Tôhoku Math. J.
45 (1993), 147–157

ON TOWERS OF LIFTINGS AND HYPERCUSPIDALITY FOR UNITARY GROUPS

TAMIR BOAZ

(Received December 9, 1991, revised October 29, 1992)

Abstract. Given a cuspidal automorphic representation on $U(2, 3)$, then its theta lift to $U(i, i)$ is cuspidal if and only if its theta lift to $U(i-1, i-1)$ is zero. Also, the theta lift of a cuspidal generic representation from $U(2, 3)$ to $U(3, 3)$ is generic. The theta lift of a cuspidal representation from $U(2, 3)$ to $U(4, 4)$ or to $U(5, 5)$ is hypercuspidal.

In this paper we study the cuspidality and hypercuspidality of some automorphic forms on the group $U(i, i)$ for $i=1$ to 5. These automorphic forms are "theta lifted" from automorphic forms belonging to the space of a cuspidal representation π for the group $U(2, 3)$.

In Chapter 1 we study the tower of liftings $\theta^i(\pi, s)$ for $i=1$ to 5 to find conditions for the cuspidality of the lift $\theta^i(\pi, s)$ in terms of the lift $\theta^{i-1}(\pi, s)$. More explicitly, Theorem 1.1 states that $\theta^i(\pi, s)$ is cuspidal if and only if $\theta^{i-1}(\pi, s)$ is zero. Moreover, $\theta^5(\pi, s)$ is nonzero, so higher theta lifts cannot be cuspidal. Therefore we stop the tower at $i=5$. These are well-known results for split groups (cf. [Ra]).

In Chapter 2 we generalize some results of [Wa], concerning the hypercuspidality of such lifts. Theorem 2.1 states that the lift $\theta^3(\pi, s)$ is already nonzero for generic representations π on $U(2, 3)$. Moreover, a Whittaker function of the lift can be expressed in terms of a Whittaker function of π. Theorem 2.2 states that $\theta^4(\pi, s)$ and $\theta^5(\pi, s)$, if cuspidal, are also hypercuspidal in the sense that all Whittaker functions disappear.

In the proof of Theorem 2.1 we use the Witt decomposition for the space of $U(2, 3)$, i.e., the existence of a maximal isotropic subspace of dimension two, and an anisotropic subspace of dimension one. In general if π is a cuspidal generic representation of $U(n, n+1)$, then the $n+1$ lift should also be generic. All theta lifts above this level should be hypercuspidal.

We conclude by remarking that the "simpler" tower, $U(1, 2)$ to $U(i, i)$, for $i=1$ to 3 is computed in [Wa]. If π on $U(1, 2)$ is generic, then the lift to $U(2, 2)$ is also generic [Wa, Theorem 4.3]. Using Theorem 2.2, it is easy to show that the theta lift of $U(1, 2)$ to $U(3, 3)$ is hypercuspidal.

1991 *Mathematics Subject Classification*. Primary 11F27.

This work was done during the author's stay in Bar-Ilan University, Ramat-Gan, Israel for his post-doctoral research in the year 1991.

ACKNOWLEDGMENT. I would like to thank Professor S. Gelbart of the Weizmann Institute of Science for his great help in writing this work.

Notation. Let K be a global field, and L a quadratic Galois extension of K. We write $L=K(i)$, and $\bar{l}$ for the Galois involution on $l\in L$. If U is an algebraic group defined over K. We write U_K for the group of its K-rational points, and U_{A_K} or U_A for the adele group.

Let W be a 5-dimensional vector space over L equipped with a Hermitian form $\langle\ ,\ \rangle_W$ having the matrix

$$Q=\begin{bmatrix} & 0 & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ 1 & & & & \end{bmatrix}$$

in the basis $\{w_1, w_2, w_0, w_{-2}, w_{-1}\}$.

Let V_i for $i=1$ to 5 be a $2i$-dimensional vector space equipped with a skew-Hermitian form $\langle\ ,\ \rangle_{V_i}$ having the matrix

$$J_i=\left[\begin{array}{c|c} & I_i \\ \hline -I_i & \end{array}\right]$$

in the basis $\{e_1, \dots, e_i, \hat{e}_1, \dots, \hat{e}_i\}$.

Let $U(2,3)$ (resp. $U(i,i)$) be the group of transformations in $GL(5)_L$ (resp. $GL(2i)_L$) preserving the form $\langle\ ,\ \rangle_W$ (resp. $\langle\ ,\ \rangle_{V_i}$). Then $U(2,3)$ and $U(i,i)$ are the groups of K-rational points of quasi-split algebraic groups defined over K, split over L. Also

$$H=U(2,3)=\{g\in GL(5)_L\,;\,\bar{g}^tQg=Q\}$$

$$U(i,i)=\{g\in GL(2i)_L\,;\,\bar{g}^tJ_ig=J_i\}\ .$$

(a) Description of parabolic subgroups of $U(2,3)$. Let P_1 be the maximal parabolic subgroup of $U(2,3)$ with Levi component $L(P_1)=\mathrm{RES}_K^L GL(1)\times U(2,1)$, having the form

$$P_1=\begin{bmatrix} d & & \\ & \begin{bmatrix} U(2,1) \end{bmatrix} & \\ & & \bar{d}^{-1} \end{bmatrix}\begin{bmatrix} 1 & a & b & c & z \\ \cdot & 1 & \cdot & \cdot & -\bar{c} \\ \cdot & \cdot & 1 & \cdot & -\bar{b} \\ \cdot & \cdot & \cdot & 1 & -\bar{a} \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix},$$

where $z = -(a\bar{c} + b\bar{b} + c\bar{a})/2 + is$ and s in K. Let N_1 be the unipotent radical of P_1.

Recall $\mathrm{RES}_K^L GL(n)$ is the disconnected quasi-split algebraic group of type $A_{n-1} \times A_{n-1}$ formed by the restriction of scalars from L to K. Note that $\mathrm{RES}_K^L GL(n)_K = GL(n)_L$ and $\mathrm{RES}_K^L GL(n)_L = GL(n)_L \times GL(n)_L$ (cf. [Ta]).

Let P_2 be the maximal parabolic subgroup of $U(2, 3)$ with Levi component $L(P_2) = \mathrm{RES}_K^L GL(2) \times U(1)$, having the form

$$P_2 = \begin{bmatrix} [g] & & \\ & u & \\ & & [J^t\bar{g}^{-1}J] \end{bmatrix} \begin{bmatrix} 1 & \cdot & b & z_1 & z_2 \\ \cdot & 1 & c & z_3 & -\bar{d} \\ \cdot & \cdot & 1 & -\bar{c} & -\bar{b} \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix},$$

where

$$z_1 = -\bar{c}b + d, \quad z_2 = -(\bar{b}b)/2 + is, \quad z_3 = -(\bar{c}c)/2 + it, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

for s, t in K. Let N_2 be the unipotent radical of P_2 (N_2 fixes w_1 and w_2).

Let N be the maximal unipotent subgroup of $U(2, 3)$. We can write N as $Z_3 . \tilde{U}$, i.e.,

$$\begin{bmatrix} 1 & a & b & z_1 & z_2 \\ \cdot & 1 & c & z_3 & -\bar{d} \\ \cdot & \cdot & 1 & -\bar{c} & -\bar{b} + \bar{a}\bar{c} \\ \cdot & \cdot & \cdot & 1 & -\bar{a} \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a & b & a_1 & a_2 \\ \cdot & 1 & c & -(\bar{c}c)/2 & 0 \\ \cdot & \cdot & 1 & -\bar{c} & -\bar{b} + \bar{a}\bar{c} \\ \cdot & \cdot & \cdot & 1 & -\bar{a} \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & d & z_4 \\ \cdot & 1 & \cdot & it & -\bar{d} \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix},$$

where

$$z_1 = d + ait - \bar{c}b + a(\bar{c}c)/2, \quad z_2 = -[(a\bar{d} + \bar{a}d) + (-b + ac)(\overline{-b + ac})]/2 + is,$$
$$z_3 = -(\bar{c}c)/2 + it, \quad z_4 = (a\bar{d} - \bar{a}d)/2 + is,$$
$$a_1 = -\bar{c}b + a(\bar{c}c)/2, \quad a_2 = -(-b + ac)(\overline{-b + ac})/2,$$

for s, t in K.

(b) Description of maximal parabolic subgroups of $U(i, i)$. Let P_i^j denote the maximal parabolic subgroup of $U(i, i)$ having a Levi component $L(P_i^j)=\mathrm{RES}_K^L GL(j)\times U(i-j, i-j)$, unipotent radical $U^{i,j}$, and of the form

$$\underbrace{\begin{array}{c} i-j\{ \\ \\ \\ \\ \end{array}\begin{bmatrix} \boxed{*} & & \boxed{*} & \\ & \boxed{g} & & \\ \boxed{*} & & \boxed{*} & \\ & & & \boxed{{}^t\bar{g}^{-1}} \end{bmatrix}}_{\mathrm{RES}_K^L GL(j)\times U(i-j,\, i-j)} \quad \underbrace{\begin{bmatrix} I_i & \begin{array}{c|c} 0 & \overbrace{\bar{X}^t}^{j} \\ \hline X & Y \end{array} \\ & I_i \end{bmatrix}}_{U_1^{i,j}}\begin{array}{l} \}i-j \\ \}j \\ \\ \end{array} \quad \underbrace{\begin{bmatrix} \begin{array}{cc} I_{i-j} & \\ Z & I_j \end{array} & 0 \\ 0 & \begin{array}{cc} I_{i-j} & -\bar{Z}^t \\ & I_j \end{array} \end{bmatrix}}_{U_2^{i,j}}$$

where $g\in GL(j)_L$ and $\bar{Y}^t=Y$. We decompose $U^{i,j}$ as $U_1^{i,j}\cdot U_2^{i,j}$.

1. Theta liftings from $U(2, 3)$ **to** $U(i, j)$. Denote by X the K-vector space $V_i\otimes W$, with its symplectic form $\langle\ ,\ \rangle=\mathrm{Real}(\langle\ ,\ \rangle_{V_i}\cdot\langle\ ,\ \rangle_W)$. Then $X=(X_1)_K+(\hat{X}_1)_K$, where $(X_1)_K=e_1W\oplus\cdots\oplus e_iW$, $(\hat{X}_1)_K=\hat{e}_1W\oplus\cdots\oplus\hat{e}_iW$. We also write W_j for e_jW and $\hat{W}_j$ for $\hat{e}_jW$. Let ψ be a nontrivial character of $K\backslash A_K$. Let $MP^{\psi}(V_i\otimes W)$ be the metaplectic group of $V_i\otimes W$. Pick s, a splitting of $U(2, 3)\times U(i, i)$ in $MP^{\psi}(V_i\otimes W)$. Let ω_ψ be the Weil representation of $MP^{\psi}(V_i\otimes W)$ (cf. [Ge-Ro-So] or [Ge-Ro]).

Let π be an automorphic cuspidal representation of $U(2, 3)$, with V_π as its vector space. We use the Schrödinger realization of ω_ψ in $S((X_1)_A)$, the Schwartz-Bruhat space of $(X_1)_A$ (cf. [Ra], [Ro], and [P.S.]). Then for f in V_π, Φ in $S((X_1)_A)$ we write

$$\theta_\Phi^1(\pi, s)f(g)=\int_{U(2,3)_K\backslash U(2,3)_A}\sum_{x\in(X_1)_K}\omega_\psi(s(g, h))\Phi(x)f(h)dh$$

for the theta lift of f to $U(i, i)$. We denote by $\theta^i(\pi, s)$ the space of such functions for all f in V_π, Φ in $S((X_1)_A)$. It is well known that $\theta^i(\pi, s)$ generates an irreducible representation for $U(i, i)$ if it is cuspidal (cf. [Ge-Ro-So]).

Suppose now $V_i\otimes W=U_1\oplus U_2$ (orthogonal sum), and $G=G_1\times G_2$ is a subgroup of $U(V_i\otimes W)$, the unitary group of $V_i\otimes W$. Suppose also G_1 (resp. G_2) acts on U_1 (resp. U_2). Then any splitting s of $U(V_i\otimes W)$ determines two splittings s_i of G_i into $MP^{\psi}(U_i)$ for $i=1, 2$. Also

$$\omega_\psi(s(g_1, g_2))=\omega_\psi^1(s_1(g_1))\otimes\omega_\psi^2(s_2(g_2))\,,$$

where ω_ψ^i is the Weil representation of $MP^{\psi}(U_i)$ for $i=1, 2$.

Picking any one of the standard maximal parabolic subgroups P_i^j of $U(i, i)$, let U_1

be $W_i \oplus \cdots \oplus W_{i-j} \oplus \hat{W}_1 \oplus \cdots \oplus \hat{W}_{i-j}$, U_2 be $W_{i-j+1} \oplus \cdots \oplus W_i \oplus \hat{W}_{i-j+1} \oplus \cdots \oplus \hat{W}_i$. Then $U(i-j, i-j)$ and H act on U_1, where H (and $GL(j)$) act on U_2. Therefore there exists a splitting s_1^i of $U(i-j, i-j) \times H$ into $MP^{\psi}(U_1)$, and a splitting s_2^j of H into $MP^{\psi}(U_2)$. Also $\omega_\psi = \omega_\psi^{j,1} \otimes \omega_\psi^{j,2}$.

THEOREM 1.1. (a) *$\theta^i(\pi, s)$ is cuspidal if and only if $\theta^{i-1}(\pi\gamma', s_1^1)$ is zero, where γ' is a character of $H_K \backslash H_A$ defined by*

$$\gamma'(h) = \{\omega_\psi^{1,2}(s_2^1(h))\Phi\}(0)\,,$$

and $\theta^{i-1}(\pi\gamma', s_1^1)$ is the lift defined by $\omega_\psi^{1,1}$.

(b) *$\theta^5(\pi, s)$ is nonzero.*

To prove the theorem we prove:

PROPOSITION 1.1. *For $j=1$ to $i-1$*

$$\int_{U_K^{i,j} \backslash U_A^{i,j}} \{\theta_\Phi^i(\pi, s) f\}(ng)dn$$

$$= \int_{H_K \backslash H_A} \sum_{(W_1 \times \cdots \times W_{i-j})_K} \omega_\psi(s(g, h))\Phi(x_1, \dots, x_{i-j}, \underbrace{0, \dots, 0}_{j}) f(h)dh\,.$$

PROOF (cf. [Ra] or [Wa]). Write $U^{i,j}$ as $U_1^{i,j} \times U_2^{i,j}$. First taking integration over $(U_1^{i,j})_K \backslash (U_1^{i,j})_A$, we have

$$\int_{(U_1^{i,j})_K \backslash (U_1^{i,j})_A} \{\theta_\Phi^i(\pi, s) f\}(u_1 g)du_1$$

$$= \int_{H_K \backslash H_A} \sum_{(X_1)_K} \int_{(U_1^{i,j})_K \backslash (U_1^{i,j})_A} \omega_\psi(s(u_1 g, h))\Phi(x_1, \dots, x_i) f(h) du_1 dh\,.$$

In writing this we have used a change in the order of integrations justified as in [Ra, Appendix to Section 1]. Suppose

$$U_1^{i,j} = \left[\begin{array}{c|c} I_i & [n_{k,l}] \\ \hline & I_i \end{array}\right]$$

with $[n_{k,l}]$ a matrix in $M_{i,i}(K \backslash A)$. Then

$$\omega_\psi(s(u_1 g, h))\Phi(x_1, \dots, x_i) =$$

$$\prod_{\substack{i \ge l > i-j \\ l > k}} \psi(\mathrm{Real}(n_{k,l}\langle x_k, x_l\rangle_W)) \prod_{i \ge k > i-j} \psi\left(\frac{1}{2} n_{k,k}\langle x_k, x_k\rangle_W\right) \cdot \Phi(x_1, \dots, x_i) .$$

Therefore the integral over $(U_1^{i,j})_K \backslash (U_1^{i,j})_A$ is zero unless

$$[\langle x_k, x_l\rangle_W]_{1 \le k,l \le i} = \left[\begin{array}{c:c} * & 0 \\ \hdashline 0 & \underbrace{0}_{j} \end{array}\right] \}j .$$

Let $(Y_1)_K$ denote the subset of $(X_1)_K$ satisfying the above matrix equation. Then

$$\int_{(U_1^{i,j})_K \backslash (U_1^{i,j})_A} \{\theta_\Phi^i(\pi, s) f\}(u_1 g) du_1 = \int_{H_K \backslash H_A} \sum_{(Y_1)_K} \omega_\psi(s(g, h))\Phi(x_1, \dots, x_i) f(h) dh .$$

Let $Sp\{w\}$ denote the L-vector space spanned by w. Then the following lemma is easily proved using Witt's theorem.

LEMMA 1.1. *There are three types of orbits of $(Y_1)_K$ under the left diagonal action of H_K and the right action of $(U_2^{i,j})_K$:*

(a) $$(x, \xi) = (x_1, \dots, x_{i-j}, \xi_j, \dots, \xi_1)$$

where $\xi_k \in Sp\{w_1, w_2\}$ for $k=1$ to j (not all in $Sp\{w_i\}$, $i=1, 2$), $x_l \in Sp\{w_0\}$ for $l=1$ to $i-j$, and (x, ξ) runs through a set of representatives of $L(P_2)$ orbits.

(b) $$(x, \xi) = (x_1, \dots, x_{i-j}, \xi_j, \dots, \xi_1)$$

where $\xi_k \in Sp\{w_1\}$ for $k=1$ to j, $x_l \in Sp\{w_2, w_0, w_{-2}\}$ for $l=1$ to $i-j$, and (x, ξ) runs through a set of representatives of $L(P_1)$ orbits.

(c) $$(x, \xi) = (x_1, \dots, x_{i-j}, 0, \dots, 0) .$$

We shall show now that integration over orbits of type (a) and (b) gives zero. The kernel of the last integrals is:

$$\sum_{(Y_1)_K} \omega_\psi(s(g, h))\Phi(x_1, \dots, x_i) = \sum_{(x,\xi)} \sum_{\delta_2} \sum_{\delta_1} \omega_\psi(s(\delta_1 g, \delta_2 h))\Phi(x, \xi)$$

where $\delta_1 \in (U_2^{i,j})_K(x, \xi) \backslash (U_2^{i,j})_K$, $\delta_2 \in H_K(x, \xi) \backslash H_K$, and $(U_2^{i,j})_K(x, \xi)$ (resp. $H_K(x, \xi)$) is the stabilizer of (x, ξ) (resp. $(x, \xi)(U_2^{i,j})_K$) in $(U_2^{i,j})_K$ (resp. H_K).

Suppose now (x, ξ) is of type (a). Then $H_K(x, \xi)$ is $(N_2)_K$. By integrating over $(U_2^{i,j})_K \backslash (U_2^{i,j})_A$ this part of the sum becomes

$$\sum_{(a)}\sum_{\delta_2}\int_{(U_2^{i,j})_K\backslash(U_2^{i,j})_A}\sum_{\delta_1}\omega_\psi(s(\delta_1 u_2 g,\delta_2 h))\Phi(x,\xi)du_2$$

$$=\sum_{(a)}\sum_{\delta_2}\int_{(U_2^{i,j})_K(x,\xi)\backslash(U_2^{i,j})_A}\omega_\psi(s(u_2 g,\delta_2 h))\Phi(x,\xi)du_2\,.$$

Next, we integrate this kernel against a cusp form f on $H_K\backslash H_A$ to get

$$\sum_{(a)}\int_{(N_2)_K\backslash H_A}\int_{(U_2^{i,j})_K(x,\xi)\backslash(U_2^{i,j})_A}\omega_\psi(s(u_2 g,h))\Phi(x,\xi)f(h)du_2\,.$$

Note that the left action of N_2 on (x, ξ) of type (a) can be written as a right action of $U_2^{i,j}$, so the inner integral is invariant under $(N_2)_K\backslash(N_2)_A$. Then the last integral reads;

$$\sum_{(a)}\int_{(N_2)_A\backslash H_A}\int_{(U_2^{i,j})_K(x,\xi)\backslash(U_2^{i,j})_A}\omega_\psi(s(ug,h))\Phi((x,\xi))du\int_{(N_2)_K\backslash(N_2)_A}f(nh)dndh\,.$$

Now f is a cusp form so this integral is zero, and we are done. For orbits of type (b) replace N_2 by N_1 and use the same reasoning. This concludes the proof of Proposition 1.1.

We continue the proof of Theorem 1.1. If $\theta_\Phi^i(\pi, s)f$ is cuspidal, then we use Proposition 1.1 for P_i^1. We write then $\Phi(x_1, \dots, x_{i-1}, 0)$ as $\Phi(x_1, \dots, x_{i-1})\Phi(0)$ and define $\gamma'(h)$ to be $\{\omega_\psi^{1,2}(s_2^1(h))\Phi\}(0)$. Conversely, we write the lift $\theta_\Phi^{i-1}(\pi\gamma', s_1^1)f$ as the integral on the right hand side of Proposition 1.1. We consider this integral as a function of $U(i-1, i-1)$ imbeded in P_i^1. We compute its zero Fourier coefficients in the direction of all standard maximal parabolic subgroups of $U(i-1, i-1)$. Then these are all integrals

$$\int_{H_K\backslash H_A}\sum_{(W_1\times\cdots\times W_l)_K}\omega_\psi(s(g,h))\Phi(x_1,\dots,x_l,0,\dots,0)f(h)dh\,,$$

where $l=1$ to $i-1$. But if all these are zero, then $\theta_\Phi^i(\pi, s)f$ is cuspidal. This concludes the proof of Part (a) of Theorem 1.1. The proof of Part (b) is a standard argument (cf. [Ra]).

2. Hypercuspidality. Let N (resp. U) denote the standard maximal unipotent subgroup of $U(2, 3)$ (resp. $U(3, 3)$). Let $\psi_{\xi,\eta}$ (resp. $\psi_{\xi,\eta,t}$) denote a nondegenerate character of N (resp. U) where $\xi, \eta \neq 0$ in L and $t \neq 0$ in K (cf. [Ge-Sh, p. 76]). For an automorphic cuspidal form f on $U(2, 3)$ (resp. $U(3, 3)$) let $W_f^{\psi_{\xi,\eta}}$ (resp. $W_f^{\psi_{\xi,\eta,t}}$) denote a $\psi_{\xi,\eta}$ (resp. $\psi_{\xi,\eta,t}$) Fourier coefficient in the direction of N (resp. U), i.e., a Whittaker function. We write $W(\pi, \psi_{\xi,\eta})$ (resp. $W(\pi, \psi_{\xi,\eta,t})$) for the space of all such Fourier coefficients, where

π is a cuspidal representation of $U(2, 3)$ (resp. $U(3, 3)$). We say that π is $\psi_{\xi,\eta}$ (resp. $\psi_{\xi,\eta,t}$) generic if $W(\pi, \psi_{\xi,\eta})$ (resp. $W(\pi, \psi_{\xi,\eta,t})$) is nonzero. We say π to be generic if π is $\psi_{\xi,\eta}$ (resp. $\psi_{\xi,\eta,t}$) generic for some $\xi, \eta \neq 0$ in L and $t \neq 0$ in K. If for all such ξ, η and t the space $W(\pi, \psi_{\xi,\eta})$ (resp. $W(\pi, \psi_{\xi,\eta,t})$ is zero, then we say that π is hypercuspidal.

THEOREM 2.1. *If π is cuspidal generic on $U(2, 3)$, then its ψ lift to $U(3, 3)$ is generic. Moreover;*

$$W^{\psi_{\xi,\eta,-t/2}}_{\theta^3_{\Phi}(\pi,s)f} = \begin{cases} 0 & \textit{if} \quad t \textit{ is not a norm} \\ \displaystyle\int_{\tilde{U}_A \backslash H_A} \omega_\psi(s(g, h))\Phi(\alpha w_0, w_2, w_1) W_f^{\psi_{\xi\alpha,\eta}}(h)dh & \textit{if} \quad t \textit{ is a norm}, \end{cases}$$

where $\alpha\bar{\alpha} = t$.

PROOF. Write $U = U_1 \cdot U_2$ where

$$U_1 = \left[\begin{array}{ccc|ccc} 1 & & & a_1 & a_2 & a_3 \\ & 1 & & \bar{a}_2 & a_4 & a_5 \\ & & 1 & \bar{a}_3 & \bar{a}_5 & a_6 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right], \quad U_2 = U_2(a, b, c) = \left[\begin{array}{ccc|ccc} 1 & & & & & \\ c & 1 & & & & \\ b & a & 1 & & & \\ \hline & & & 1 & -\bar{c} & z \\ & & & & 1 & -\bar{a} \\ & & & & & 1 \end{array}\right],$$

with $a_1, a_4, a_6 \in K$ and $\bar{z} = ac - b$. Next we integrate the theta lift of f over $(U_1)_K \backslash (U_1)_A$ against the character $\psi_{\xi,\eta,-(t/2)}(u_1) = \psi((-t/2)a_1)$. This integral is zero unless

$$[\langle x_k, x_l \rangle_W]_{1 \le k, l \le 3} = \begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $(Y_1)_K$ denote the subset of $(X_1)_K$ satisfying the above matrix equation. Then

$$W^{\psi_{\xi,\eta,-(t/2)}}_{\theta^3_{\Phi}(\pi,s)f} = \int_{H_K \backslash H_A} \int_{(U_2)_K \backslash (U_2)_A} \sum_{(Y_1)_K} \omega_\psi(s(ug, h))\Phi(x_1, x_2, x_3) f(h) \psi_{\xi,\eta}(u) du dh .$$

The orbits of $(Y_1)_K$ under the diagonal action of H_K can have the following representatives:

$$\left.\begin{array}{ll}\text{(a)} & (x,\xi)=(w_{-2}+(t/2)w_2,\ w_1,\ 0)\\ \text{(b)} & (x,\xi)=(w_{-2}+(t/2)w_2,\ lw_1,\ w_1)\\ \text{(c)} & (x,\xi)=(w_{-2}+(t/2)w_2,\ 0,\ 0)\end{array}\right\}\ \text{if } t \text{ is not a norm},$$

$$\left.\begin{array}{ll}\text{(d)} & (x,\xi)=(\alpha w_0,\ w_1,\ 0)\\ \text{(e)} & (x,\xi)=(\alpha w_0,\ lw_1,\ w_1)\\ \text{(f)} & (x,\xi)=(\alpha w_0,\ 0,\ 0)\\ \text{(g)} & (x,\xi)=(\alpha w_0,\ w_2,\ w_1)\end{array}\right\}\ \text{if } t \text{ is a norm},$$

where $\alpha\bar{\alpha}=t$ and l in L. This is clear by using the action of $(P_1)_K$ and $(P_2)_K$ on $(Y_1)_K$. For orbits of type (a), (c), (d) and (f), the action of $(U_2)_K\backslash(U_2)_A$ gives an integration of an additive character over $L\backslash A_L$. For orbits of type (b) and (e), observe that

$$\int\limits_{(U_2)_K\backslash(U_2)_A}\ \int\limits_{H_K(x,\xi)\backslash H_A} \omega_\psi(s(u_2(a,b,c)g,h))\Phi(x,lw_1,w_1)\psi_\eta(a)\psi_\xi(c)\,dadcdb$$

$$=\int\limits_{(U_2)_K\backslash(U_2)_A}\ \int\limits_{H_K(x,\xi)\backslash H_A} \omega_\psi(s(g,h))\Phi(x+(cl+b)w_1,(l+a)w_1,w_1)\psi_\eta(a)\psi_\xi(c)\,dadcdb\,.$$

Next we change the variable c (if $l\neq 0$), so we end with an integration of an additive character over $L\backslash A_L$. As for orbits of type (g), write N as $Z_3\tilde{U}$, and note that $\tilde{U}_K$ stabilizes $(\alpha w_0, w_2, w_1)$. Now the integral reads;

$$\int\limits_{(U_2)_K\backslash(U_2)_A}\ \int\limits_{\tilde{U}_K\backslash H_A} \omega_\psi(s(u_2g,h))\Phi(\alpha w_0,w_2,w_1)\psi_{\xi,\eta}(u_2)f(h)\,dhdu_2\,.$$

Observe also that

$$\omega_\psi(s(u_2(a,b,c)g,h))\Phi(\alpha w_0,w_2,w_1)=\Phi(\alpha w_0+cw_2+bw_1,\ w_2+aw_1,\ w_1)\,,$$

and

$$\omega_\psi(s(g,z_3h))\Phi(\alpha w_0,w_2,w_1)=\Phi(\alpha w_0+\alpha cw_2+\alpha bw_1,\ w_2+aw_1,\ w_1)\,.$$

By using the above equations and a change of variables, the last integral reads;

$$\int\limits_{(Z_3)_K\backslash(Z_3)_A}\ \int\limits_{\tilde{U}_K\backslash H_A} \omega_\psi(s(g,zh))\Phi(\alpha w_0,w_2,w_1)\psi_{\xi\alpha,\eta}(z)f(h)\,dzdh$$

$$=\int\limits_{\tilde{U}_A\backslash H_A} \omega_\psi(s(g,h))\Phi(\alpha w_0,w_2,w_1)\int\limits_{\tilde{U}_K\backslash\tilde{U}_A}\ \int\limits_{(Z_3)_K\backslash(Z_3)_A} \psi_{\xi\alpha,\eta}(z)f(z^{-1}h)\,dzdh\,.$$

Note that the last integral cannot be zero for all Φ in $S((X_1)_A)$ (see also [Wa, Theorem 3.1]). So this concludes the proof of Theorem 2.1.

COROLLARY 2.1. *Let π be a cuspidal representation of $U(2, 3)$. If $\theta^4(\pi, s)$ (resp. $\theta^5(\pi, s)$) is cuspidal, then π must be hypercuspidal.*

PROOF. If $\theta^4(\pi, s)$ (resp. $\theta^5(\pi, s)$) is cuspidal then $\theta^5(\pi\gamma', s_1^1)=0$ (resp. $\theta^3(\pi\delta', s_1^2)=0$) where $\gamma'(h)=\omega_\psi^{1,2}(s_2^1(h))\Phi(0)$ (resp. $\delta'(h)=\omega_\psi^{2,2}(s_2^2(h))\Phi(0, 0)$) (cf. Proposition 1.1). So $\pi\gamma'$ (resp. $\pi\delta'$) cannot be generic (cf. Theorem 2.1).

THEOREM 2.2. *Let π be a cuspidal representation of $U(2, 3)$. Assume $\theta^4(\pi, s)$ (resp. $\theta^5(\pi, s)$) is cuspidal. Then it is hypercuspidal.*

PROOF. Consider $\theta^4(\pi, s)$. Write the maximal unipotent subgroup of $U(4, 4)$ as $U_1 \cdot U_2$ (as done previously for $U(3, 3)$). The integral of the theta lift over $(U_1)_K\backslash(U_1)_A$ against $\psi_{\xi,\eta,\delta,-(t/2)}$ is zero unless

$$[\langle x_k, x_l\rangle_W]_{1\le k,l\le 4} = \left[\begin{array}{c:c} t & 0 \\ \hdashline 0 & \underbrace{0}_{3} \end{array}\right]\Big\} 3 .$$

Let $(Y_1)_K$ denote the subset of $(X_1)_K$ satisfying the above matrix equation. Then

$$W^{\psi_{\xi,\eta,\delta,-(t/2)}}_{\theta^4_\Phi(\pi,s)f}(g)=\int_{(U_2)_K\backslash(U_2)_A}\int_{H_K\backslash H_A}\sum_{(Y_1)_K}\omega_\psi(s(ug, h))\Phi(x_1, x_2, x_3, x_4)f(h)\psi_{\xi,\eta,\delta}(u)du dh .$$

The orbits of $(Y_1)_K$ under the left diagonal action of H_K can have the following representatives:

(a) $(x, \xi_1, \xi_2, \xi_3)=(\alpha w_0, aw_1+bw_2, w_1, w_2)$

(b) $(x, \xi_1, \xi_2, \xi_3)=(\alpha w_0, w_1, aw_1, bw_1)$

(c) $(x, \xi_1, \xi_2, \xi_3)=(w_{-2}+(t/2)w_2, w_1, aw_1, bw_1)$

(d) $(x, \xi_1, \xi_2, \xi_3)=(w_{-2}+(t/2)w_2, 0, 0, 0)$,

where a, b in L, $\alpha\bar{\alpha}=t$, and ξ_1, ξ_2, ξ_3 can appear in any order. It is now left to show that the integral corresponding to each of the above orbits vanishes. If $\xi_i=0$ for some i, then the conclusion is easy. Otherwise we end as we do in Theorem 2.1 for orbits of type (b) and (e).

A similar argument holds for the theta lift $\theta^5(\pi, s)$.

REFERENCES

[Ge-Ro-So] S. GELBART, J. ROGAWSKI AND D. SOUDRY, On periods for the unitary group $U(3)$, preprint.

[Ge-Ro] S. GELBART AND J. ROGAWSKI, L-functions and Fourier-Jacobi coefficients for the unitary group $U(3)$, Invent. Math. 105 (1991), 445–472.

[Ge-Sh] S. GELBART AND F. SHAHIDI, Analytic properties of automorphic L-functions, Academic Press, 1988.

[PS] I. PIATETSKI-SHAPIRO, On the Saito Kurokawa lifting, Invent. Math. 71 (1983), 309–338.

[Ra] S. RALLIS, On the Howe duality conjecture, Composition Math. 51 (1984), 333–399.

[Ro] R. RAO, On some explicit formulas in the theory of the Weil representation, unpublished manuscript, 1978.

[Ta] B. TAMIR, On L-functions and intertwining operators for unitary groups, Israel J. of Math. 73 (1991), 161–188.

[Wa] T. WATANABE, Hypercuspidality of automorphic cuspidal representations of the unitary group $U(2, 2)$, Tôhoku Math. J. 39 (1987), 259–279.

P.O. BOX 788
JERUSALEM 91007
ISRAEL