ii).

(iii) Consider the Weil operator C_s on $\boldsymbol{W}_{\mathbb{R},s}$. It is decomposed as $C_s = \bigoplus_{i=1}^{t} C_s^{(i)}$ according to (5.1). By Lemma (5.12), after a suitable renumbering of i, one may assume that

$$C_s = \left(\sum_{i=1}^{t'} 1 \otimes I_s^{(i)} \right) + \left(\sum_{i=t'+1}^{t} I'^{(i)} \otimes 1 \right). \tag{6.18}$$

Note that $C_s \in \mathscr{D}(\boldsymbol{W}_{\mathbb{R},s}, Q_s)$ (cf. (6.3)). Now set

$$H_{0,s} = \frac{1}{2} \left(\sum_{i=1}^{t'} (I_s^{(i)} - \mu^i) \right), \tag{6.19}$$

$$H'_0 = \frac{1}{2} \left(\sum_{i=t'+1}^{t} (I'^{(i)} - \mu^i) \right), \tag{6.20}$$

where

$$\mu^i = \begin{cases} 0 & \text{if } D^{(i)} = \mathbb{R}, \mathbb{H}, \\ \sqrt{-1}(p_i - q_i)/(p_i + q_i) & \text{if } D^{(i)} = \mathbb{C} \text{ and } \sqrt{-1}\boldsymbol{h}_S^{(i)} \text{ has signature } (p_i, q_i). \end{cases}$$

Then it is easy to see that $H_{0,s}$ (resp. H'_0) defines an H-element of $G_{\mathbb{R}}$ (resp. $G'_{\mathbb{R}}$). Moreover one can also check the condition (6.2) for $\rho_{\mathbb{R}}$, $H_{0,s}$, C_s (resp. $\rho'_{\mathbb{R}}$, H'_0, C_s).

(6.21) THEOREM. *In the notation in Proposition* (5.12), *let* H'_0 *denote the element of Lie algebra* $\mathfrak{g}'_{\mathbb{R}}$ *if* $G'_{\mathbb{R}}$ *defined in Theorem* (6.17). *Let* $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}' \oplus \mathfrak{p}'$ *be the corresponding decomposition of the Lie algebra, and* $\mathfrak{p}'_{\mathbb{C}} = \mathfrak{p}'^{+} \oplus \mathfrak{p}'^{-}$ *the decomposition of the complexification of* $\mathfrak{p}'$ *with respect to the complex structure* $\mathrm{ad}_{\mathfrak{p}'}(H'_0)$. *Then we have an isomorphism of the* $\mathbb{C}$*-vector spaces*

$$(\mathrm{End}^{Q}(\boldsymbol{R}_1 f_* \mathbb{Z}_X) \otimes_{\mathbb{Z}} \mathbb{C})^{-1,1} \cong \mathfrak{p}'^{+}. \tag{6.22}$$

PROOF. First, let us remark that from (3.13), (4.8) and (6.5), there exists an isomorphism

$$\mathrm{End}^Q(\boldsymbol{R}_1 f_* \mathbb{Z}_X) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathfrak{g}'_{\mathbb{R}} .$$

The Hodge structures on both hand sides of this isomorphism are induced by the Weil operator on $\boldsymbol{W}_{\mathbb{R},s}$ (cf. (6.18) and (6.20)). q.e.d.

(6.23) COROLLARY. *A primary $\mathbb{Z}$-VHS $\boldsymbol{W}_{\mathbb{Z}}$ is rigid if and only if the Lie group $G'_{\mathbb{R}} = R_{F/\mathbb{Q}}(SU(U, h'))_{\mathbb{R}}$ is compact.*

PROOF. In view of Corollary (2.4) and (6.21), $\boldsymbol{W}_{\mathbb{Z}}$ is rigid if and only if $\mathfrak{p}'^{+} = \{0\}$, which is equivalent to the compactness of $G'_{\mathbb{R}}$.

Let us fix a point $s \in S$, and consider the monodromy representation

$$\mu_s : \pi_1(S, s) \to Sp(\boldsymbol{W}_{\mathbb{Q},s}, Q) . \tag{6.24}$$

Then μ_s factors through ρ, i.e., there exists a homomorphism

$$\mu_{1,s} : \pi_1(S, s) \to G_{\mathbb{Q}} = R_{F/\mathbb{Q}}(SU(\boldsymbol{V}_s, \boldsymbol{h}_s)) , \tag{6.25}$$

such that $\mu_s = \rho \cdot \mu_{1,s}$.

(6.26) PROPOSITION. *If $G_{\mathbb{R}}$ is compact, the $\mathbb{Z}$-VHS $\boldsymbol{W}_{\mathbb{Z}}$ is locally trivial, and hence the corresponding abelian scheme $f: X \to S$ becomes isomorphic to the product $S' \times X_s$ after a finite base change $p: S' \to S$.*

PROOF. Since the image of $\mu_{1,s}$ is contained in a discrete subgroup of $G_{\mathbb{R}}$, the compactness of $G_{\mathbb{R}}$ implies the finiteness of the image of $\pi_1(S, s)$ under $\mu_{1,s}$.

(6.27) COROLLARY. *Let $f: X \to S$ be an abelian scheme such that the corresponding $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}} = \boldsymbol{R}_1 f_* \mathbb{Q}_X$ is primary (e.g., the generic fiber of f is simple). Let $G_{\mathbb{Q}}$ and $G'_{\mathbb{Q}}$ be the $\mathbb{Q}$-algebraic groups defined in* (6.5), *and set $G^{(i)}_{\mathbb{R}} = SU(\boldsymbol{V}^{(i)}_s, \boldsymbol{h}^{(i)}_s)$ and $G'^{(i)}_{\mathbb{R}} = SU(U^{(i)}, h'^{(i)})$ as in* (6.14) *and* (6.15). *Then we have the following*:

(i) *If $G^{(i)}_{\mathbb{R}}$ (resp. $G'^{(i)}_{\mathbb{R}}$) is non-compact, then the group $G'^{(i)}_{\mathbb{R}}$ (resp. $G^{(i)}_{\mathbb{R}}$) is compact. Therefore, if $f: X \to S$ in non-isotrivial, then $G_{\mathbb{R}}$ is non-compact, and hence at least one of $\{G'^{(i)}_{\mathbb{R}}\}$ is compact, i.e., $G'_{\mathbb{R}}$ has compact factors.*

(ii) *If $f: X \to S$ is non-rigid, then $G'_{\mathbb{R}}$ is non-compact, and hence at least one of $\{G^{(i)}_{\mathbb{R}}\}$ is compact, i.e., $G_{\mathbb{R}}$ has compact factors.*

(iii) *In particular, when $f: X \to S$ is non-isotrivial and non-rigid, one has $t = [F^+ : \mathbb{Q}] \geq 2$ and $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$ have both compact and non-compact factors.*

7. The Satake classification. In view of Theorem (6.21) and Corollary (6.27), the classification of non-rigid primary $\mathbb{Q}$-VHS, or the corresponding abelian schemes, can be reduced to that of the certain types of $\mathbb{Q}$-symplectic representations. Namely, in the notation of §6, if a primary $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}}$ over S is non-rigid if and only if $G'_{\mathbb{R}}$ is non-compact, and if it is non-isotrivial then $G_{\mathbb{R}}$ must be non-compact.

Satake [S2] classified $\mathbb{Q}$-primary symplectic representations of $\mathbb{Q}$-Hermitian pair $(G_{\mathbb{Q}}, H_0)$. We refer to his results in [S2] and to [S1, Ch. IV].

First, from Theorem (6.17), (iii), and Corollary (6.26) and the argument in [S1, Ch. IV, §6], we can deduce the following:

(7.1) THEOREM. *Let $W_{\mathbb{Q}}$ be a primary $\mathbb{Q}$-VHS of weight -1 of type $(0, -1)$ and $(0, -1)$ over S, V, D, F as defined in* (3.7), *and ι_0, $\boldsymbol{h}$ as in* (4.6). *Assume that $W_{\mathbb{Q}}$ is not isotrivial. Then the one of the following cases occurs*:

(R1) *$D=F$ is a totally real algebraic number field and with ι the identity, and h is a symplectic form on V_s $(\varepsilon=-1)$.*

(R2, ε) *D is a quaternion algebra over a totally real algebraic number field F and ι is the standard involution, while h is a (D, ε)-Hermitian form V with respect to ι, where $\varepsilon=\pm 1$.*

(C) *F is a* CM *field, i.e., a purely imaginary quadratic extension of a totally real algebraic number field F^+, D is a central division algebra over F, ι an involution of D of the second kind, and h a (D, ε)-Hermitian form with respect to ι, where $\varepsilon=\pm 1$.*

Moreover in each case, under the notation of §5, one has the following explicit descriptions of $F^{(i)}$, D^{τ_i}, $D^{(i)}$, $V_s^{(i)}$, $\boldsymbol{h}_s^{(i)}$, $U^{(i)}$, $h'^{(i)}$, $G_{\mathbb{R}}$, $G'_{\mathbb{R}}$ for the cases (R1), (R2, ε), (C) respectively.

(7.2) THEOREM (cf. [S1, Ch. IV, §6]). *Let $W_{\mathbb{Q}}$ be as in Theorem* (7.1), *$F^{(i)}$, D^{τ_i}, $D^{(i)}$, $V^{(i)}$, $h^{(i)}$, $U^{(i)}$, $h'^{(i)}$ as in §5, and $G_{\mathbb{R}}$, $G'_{\mathbb{R}}$ as in* (6.5). *Assume that $W_{\mathbb{Q}}$ is not isotrivial. Then for each of the cases* (R1), (R2, ε), (C), *we have the following*:

(R1) $(\varepsilon=-1) D=F=F^+$. *Set* $\dim_F V_s=n$, $\dim_F U=m$. *Then one has*

$$F^{(i)}\cong D^{\tau_i}\cong D^{(i)}\cong\mathbb{R}, \quad V_s^{(i)}\cong\mathbb{R}^n, \quad U^{(i)}\cong\mathbb{R}^m,$$

$\boldsymbol{h}_s^{(i)}$: *an $\mathbb{R}$-symplectic form on V_s^i, $(\eta_i=1)$ for $1\le i\le t=d$,*

$\boldsymbol{h}'^{(i)}$: *a positive definite $\mathbb{R}$-symmetric form on U^i, $(\eta_i=1)$ for $1\le i\le t=d$,*

and

$$C_s^{(i)}=1_{U^{(i)}}\otimes I_s^{(i)}$$

$$G_{\mathbb{R}}\cong\underbrace{Sp_{n/2}(\mathbb{R})\times\cdots\times Sp_{n/2}(\mathbb{R})}_{d\times(\mathrm{III})_{n/2}}. \tag{7.3}$$

$$G'_{\mathbb{R}}\cong\underbrace{SO_m(\mathbb{R})\times\cdots\times SO_m(\mathbb{R})}_{d\times \text{compact}}. \tag{7.4}$$

(R2, ε) *We have $F=F^+$, and D is a quaternion algebra over F. Set* $\operatorname{rank}_D V=n$, $\operatorname{rank}_D U=m$. *Then one has $F^{(i)}=\mathbb{R}$. After a suitable renumbering of $\{\tau_i\}$, we may assume that for some t', $0\le t'\le t$,*

$$D^{\tau_i} \cong \begin{cases} \mathbb{H} & 1 \le i \le t' \\ M_2(\mathbb{R}) & t'+1 \le i \le t, \end{cases} \qquad D^{(i)} \cong \begin{cases} \mathbb{H} & 1 \le i \le t' \\ \mathbb{R} & t'+1 \le i \le t. \end{cases}$$

Then one has

$$V_s^{(i)} \cong \begin{cases} \mathbb{H}^n \\ \mathbb{R}^{2n} \end{cases}, \qquad U^{(i)} \cong \begin{cases} \mathbb{H}^m \\ \mathbb{R}^{2m} \end{cases}, \qquad W_s^{(i)} \cong \begin{cases} \mathbb{H}^n \otimes_{\mathbb{H}} \mathbb{H}^m & 1 \le i \le t' \\ \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{R}^{2m} & t'+1 \le i \le t. \end{cases}$$

$(\varepsilon = 1)$

$$h_s^{(i)} = \begin{cases} \text{a positive definite } \mathbb{H}\text{-symmetric form } (\eta_i = 1) & 1 \le i \le t', \\ \text{an } \mathbb{R}\text{-symplectic form } (\eta_i = -1) & t'+1 \le i \le t, \end{cases}$$

$$h'^{(i)} = \begin{cases} \text{an } \mathbb{H}\text{-symplectic form } (\eta_i = 1) & 1 \le i \le t', \\ \text{a positive definite } \mathbb{R}\text{-symmetric form } (\eta_i = -1) & t'+1 \le i \le t, \end{cases}$$

$$C_s = \begin{cases} I'^{(i)} \otimes 1_{V_s^{(i)}} & 1 \le i \le t', \\ 1_{U^{(i)}} \otimes I_s^{(i)} & t'+1 \le i \le t, \end{cases}$$

(7.5) $$G_{\mathbb{R}} = \underbrace{SU_n(\mathbb{H}) \times \cdots \times SU_n(\mathbb{H})}_{t' \times \text{compact}} \times \underbrace{Sp_n(\mathbb{R}) \times \cdots \times Sp_n(\mathbb{R})}_{(t-t') \times (\text{III})_n},$$

(7.6) $$G'_{\mathbb{R}} = \underbrace{SU_m(\mathbb{H})^- \times \cdots \times SU_m(\mathbb{H})^-}_{t' \times (\text{II})_m} \times \underbrace{SO_{2m}(\mathbb{R}) \times \cdots \times SO_{2m}(\mathbb{R})}_{(t-t') \times \text{compact}}.$$

$(\varepsilon = -1)$

$$h_s^{(i)} = \begin{cases} \text{an } \mathbb{H}\text{-symplectic form } (\eta_i = 1) & 1 \le i \le t', \\ \text{a positive definite } \mathbb{R}\text{-symmetric form } (\eta_i = -1) & t'+1 \le i \le t, \end{cases}$$

$$h'^{(i)} = \begin{cases} \text{a positive definite } \mathbb{H}\text{-symmetric form } (\eta_i = 1) & 1 \le i \le t', \\ \text{an } \mathbb{R}\text{-symplectic form } (\eta_i = -1) & t'+1 \le i \le t, \end{cases}$$

$$C_s = \begin{cases} 1_{U^{(i)}} \otimes I_s^{(i)} & 1 \le i \le t', \\ I'^{(i)} \otimes 1_{V_s^{(i)}} & t'+1 \le i \le t, \end{cases}$$

(7.7) $$G_{\mathbb{R}} = \underbrace{SU_n(\mathbb{H})^- \times \cdots \times SU_n(\mathbb{H})^-}_{t' \times (\text{II})_n} \times \underbrace{SO_{2n}(\mathbb{R}) \times \cdots \times SO_{2n}(\mathbb{R})}_{(t-t') \times \text{compact}}.$$

(7.8) $$G'_{\mathbb{R}} = \underbrace{SU_m(\mathbb{H}) \times \cdots \times SU_m(\mathbb{H})}_{t' \times \text{compact}} \times \underbrace{Sp_m(\mathbb{R}) \times \cdots \times Sp_m(\mathbb{R})}_{(t-t') \times (\text{III})_m}.$$

(C) $(\varepsilon = \pm 1)$ *F is a purely imaginary quadratic extension of F^+, so $t = [F:\mathbb{Q}]/2$. We set $[D:F] = r^2$, $\mathrm{rank}_D V = n$, and $\mathrm{rank}_D U = m$. Then one has*

$$F^{(i)} \cong D^{(i)} \cong \mathbb{C}, \qquad D^{\tau_i} \cong M_r(\mathbb{C}),$$

$$V_s^{(i)} \cong \mathbb{C}^{nr}, \quad U^{(i)} \cong \mathbb{C}^{mr}, \quad W^{\tau_i} \cong \mathbb{C}^{mr} \otimes_{\mathbb{C}} \mathbb{C}^{nr}.$$

We may assume that for t', $0 \leq t' \leq t$,

$$h_s^{(i)} = \begin{cases} \mathbb{C}\text{-symplectic form with signature } (p_i, q_i) & 1 \leq i \leq t' \quad (p_i \geq q_i > 0)\,, \\ \text{positive definite } \mathbb{C}\text{-Hermitian form} & t'+1 \leq i \leq t\,, \end{cases}$$

$$h'^{(i)} = \begin{cases} \text{positive definite } \mathbb{C}\text{-Hermitian form} & 1 \leq i \leq t'\,, \\ \mathbb{C}\text{-symplectic form with signature } (p'_i, q'_i) & t'+1 \leq i \leq t\,, \quad (p'_i \geq q'_i > 0)\,, \end{cases}$$

$$G_{\mathbb{R}} \cong \prod_{i=1}^{t'} \underbrace{SU(p_i, q_i, \mathbb{C})}_{(\mathrm{I})_{p_i q_i}} \times \underbrace{SU_{nr}(\mathbb{C}) \times \cdots \times SU_{nr}(\mathbb{C})}_{(t-t') \times \text{compact}} \tag{7.9}$$

$$G'_{\mathbb{R}} \cong \underbrace{SU_{nr}(\mathbb{C}) \times \cdots \times SU_{nr}(\mathbb{C})}_{t' \times \text{compact}} \times \prod_{i=1}^{t'} \underbrace{SU(p'_i, q'_i, \mathbb{C})}_{(\mathrm{I})_{p'_i p'_i}} \tag{7.10}$$

8. Geometric results. The following theorem is a consequence of Corollary (6.27) and Theorem (7.2).

(8.1) THEOREM. *Let* $f: X \to S$ *be an abelian scheme such that the corresponding* $\mathbb{Q}$-VHS $W_{\mathbb{Q}} = R_1 f_* \mathbb{Q}_X$ *is primary (e.g., the generic fiber* X_η *of* f *is simple). Let* V, $D = \mathrm{End}(V)$ *and* U *be as in* (3.8), *and* $W_{\mathbb{Q}} = U \otimes_D V$ *the tensor product decomposition of* $W_{\mathbb{Q}}$ *as in* (3.11). *Set* $\mathrm{rank}_D U = m$, $\mathrm{rank}_D V = n$, *and* $t = [F^+ : \mathbb{Q}]$ *(see* §3 *and* §5 *for notation). Assume that* $f: X \to S$ *is non-isotrivial and non-rigid.*

(i) *If the center* F *of* D *is totally real (i.e.,* $F = F^+$*), then* D *is a quaternion algebra over* $F = F^+$ *such that*

$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong \underbrace{\mathbb{H} \times \cdots \times \mathbb{H}}_{t'} \times \underbrace{M_2(\mathbb{R}) \times \cdots \times M_2(\mathbb{R})}_{t-t'}\,.$$

Hence if one denotes by $r(f)$ *the relative dimension of* $f: X \to S$, *one has*

$$r(f) = \frac{1}{2} \cdot \mathrm{rank}_{\mathbb{Q}}\, U \otimes_D V = 2tmn\,. \tag{8.2}$$

Here one must have $t' > 0$ *and* $t - t' > 0$, *hence in particular* $t = [F: \mathbb{Q}] \geq 2$. *Moreover one of the following cases occurs (see Theorem* (7.2)*):*

$$\textit{Case } (\mathrm{R2}, 1) \quad n \geq 1 \quad \textit{and} \quad m \geq 2\,,$$

$$\textit{Case } (\mathrm{R2}, -1) \quad n \geq 2 \quad \textit{and} \quad m \geq 1\,.$$

In particular, the relative dimension $r(f)$ *is even, and* ≥ 8.

(ii) *If the center* F *of* D *is a* CM *field (i.e.,* $[F: F^+] = 2$*), then* D *is a central simple division algebra over* F *such that* $[D: F] = r^2$ *and*

$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong \underbrace{M_r(\mathbb{C}) \times \cdots \times M_r(\mathbb{C})}_{t}\,.$$

In this case, one has

(8.3) $$r(f)=\frac{1}{2}2tnmr^2=t(nr)(mr)\,.$$

Moreover, the bilinear forms $\boldsymbol{h}_s$ *must be indefinite at a place* $\tau_i\colon F^+\hookrightarrow\mathbb{R}$ *and definite at some other places. And* h' *satisfies the condition in* (C) *in Theorem* (7.2). *Hence* $t=[F^+:\mathbb{Q}]=[F:\mathbb{Q}]/2\geq 2$, $nr\geq 2$, $mr\geq 2$. *In particular,* $r(f)\geq 8$.

(8.4) COROLLARY. *Let* $f\colon X\to S$ *be an abelian scheme which has no isotrivial factors. The abelian scheme is rigid, if the relative dimension* $r(f)$ *of* f *is less than* 8.

(8.5) COROLLARY. *Let* $f\colon X\to S$ *be an abelian scheme whose generic fiber* X_η *is simple. Assume that* f *has no-isotrivial factor and the relative dimension of* f *is a prime integer. Then* $f\colon X\to S$ *is rigid.*

The following theorem is a consequence of Corollary (6.27), and we call it the *monodromy theorem.*

(8.6) THEOREM. *Let* $f\colon X\to S$ *be an abelian scheme such that the corresponding* $\mathbb{Q}$-VHS $\boldsymbol{W}_{\mathbb{Q}}=\boldsymbol{R}_1f_*\mathbb{Q}_X$ *is primary* (*e.g., the generic fiber of* f *is simple*). *Assume that* S *is non-compact and a local monodromy around a point in the boundary has infinite order. Then* $f\colon X\to S$ *is rigid.*

PROOF. The image of the monodromy representation of $\pi_1(S,s)$ lies in $G_{\mathbb{Q}}$ (see (6.25)). If $f\colon X\to S$ is non-rigid, from Corollary (6.27), $G_{\mathbb{R}}$ has a compact factor, hence, in particular, the $\mathbb{Q}$-rank of $G_{\mathbb{Q}}$ is zero. On the other hand, it is known that the monodromy of the $\mathbb{Z}$-VHS around the boundary divisor is quasi-unipotent, a contradiction to the assumption.

9. Examples of non-rigid abelian schemes. In this section, we will give examples of non-rigid abelian schemes and show that Theorem (8.1) is the best possible, i.e., in both cases (i) and (ii) in Theorem (8.1), one can give examples of abelian schemes with a given relative dimension. Such examples shall be obtained as *Kuga fiber spaces of abelian varieties*, which are constructed from $\mathbb{Q}$-symplectic representations of $\mathbb{Q}$-algebraic groups.

(9.1) Kuga fiber spaces. Let $(G_{\mathbb{Q}}, H_0)$ be a $\mathbb{Q}$-Hermitian pair and $(W_{\mathbb{Q}}, \rho_{\mathbb{Q}}, Q_{\mathbb{Q}}, I)$ a $\mathbb{Q}$-symplectic representation of $(G_{\mathbb{Q}}, H_0)$ (see Definition (6.1)). By a lattice $W_{\mathbb{Z}}$ of $W_{\mathbb{Q}}$, we mean a free $\mathbb{Z}$-submodule $W_{\mathbb{Z}}$ of $W_{\mathbb{Q}}$ such that $W_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{Q}\cong W_{\mathbb{Q}}$ and

$$Q_{\mathbb{Q}}(W_{\mathbb{Z}}, W_{\mathbb{Z}})\subset\mathbb{Z}\,.$$

Such a quintuple $(W_{\mathbb{Q}}, \rho_{\mathbb{Q}}, Q_{\mathbb{Q}}, I, W_{\mathbb{Z}})$ is called a *Kuga quintuple*. Let K be the maximal compact subgroup of $G_{\mathbb{R}}$ determined by H_0, and denote by $\mathscr{D}:=G_{\mathbb{R}}/K$ the corresponding Hermitian symmetric space. The representation $\rho_{\mathbb{Q}}\colon G_{\mathbb{Q}}\to Sp(W_{\mathbb{Q}}, Q_{\mathbb{Q}})$ induces a representation $\rho_{\mathbb{R}}\colon G_{\mathbb{R}}\to Sp(W_{\mathbb{R}}, Q_{\mathbb{R}})$, and an equivariant holomorphic embedding $h: \mathscr{D} \hookrightarrow \mathscr{D}(W_{\mathbb{R}}, Q_{\mathbb{R}}) \cong Sp(W_{\mathbb{R}}, Q_{\mathbb{R}})/K'$ with respect to $\rho_{\mathbb{R}}$. Note that $\mathscr{D}(W_{\mathbb{R}}, Q_{\mathbb{R}})$ is isomorphic to the Siegel upper half plane $(\mathrm{III})_k$ where $k=(1/2)\dim W_{\mathbb{R}}$.

The lattice $W_{\mathbb{Z}}$ determines an arithmetic subgroup $\Gamma_{W_{\mathbb{Z}}}=\{g \in Sp(W_{\mathbb{R}}, Q_{\mathbb{R}} \mid gW_{\mathbb{Z}}= W_{\mathbb{Z}}\}$, and a subgroup $\rho_{\mathbb{Q}}^{-1}(\Gamma_{W_{\mathbb{Z}}})$ of $G_{\mathbb{R}}$ becomes arithmetic. There exists a torsion-free subgroup $\Gamma \subset \rho_{\mathbb{Q}}^{-1}(\Gamma_{W_{\mathbb{Z}}})$ of a finite index, so that the quotient space $\Gamma \backslash \mathscr{D}$ becomes a smooth quasi-projective variety (cf. [Ba-Bo]). It is well-known (or easy to see) that there exists a universal $\mathbb{Z}$-VHS $\phi: \mathscr{W}_{\mathbb{Z}} \to \mathscr{D}(W_{\mathbb{R}}, Q_{\mathbb{R}})$ of weight -1 and of type $(-1, 0)$, $(0, -1)$, whose typical fibers are isomorphic to $W_{\mathbb{Z}}$. Moreover, there exists the corresponding universal family of abelian varieties. Via the equivariant embedding $h: \mathscr{D} \hookrightarrow \mathscr{D}(W_{\mathbb{R}}, Q_{\mathbb{R}})$, one can pull back the $\mathbb{Z}$-VHS ϕ to a $\mathbb{Z}$-VHS over $\mathscr{D}$, and moreover descends it to a $\mathbb{Z}$-VHS over the quotient variety $M_\Gamma := \Gamma \backslash \mathscr{D}$. Hence one obtains the corresponding abelian scheme $f: X_\Gamma \to M_\Gamma = \Gamma \backslash \mathscr{D}$ (see (1.10)).

(9.2) Definition (cf. [S1, Ch. IV, §7]). The abelian scheme $f: X_\Gamma \to M_\Gamma = \Gamma \backslash \mathscr{D}$ constructed from a given Kuga quintuple and a torsion-free subgroup $\Gamma \subset G_{\mathbb{Q}}$ is called the *Kuga fiber space* of abelian varieties.

(9.3) Remark. The fiber spaces of abelian varieties above have been studied from many points of view by many people such as Kuga, Shimura, Satake, Mumford, et al. The reader can find many references about Kuga fiber spaces in §7 of Ch. IV and the References in Satake [S1].

(9.4) Quaternion algebras. We shall quickly review the theory of quaternion algebras following Satake [S1, Appendix, §2]. Let F be a field of characteristic different from 2. A *quaternion algebra* over F is, by definition, a central simple algebra over F with $[D: F]=4$. If D is not a division albgebra, one has $D \cong M_2(F)$, in which case D is called a *split* quaternion algebra.

For given $\alpha, \beta \in F^\times$, one can define a quaternion algebra $D(\alpha, \beta)$ as an algebra with the unit element 1 over F generated by two elements x_1, x_2 satisfying

$$x_1^2=\alpha, \quad x_2^2=\beta, \quad x_1x_2=-x_2x_1 .$$

Let $\mathscr{B}(F)$ denote the Brauer group of F, and $\mathrm{Cl}(D) \in \mathscr{B}(F)$ the Brauer class of D. Since $[D: F]=2^2$, $\mathrm{Cl}(D)$ lies in the subgroup ${}_2\mathscr{B}(F)$ of $\mathscr{B}(F)$ consisting of the elements of order at most 2.

If F is a local field, the Brauer group is

$$\mathscr{B}(F) \cong \begin{cases} 1 & \text{if } F \cong \mathbb{C}, \\ \mathbb{Z}/2 & \text{if } F \cong \mathbb{R}, \\ \mathbb{Q}/\mathbb{Z} & \text{otherwise}. \end{cases}$$

In these cases, $\mathrm{Cl}(D(\alpha, \beta))$ is given by the Hilbert symbol $(\alpha, \beta)_F$, that is,

$$\mathrm{Cl}(D(\alpha,\beta))=(\alpha,\beta)_F=\begin{cases}1 & \text{if } \alpha x^2+\beta y^2=1 \text{ has a solution in } F,\\ -1 & \text{otherwise}.\end{cases}$$

Note that $\mathrm{Cl}(D(\alpha,\beta))=1$ if and only if $D(\alpha,\beta)$ splits.

Now let F be an algebraic number field, $\Omega(F)$ the set of all valuations of F. Consider the quaternion algebra $D(\alpha,\beta)$ for $\alpha,\beta\in F^\times$. For a valuation $v\in\Omega(F)$, denote by F_v the completion of F with respect to v, and set

$$D(\alpha,\beta)_v=D(\alpha,\beta)\otimes_F F_v .$$

Then the Hilbert reciprocity law says that for all most all $v\in\Omega(F)$, one has $\mathrm{Cl}(D(\alpha,\beta)_v)=1$, and

$$\prod_{v\in\Omega(F)}\mathrm{Cl}(D(\alpha,\beta)_v)=1 . \tag{9.5}$$

Conversely, if T is a finite subset of $\Omega(F)$ consisting of an even number of discrete or real valuations of F, then there exist $\alpha,\beta\in F^\times$ such that

$$\mathrm{Cl}(D(\alpha,\beta)_v)=\begin{cases}-1 & \text{if } v\in T,\\ 1 & \text{if } v\in\Omega(F)-T.\end{cases}$$

(See [O'M, Theorem [71 : 19]].)

From this fact, one can see the following:

(9.6) Proposition. *For an arbitrary positive integer t and an integer t' satisfying $0\le t'\le t$, there exist a totally real number field F of degree t and a quaternion algebra D such that*

$$D\otimes_{\mathbb{Q}}\mathbb{R}\cong\underbrace{\mathbb{H}\times\cdots\times\mathbb{H}}_{t'}\times\underbrace{M_2(\mathbb{R})\times\cdots\times M_2(\mathbb{R})}_{t-t'} . \tag{9.7}$$

Proof. It is well-known that there is a totally real number field F of arbitrary degree. The existence of the quaternion algebra over F with arbitrary spliting type follows from the converse of the Hilbert reciprocity theorem.

(9.8) Examples of type (R2, ± 1). Let t be an arbitrary positive integer, t' an integer such that $0\le t'\le t$, and let F and D be as in Proposition (9.5). We denote by ι_0 the standard involution of D. For positive integers n and m, set

$$V:=D^n,\quad U:=D^m .$$

We regard V as a left D-module, which can also be regarded as a right $\bar{D}$-module via the action

$$v\cdot\alpha=\alpha^{\iota_0}\cdot v$$

for $v\in V$ and $\alpha\in D$. We regard U as a right D-module.

Taking $s \in D^\times$ an element skew with respect to ι_0, i.e., $s^{\iota_0} = -s$, we define a $(\bar{D}, 1)$-Hermitian form h on V and a $(D, -1)$-Hermitian form h' on U by

$$h(x, y) = \sum_{i=1}^{n} x_i \cdot y_i^{\iota_0}, \quad h'(x, y) = \sum_{i=1}^{m} x_i^{\iota_0} \cdot s \cdot y_i. \tag{9.9}$$

Now consider the $\mathbb{Q}$-algebraic groups

$$G_{\mathbb{Q}} = R_{F/\mathbb{Q}}(SU(V, h)), \quad G'_{\mathbb{Q}} = R_{F/\mathbb{Q}}(SU(U, h')), \tag{9.10}$$

and denote by $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$ the corresponding $\mathbb{R}$-groups, respectively. From Lemma (5.18) and Theorem (7.2), one has the decompositions

$$G_{\mathbb{R}} = \underbrace{SU_n(\mathbb{H}) \times \cdots \times SU_n(\mathbb{H})}_{t' \times \text{compact}} \times \underbrace{Sp_n(\mathbb{R}) \times \cdots \times Sp_n(\mathbb{R})}_{(t-t') \times (\mathrm{III})_n}, \tag{9.11}$$

$$G'_{\mathbb{R}} = \underbrace{SU_m(\mathbb{H})^- \times \cdots \times SU_m(\mathbb{H})^-}_{t' \times (\mathrm{II})_m} \times \underbrace{SO_{2m}(\mathbb{R}) \times \cdots \times SO_{2m}(\mathbb{R})}_{(t-t') \times \text{compact}}. \tag{9.12}$$

Note that $G_{\mathbb{R}}$ is non-compact if and only if $t - t' > 0$, while $n \geq 1$, and $G'_{\mathbb{R}}$ is non-compact if and only if $t' > 0$ and $m \geq 2$. Setting

$$W_{\mathbb{Q}} = U \otimes_D V, \qquad Q := \mathrm{Tr}_{D/\mathbb{Q}}(h' \otimes h), \tag{9.13}$$

one has natural representations

$$\rho_1 : G_{\mathbb{Q}} \to Sp(W_{\mathbb{Q}}, Q), \qquad \rho_2 : G'_{\mathbb{Q}} \to Sp(W_{\mathbb{Q}}, Q).$$

One can get a complex structure I on $W_{\mathbb{R}}$ (see (R2, 1) in Theorem (7.2)), so that $(W_{\mathbb{Q}}, Q, \rho_1, I)$ become a $\mathbb{Q}$-symplectic representation of $G_{\mathbb{Q}}$ (with respec to some H-element H_0) of type (R2, 1) (see §6 and (7.2)). Therefore we obtain a $\mathbb{Z}$-VHS $\phi : \mathscr{W}_{\mathbb{Z}} \to M_\Gamma$ where $M_\Gamma = \Gamma \backslash \mathscr{D}$ is an arithmetic quotient of the Hermitian symmetric space

$$\mathscr{D} := G_{\mathbb{R}}/K \cong \underbrace{(\mathrm{III})_n \times \cdots \times (\mathrm{III})_n}_{t-t' \text{ times}}, \tag{9.14}$$

and the corresponding Kuga fiber space $f : X_\Gamma \to M_\Gamma$. Similarly, $(W_{\mathbb{Q}}, Q, \rho_2, I)$ becomes a $\mathbb{Q}$-symplectic representation of $G'_{\mathbb{R}}$ of type (R2, -1). Hence we obtain a $\mathbb{Z}$-VHS $\phi' : \mathscr{W}'_{\mathbb{Z}} \to M'_{\Gamma'}$ where $M'_{\Gamma'} = \Gamma' \backslash \mathscr{D}'$ is an arithmetic quotient of the Hermitian symmetric space

$$\mathscr{D}' := G'_{\mathbb{R}}/K' \cong \underbrace{(\mathrm{II})_m \times \cdots \times (\mathrm{II})_m}_{t' \text{ times}}, \tag{9.15}$$

and the corresponding Kuga fiber space $f' : X'_\Gamma \to M'_{\Gamma'}$.

(9.16) Definition-Proposition. *The Kuga fiber space $f : X_\Gamma \to M_\Gamma$ (resp. $f' : X'_\Gamma \to M'_{\Gamma'}$) as above is said to be of type* (R2, 1) (*resp.* (R2, -1)). *The relative dimension f (resp.*

f') is $2tnm$, and the fiber space has no isotrivial factors if $\dim M_\Gamma > 0$ (*resp.* $\dim M'_{\Gamma'} > 0$).

Moreover, the data $(W_\mathbb{Q}, Q, \rho_1 \otimes \rho_2, I)$ become a $\mathbb{Q}$-symplectic representation of the product group $G_\mathbb{Q} \times G'_\mathbb{Q}$, and therefore we obtain a $\mathbb{Z}$-VHS $\tilde{\Phi}: \hat{\mathscr{W}}_\mathbb{Z} \to M_\Gamma \times M'_{\Gamma'}$, and the corresponding Kuga fiber space $\tilde{f}: (X_{\Gamma\times\Gamma'})^\wedge \to M_\Gamma \times M'_{\Gamma'}$.

If we take a suitable point $[o] \in M'_\Gamma$, the family $\tilde{f}_{|o\times M_\Gamma}: (X_{\Gamma\times\Gamma'})^\wedge_{|o\times M_\Gamma} \to [o] \times M_\Gamma$ is isomorphic to the original Kuga fiber space $f: X_\Gamma \to M_\Gamma$, and the family $\tilde{f}: (X_{\Gamma\times\Gamma'})^\wedge \to M_\Gamma \times M'_{\Gamma'}$ can be regarded as a deformation of the original abelian scheme f of type (R2, 1) with the parameter space M'_Γ. We can interchange the roles of M_Γ and $M'_{\Gamma'}$ and regard M_Γ as the parameter space for the deformation of the Kuga fiber space $f': X'_\Gamma \to M'_{\Gamma'}$ of type (R2, -1).

From these facts, we can deduce the following:

(9.17) THEOREM. *Let t, n, m be arbitrary positive integers, and t' an integer such that $0 \leq t' \leq t$.*

(i) *One has $\mathbb{Q}$-algebraic groups $G_\mathbb{Q}$ and $G'_\mathbb{Q}$ of Hermitian type defined in* (9.10) *and $\mathbb{Q}$-symplectic representations $(W_\mathbb{Q}, Q, \rho_1, I)$ for $G_\mathbb{Q}$ of type* (R2, 1) *and $(W_\mathbb{Q}, Q, \rho_2, I)$ of $G'_\mathbb{Q}$ of type* (R2, -1) *such that $\dim_\mathbb{Q} W_\mathbb{Q} = 4tmn$.*

Hence we obtain a Kuga fiber space $f: X_\Gamma \to M_\Gamma$ of type (R2, 1) *and a Kuga fiber space $f': X'_\Gamma \to M'_{\Gamma'}$ of type* (R2, -1), *where $M_\Gamma = \Gamma\backslash\mathscr{D}$ and $M'_{\Gamma'} = \Gamma'\backslash\mathscr{D}'$ are arithmetic quotients of the Hermitian symmetric spaces $\mathscr{D}$ and $\mathscr{D}'$ in* (9.14) *and* (9.15), *respectively.*

(ii) *Moreover from the tensor product representation $\rho_1 \otimes \rho_2$ one obtains a Kuga fiber space $\tilde{f}: (X_{\Gamma\times\Gamma'})^\wedge \to M_\Gamma \times M'_{\Gamma'}$, which Kuga fiber space $\tilde{f}$ gives the deformation of both Kuga fiber spaces f and f'.*

(iii) *If* $\dim M_\Gamma > 0$, *i.e., if $t - t' > 0$ and $n \geq 1$, then $f: X_\Gamma \to M_\Gamma$ has no isotrivial factor, and if* $\dim M'_{\Gamma'} > 0$, *i.e., if $t' > 0$ and $m \geq 2$, f is non-rigid. Conversely, if* $\dim M'_{\Gamma'} > 0$, *f' has no isotrivial factor, and if* $\dim M_\Gamma > 0$, *then f' is non-rigid.*

(9.18) COROLLARY. *For all even integer $r \geq 8$, there exists a non-rigid abelian scheme of relative dimension r of type* (R2, ± 1) *with no isotrivial factor.*

(9.19) REMARK. In Theorem (9.17), one has $\dim M_\Gamma = (t - t') \times n(n+1)/2$ and $\dim M'_{\Gamma'} = t' \times m(m-1)/2$. Following Deligne's suggestion, Faltings [F] gave an example of non-rigid abelian schemes over a modular curve of relative dimension 8 which has no isotrivial factor. In our notation, his example corresponds to a Kuga fiber space of type (R2, 1) with $t = [F: \mathbb{Q}] = 2$, $t' = 1$, (i.e., $D \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{H} \times M_2(\mathbb{R})$ and $n = 1$ and $m = 2$).

(9.20) Examples of type (C). In this subsection, we shall give a non-rigid abelian scheme of type (C) in Theorem (7.2). Let t be an arbitrary positive integer, and F a CM field of degree $2t$, i.e., a purely imaginary quadratic extension of a totally real field F^+ of degree t. The complex conjugation of F will be denoted by ι_0, and let $\{\tau_i: F^+ \hookrightarrow \mathbb{R}\}_{i=1}^t$ be the set of all distinct embeddings of F^+ into $\mathbb{R}$.

For positive integers n and m, set

$$V := F^{\oplus n}, \quad U := F^{\oplus m}.$$

Taking two sequences $\{\alpha_j\}_{j=1}^n$ and $\{\beta_k\}_{k=1}^m$ of non-zero elements in F^+, we can define $((F, 1)$-)Hermitian forms h and h' on V and U by

$$h(x, y) = \sum_{j=1}^{n} x_j^{\iota_0} \cdot \alpha_j \cdot y_j, \tag{9.21}$$

$$h'(x', y') = \sum_{k=1}^{m} x_k'^{\iota_0} \cdot \beta_k \cdot y_k'. \tag{9.22}$$

Since one has isomorphisms

$$V^{(i)} := V \otimes_{F^+, \tau_i} \mathbb{R} \cong \mathbb{C}^n, \quad U^{(i)} := U \otimes_{F^+, \tau_i} \mathbb{R} \cong \mathbb{C}^m,$$

one obtains a Hermitian form $h^{(i)}$ on $V^{(i)}$ induced by h, as well as a Hermitian form $h'^{(i)}$ on $U^{(i)}$ induced by h'.

(9.23) LEMMA. *Assume that for every i, $1 \le i \le t$, a pair of non-negative integers (p_i, q_i) such that $p_i + q_i = n$ is given. Then we can choose $\{\alpha_j\}_{j=1}^n$ in such a way that the corresponding Hermitian form $h(x, y)$ in* (9.21) *induces a Hermitian form $h^{(i)}$ on $V^{(i)}$ with the pre-assigned signature $(+, -) = (p_i, q_i)$. The same is true for h'.*

PROOF. The induced Hermitian form $h^{(i)}$ has the form

$$h^{(i)}(x, y) = \sum_{j=1}^{n} x_j^{\iota_0} \cdot \tau_i(\alpha_j) \cdot y_j.$$

Then the assertion follows from a well-known result in number theory, that is, there exists a non-zero element $\alpha \in F^+$ such that $\tau_i(\alpha)$ for every i has a pre-assigned sign.

Now let us take an element $\theta \in F$ such that $\theta^{\iota_0} = -\theta$. Setting

$$W_{\mathbb{Q}} = U \otimes_F V, \quad Q_{\mathbb{Q}} = \mathrm{Tr}_{F/\mathbb{Q}}((\theta \cdot h') \otimes h),$$

we obtain a $\mathbb{Q}$-symplectic vector space $(W_{\mathbb{Q}}, Q_{\mathbb{Q}})$ of dimension $2tnm$. Define $\mathbb{Q}$-algebraic groups $G_{\mathbb{Q}}$ and $G'_{\mathbb{Q}}$ as in (9.10), and assume that we are given an integer t', $0 \le t' \le t$, pairs of integers (p_i, q_i), $p_i + q_i = n$ for $1 \le i \le t'$, (p'_i, q'_i), $p'_i + q'_i = m$ for $t' + 1 \le i \le t$. Then by Lemma (9.23), we may assume that the Hermitian forms h and h' satisfy

$$h^{(i)} = \begin{cases} \text{a } \mathbb{C}\text{-Hermitian form with signature } (p_i, q_i) & 1 \le i \le t' \quad (p_i \ge q_i > 0), \\ \text{a positive definite } \mathbb{C}\text{-Hermitian form} & t' + 1 \le i \le t, \end{cases}$$

$$h'^{(i)} = \begin{cases} \text{a positive definite } \mathbb{C}\text{-Hermitian form} & 1 \le i \le t', \\ \text{a } \mathbb{C}\text{-Hermitian form with signature } (p'_i, q'_i) & t' + 1 \le i \le t, \quad (p'_i \ge q'_i > 0). \end{cases}$$

Then the groups $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$ of $\mathbb{R}$-valued points of $G_{\mathbb{Q}}$ and $G'_{\mathbb{Q}}$ are given by

$$G_{\mathbb{R}} \cong \prod_{i=1}^{t'} \underbrace{SU(p_i, q_i, \mathbb{C})}_{(\mathrm{I})_{p_i q_i}} \times \underbrace{SU_n(\mathbb{C}) \times \cdots \times SU_n(\mathbb{C})}_{(t-t') \times \text{compact}}, \tag{9.24}$$

$$G'_{\mathbb{R}} \cong \underbrace{SU_m(\mathbb{C}) \times \cdots \times SU_m(\mathbb{C})}_{t' \times \text{compact}} \times \prod_{i=1}^{t'} \underbrace{SU(p'_i, q'_i, \mathbb{C})}_{(\mathrm{I})_{p'_i q'_i}}. \tag{9.25}$$

As in (9.8), one can obtain representations

$$\rho_1 : G_{\mathbb{Q}} \to Sp(W_{\mathbb{Q}}, Q_{\mathbb{Q}}), \tag{9.26}$$

$$\rho_2 : G'_{\mathbb{Q}} \to Sp(W_{\mathbb{Q}}, Q_{\mathbb{Q}}), \tag{9.27}$$

so that for a suitable complex structure I on $W_{\mathbb{R}}$, the data $(W_{\mathbb{Q}}, Q_{\mathbb{Q}}, \rho_i, I)$ for $i=1,2$ become $\mathbb{Q}$-symplectic representations of $G_{\mathbb{Q}}$ and $G'_{\mathbb{Q}}$, respectively. From (9.24) and (9.25), one can see that the corresponding Hermitian symmetric spaces are given by

$$\mathscr{D} = G_{\mathbb{R}}/K \cong \prod_{i=1}^{t'} (\mathrm{I})_{p_i, q_i}, \tag{9.28}$$

$$\mathscr{D}' = G'_{\mathbb{R}}/K' \cong \prod_{i=t'+1}^{t} (\mathrm{I})_{p'_i, q'_i}. \tag{9.29}$$

Choosing suitable torsion-free arithmetic subgroups $\Gamma \subset G_{\mathbb{Q}}$ and $\Gamma' \subset G'_{\mathbb{Q}}$, one obtains Kuga fiber spaces

$$f : X_{\Gamma} \to M_{\Gamma} = \Gamma \backslash \mathscr{D}, \tag{9.30}$$

$$f' : X'_{\Gamma'} \to M'_{\Gamma'} = \Gamma' \backslash \mathscr{D}'. \tag{9.31}$$

As in (9.8), one can also obtain a Kuga fiber space $\tilde{f} : (X_{\Gamma \times \Gamma'})^{\wedge} \to M_{\Gamma} \times M'_{\Gamma'}$ induced by the $\mathbb{Q}$-symplectic representation $(W_{\mathbb{Q}}, Q_{\mathbb{Q}}, \rho_1 \otimes \rho_2, I)$ of $G_{\mathbb{Q}} \times G'_{\mathbb{Q}}$. Therefore one has the following:

(9.32) THEOREM. *Let t, n and m be arbitrary positive integers, and t' an integer such that $0 \le t' \le t$. Assume that the signatures (p_i, q_i) for $1 \le i \le t'$ and (p'_i, q'_i) for $t'+1 \le i \le t$ are given.*

(i) *There exists Kuga fiber spaces $f : X_{\Gamma} \to M_{\Gamma}$ and $f' : X'_{\Gamma'} \to M'_{\Gamma'}$ constructed from $\mathbb{Q}$-symplectic representations* (9.26) *and* (9.27) *of type* (C), *whose relative dimensions are equal to tnm. Here M_{Γ} (resp. $M'_{\Gamma'}$) is an arithmetic quotient of a product of Hermitian symmetric domains of type $(\mathrm{I})_{p_i, q_i}$ in* (9.27) *(resp. $(\mathrm{I})_{p'_i, q'_i}$ in* (9.28)).

(ii) *From the tensor representation $\rho_1 \otimes \rho_2$, one obtains a Kuga fiber space $\tilde{f} : (X_{\Gamma \times \Gamma'})^{\wedge} \to M_{\Gamma} \times M'_{\Gamma}$. This gives deformations of f and f'.*

(iii) *If $\dim M_{\Gamma} > 0$, i.e., $t' > 0$, then the Kuga fiber space $f : X_{\Gamma} \to M_{\Gamma}$ has no isotrivial factor, and if $\dim M'_{\Gamma'} > 0$, i.e., $t - t' > 0$, then f is non-rigid. Conversely, if $\dim M'_{\Gamma'} > 0$, then f' has no isotrivial factor, and if $\dim M_{\Gamma} > 0$, then f' is non-rigid.*

(9.33) REMARK. In order to obtain a non-rigid Kuga fiber space of type (C) which has no isotrivial factor, one must have $t \geq 2$, $n \geq 2$, and $m \geq 2$. Hence the minimal relative dimension $r(f)$ for non-rigid Kuga fiber space is 8, when $t=n=m=2$. In this case, one has $G_{\mathbb{R}} \cong SU(1, 1, \mathbb{C}) \times SU(2, \mathbb{C})$ and $G'_{\mathbb{R}} \cong SU(2, \mathbb{C}) \times SU(1, 1, \mathbb{C})$.

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