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WEIGHTS FOR THE ERGODIC MAXIMAL OPERATOR AND A. E. CONVERGENCE OF THE ERGODIC AVERAGES FOR FUNCTIONS IN LORENTZ SPACES

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Abstract. In this paper, we deal with an invertible null-preserving transformation into itself of a finite measure space. We prove that the uniform boundedness of the ergodic averages in a reflexive Lorentz space implies a.e. convergence. In order to do this, we study the "good weights" for the maximal operator associated to an invertible measure preserving transformation.

1. Introduction and results. Let T be an invertible measure preserving transformation on a measure space $(X, \mathcal{M}, \mu)$. Let $T_{n,m}$ and M be the ergodic averages and the maximal operator defined, respectively, by

$$T_{n,m}f(x)=\frac{1}{n+m+1}\sum_{j=-n}^{m} f(T^j x) \quad \text{and} \quad Mf=\sup_{n,m\geq 0} T_{n,m}|f|\,.$$

Martín-Reyes [6] studied the good weights for M to be bounded in L_p $(1<p<\infty)$ and from L_p to $L_{p,\infty}$ $(1\leq p<\infty)$. He proved that M is bounded from $L_p(v)$ to $L_{p,\infty}(u)$ if and only if (u, v) satisfies $A_p(T)$, which means for $p>1$

$$\left(\sum_{i=0}^{k} u(T^i x)\right)\left(\sum_{i=0}^{k} v^{1-p'}(T^i x)\right)^{p-1}\leq C(k+1)^p \quad \text{a.e.}$$

with C independent of k and x and $pp'=p+p'$, and for $p=1$

$$Mu(x)\leq Cv(x) \quad \text{a.e.}$$

Moreover, he proved that, for $u=v$ and $p>1$, $A_p(T)$ is also equivalent to the boundedness of M in $L_p(u)$. Then, he used these results to obtain theorems about convergence a. e. of the ergodic averages of functions in weighted L_p-spaces.

Gallardo [2], [3] has generalized these results to Orlicz spaces.

Our purpose is to extend the L_p results to $L_{p,q}$ spaces. In this paper, we characterize

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the pairs of non-negative measurable functions (u, v) such that M is bounded from $L_{p,q}(v)$ to $L_{p,\infty}(u)$. By $L_{p,q}(v)$ we design the Lorenz space of all measurable functions f such that $\|f\|_{p,q;v}<\infty$, where

$$\|f\|_{p,q;v}=\left(q\int_0^\infty\left(\int_{\{x/|f(x)|>y\}} v\,d\mu\right)^{q/p} y^{q-1}dy\right)^{1/q} \quad \text{if} \quad 1\le q<\infty$$

and

$$\|f\|_{p,\infty;v}=\sup_{y>0} y\left(\int_{\{x/|f(x)|>y\}} v\,d\mu\right)^{1/p}.$$

The condition we give is analogous to the condition $A_{p,q}$ in [1] in the same way that $A_p(T)$ in [6] is analogous to Muckenhoupt's A_p.

A pair (u, v) of non-negative functions on X satisfies $A_{p,q}(T)$ ($1<p<\infty$ and $1\le q\le\infty$ or $p=q=1$) if there exists $C>0$ such that

$$\|\chi_{[0,k]}\|_{p,q;u^x}\cdot\|\chi_{[0,k]}(v^x)^{-1}\|_{p',q';v^x}\le C(k+1)$$

for every $k\in\boldsymbol{N}$ and a. e. $x\in X$, where the norms are in the integers, p' and q' denote the conjugate exponents of p and q, respectively, and f^x means the function on $\boldsymbol{Z}$ defined by $f^x(i)=f(T^ix)$.

After obtaining the maximal inequalities, we will prove that for a null-preserving invertible transformation on a finite measure space, the uniform weak type of the ergodic averages $T_{n,m}$ suffices to get the a. e. convergence of $T_{n,n}f$ for every f in $L_{p,q}$.

The proofs of our results follow the techniques in [6] adapted to the $L_{p,q}$ context. Besides, to make transference in $L_{p,q}$ we will need straightforward versions of Theorems 1 and 2 in [1] for two weights. It is remarkable that the process of transference in $L_{p,q}$ is not as easy as it is in L_p. We only work in the case $q\le p$ and the difficulties are solved by means of Minkowski's integral inequality.

Throughout this paper, C will denote a positive constant, not necessarily the same at each occurrence.

Our results are the following:

THEOREM 1. *Let $1\le q\le p<\infty$ and u, v be positive measurable functions. The following statements are equivalent*:

(a) $\|Mf\|_{p,\infty;u}\le C\|f\|_{p,q;v}$

(b) $\sup_{n,m\ge 0}\|T_{n,m}f\|_{p,\infty;u}\le C\|f\|_{p,q;v}$

(c) (u, v) satisfies $A_{p,q}(T)$.

THEOREM 2. *Let $1<p<\infty$ and $1<q\le\infty$. Let w be a positive measurable function. The following statements are equivalent*:

(a) $\|Mf\|_{p,\infty;w}\le C\|f\|_{p,q;w}$

(b) $\sup_{n,m\ge 0}\|T_{n,m}f\|_{p,\infty;w}\le C\|f\|_{p,q;w}$

(c) $\|Mf\|_{p,q;w} \leq C\|f\|_{p,q;w}$
(d) $\sup_{n,m\geq 0}\|T_{n,m}f\|_{p,q;w} \leq C\|f\|_{p,q;w}$
(e) $w \in A_{p,q}(T)$
(f) $w \in A_p(T)$.

THEOREM 3. *Let $1<p<\infty$ and $1<q<\infty$. Let $(X, \mathscr{F}, v)$ be a finite measure space and let $T\colon X\to X$ be a null-preserving invertible transformation. The following statements are equivalent*:

(a) $\|Mf\|_{p,\infty} \leq C\|f\|_{p,q}$
(b) $\sup_{n,m\geq 0}\|T_{n,m}f\|_{p,\infty} \leq C\|f\|_{p,q}$
(c) $\|Mf\|_{p,q} \leq C\|f\|_{p,q}$
(d) $\sup_{n,m\geq 0}\|T_{n,m}f\|_{p,q} \leq C\|f\|_{p,q}$.

Moreover, if one of the above conditions holds, then $\{T_{n,n}f\}$, $\{T_{0,n}f\}$ and $\{T_{n,0}f\}$ converge a.e. for every $f \in L_{p,q}$.

2. Proof of Theorem 1. The case $p=q=1$ is taken care of by [6, Theorem 2.26]. Hence we here consider only the case $1<p<\infty$. The implication (a) $\Rightarrow$ (b) is obvious. To prove (b) $\Rightarrow$ (c) we will need two lemmas:

LEMMA 1 (see [6]). *Let k be a natural number. Then, there exists a countable family $\{B_i\colon i \in N\}$ of measurable sets such that the following are satisfied*:

(i) $X = \bigcup_i B_i$.
(ii) $B_i \cap B_j = \varnothing$ *if* $i \neq j$.
(iii) *For every i, there exists a natural number $s(i)$ with $0 \leq s(i) \leq k$ such that the sets $\{T^{-j}B_i : 0 \leq j \leq s(i)\}$ are pairwise disjoint and such that if $s(i)<k$ then $T^{-1-s(i)}A = A$ for every subset A of B_i. Consequently, for every subset A of B_i*

$$\sum_{j=0}^{k} \chi_{T^{-j}A} \leq C(i) \sum_{j=0}^{s(i)} \chi_{T^{-j}A} \leq 2 \sum_{j=0}^{k} \chi_{T^{-j}A}\,,$$

where $C(i)$ is the least integer satisfying $(k+1)(1+s(i))^{-1} \leq C(i)$.

LEMMA 2. *Let $k \in N$ and let B be a measurable set. For every $x \in B$ and $n \in \boldsymbol{Z}$, let $H_n^x = \{i \in [0,k]/v^{-1}(T^i x) > 3^n\}$. Let $\boldsymbol{A}$ be the collection of all nonincreasing sequences in $\boldsymbol{Z} \cup \{-\infty\}$ with no more than 2^{k+1} different elements and at least an element in $\boldsymbol{Z}$. If $\alpha = \{a_n\} \in \boldsymbol{A}$, let A_α be the set defined by*

$$A_\alpha = \left\{ x \in B / H_n^x = \varnothing \text{ if } a_n = -\infty \text{ and } 2^{a_n} < \sum_{i \in H_n^x} v(T^i x) \leq 2^{a_n+1} \text{ if } a_n \neq -\infty \right\}.$$

Then $\{A_\alpha\}_{\alpha \in \boldsymbol{A}}$ is a countable family such that their elements are pairwise disjoint and $B = \bigcup_{\alpha \in \boldsymbol{A}} A_\alpha$.

PROOF OF LEMMA 2. It is clear that $\boldsymbol{A}$ is a countable family and that $\alpha \neq \beta$ in $\boldsymbol{A}$ implies $A_\alpha \cap A_\beta = \varnothing$. To see that $B = \bigcup_{\alpha \in \boldsymbol{A}} A_\alpha$ let $x \in B$ and let, for every $n \in \boldsymbol{Z}$ with

$H_n^x \neq \varnothing$, a_n be the only integer such that

$$2^{a_n} < \sum_{i \in H_n^x} v(T^i x) \leq 2^{a_n+1}\,.$$

If $H_n^x = \varnothing$, let $a_n = -\infty$. Then, the sequence $\alpha = \{a_n\}$ is nonincreasing (since $H_{n-1}^x \supset H_n^x$), it contains no more than 2^{k+1} different elements (since there are no more than 2^{k+1} different H_n^x) and $x \in A_\alpha$.

(b) $\Rightarrow$ (c) Let $k \in N$ and $\{B_i\}$ be the sequence associated to X and k by Lemma 1. Fix B_i. By Lemma 2, $B_i = \bigcup_{\alpha \in A} A_\alpha$. Fix one of these A_α and consider, for every $(n_0, n_1, \ldots, n_k) \in Z^{k+1}$, the set

$$H_{n_0, n_1, \ldots, n_k} = \{x \in A_\alpha / 2^{n_i} < v(T^i x) \leq 2^{n_i+1},\ i = 0, 1, \ldots, k\}\,.$$

It is clear that the $H_{n_0, n_1, \ldots, n_k}$ are measurable, their union is A_α and they are pairwise disjoint. Fix $H_{n_0, n_1, \ldots, n_k}$ and let A be a measurable set of $H_{n_0, n_1, \ldots, n_k}$.

Step 1. Let $R = \bigcup_0^{s(i)} T^j A$. First we will prove the relation

$$\|\chi_R\|_{p,q;u} \|\chi_R v^{-1}\|_{p',q';v} \leq C\mu(R) \tag{1.1}$$

with C independent of k, $s(i)$ and R.

In order to do this, let us see that

$$\|\chi_R v^{-1}\|_{p',q';v} \leq C\mu(A)^{1/p'} \|\chi_{[0,s(i)]} w^{-1}\|_{p',q';w}\,, \tag{1.2}$$

where w is defined in Z by $w(j) = 2^{n_j}\chi_{[0,s(i)]}$ and the (p', q')-norm on the right hand side (1.2) is a norm in the integers.

By the definitions of $\|\ \ \|_{p',q';v}$ and $H_{n_0, n_1, \ldots, n_k}$ we have for $q' < \infty$:

$$\begin{aligned}
\|\chi_R v^{-1}\|_{p',q';v} &= \left(q' \int_0^\infty \left(\int_A \sum_{\{j \in [0,s(i)]/(v^x)^{-1}(j) > y\}} v^x(j)\,d\mu\right)^{q'/p'} y^{q'-1}\,dy\right)^{1/q'} \\
&\leq \left(q' \int_0^\infty \left(\int_A \sum_{\{j \in [0,s(i)]/2^{-n_j} > y\}} 2^{n_j+1}\,d\mu\right)^{q'/p'} y^{q'-1}\,dy\right)^{1/q'} \\
&= 2^{1/p'}\mu(A)^{1/p'} \left(q' \int_0^\infty \left(\sum_{\{j \in [0,s(i)]/2^{-n_j} > y\}} 2^{n_j}\right)^{q'/p'} y^{q'-1}\,dy\right)^{1/q'} \\
&= C\mu(A)^{1/p'} \|\chi_{[0,s(i)]} w^{-1}\|_{p',q';w}
\end{aligned}$$

and for $q' = \infty$:

$$\begin{aligned}
\|\chi_R v^{-1}\|_{p',\infty;v} &= \sup_{y>0} y \left(\int_A \sum_{\{j \in [0,s(i)]/(v^x)^{-1}(j) > y\}} v^x(j)\,d\mu\right)^{1/p'} \\
&\leq \sup_{y>0} y \left(\int_A \sum_{\{j \in [0,s(i)]/2^{-n_j} > y\}} 2^{n_j+1}\,d\mu\right)^{1/p'}
\end{aligned}$$

$$=2^{1/p'}\mu(A)^{1/p'}\sup_{y>0} y\left(\sum_{\{j/(\chi_{[0,s(i)]}w^{-1})(j)>y\}} w(j)\right)^{1/p'}$$

$$=C\mu(A)^{1/p'}\|\chi_{[0,s(i)]}w^{-1}\|_{p',\infty;w}\,.$$

Therefore, (1.2) is proved.

Now we use an argument of duality: there exists $w'\geq 0$ with $\|w'\|_{p,q;w}=1$ such that

$$C\|\chi_{[0,s(i)]}w^{-1}\|_{p',q';w}\leq \sum_{j=0}^{s(i)} w'(j)\,. \tag{1.3}$$

From (1.2) and (1.3) it follows that

$$\|\chi_R v^{-1}\|_{p',q';v}\leq C\mu(A)^{1/p'}\sum_{j=0}^{s(i)} w'(j)\,. \tag{1.4}$$

Let f be the function defined on X by

$$f(x)=\sum_{j=0}^{s(i)} w'(j)\chi_{T^jA}(x)\,.$$

The function f verifies:

$$\left\{x\in X/T_{s(i),s(i)}f(x)>C\frac{\sum_{j=0}^{s(i)} w'(j)}{s(i)+1}\right\}\supset R\,. \tag{1.5}$$

From our hypothesis about M we obtain:

$$u(R)\leq C\frac{(s(i)+1)^p}{\left(\sum_{j=0}^{s(i)} w'(j)\right)^p}\|f\|^p_{p,q;v}\,. \tag{1.6}$$

Let us compute $\|f\|_{p,q;v}$:

$$\|f\|_{p,q;v}=\left(q\int_0^\infty\left(\int_{\left\{x/\sum_{j=0}^{s(i)} w'(j)\chi_{T^jA}(x)>y\right\}} v\,d\mu\right)^{q/p} y^{q-1}dy\right)^{1/q}$$

$$=\left(q\int_0^\infty\left(\int_A\sum_{j=0}^{s(i)} v(T^jx)\chi_{\{z/w'(j)\chi_{T^jA}(z)>y\}}(T^jx)d\mu(x)\right)^{q/p} y^{q-}\right.$$

$$=\left(q\int_0^\infty\left(\sum_{\{j\in[0,s(i)]/w'(j)>y\}}\int_A v^x(j)d\mu\right)^{q/p} y^{q-1}dy\right)^{1/q}$$

$$\leq\left(q\int_0^\infty\left(\sum_{\{j\in[0,s(i)]/w'(j)>y\}}\mu(A)2^{n_j+1}\right)^{q/p} y^{q-1}dy\right)^{1/q}$$

$$=\mu(A)^{1/p}2^{1/p}\left(q\int_0^\infty\left(\sum_{\{j\in[0,s(i)]/w'(j)>y\}}w(j)\right)^{q/p}y^{q-1}dy\right)^{1/q}$$

$$=C\mu(A)^{1/p}\|w'\|_{p,q;w}=C\mu(A)^{1/p}.$$

This inequality together with (1.6) and (1.4) give

$$u(R)\le C\frac{(s(i)+1)^p}{\|\chi_R v^{-1}\|^p_{p',q';v}}\mu(A)^p. \tag{1.7}$$

Raising this to $1/p$ and keeping in mind that $(s(i)+1)\mu(A)=\mu(R)$ we obtain (1.1).

Step 2. Relation (1.1) can be written as

$$\left(\int_A\sum_{j=0}^{s(i)}u(T^jx)d\mu\right)^{1/p}\left(q'\int_0^\infty\left(\int_A\sum_{\{j\in[0,s(i)]/v^{-1}(T^jx)>y\}}v(T^jx)\right)^{q'/p'}y^{q'-1}dy\right)^{1/q'} \tag{1.8}$$
$$\le C(s(i)+1)\mu(A).$$

By means of Lemma 1, we are going to prove that (1.8) remains valid with k in palce of $s(i)$.

Let us consider the first factor on the left hand side of (1.8). By Lemma 1

$$\left(\int_A\sum_{j=0}^{s(i)}u(T^jx)d\mu\right)^{1/p}\ge\left(\frac{1}{C(i)}\int_A\sum_{j=0}^{k}u(T^jx)d\mu\right)^{1/p}. \tag{1.9}$$

Let us dominate from below the second factor on the left hand side of (1.8):

$$\int_A\sum_{\{j\in[0,s(i)]/v^{-1}(T^jx)>y\}}v(T^jx)d\mu=\sum_{j=0}^{s(i)}\int_{T^jA}\chi_{\{z/v^{-1}(T^jz)>y\}}(T^{-j}x)v(x)d\mu \tag{1.10}$$
$$=\sum_{j=0}^{s(i)}\int_{T^jA}\chi_{\{z/v^{-1}(z)>y\}}(x)v(x)d\mu\ge\frac{1}{C(i)}\sum_{j=0}^{k}\int_{T^jA}\chi_{\{z/v^{-1}(z)>y\}}(x)v(x)d\mu$$
$$=\frac{1}{C(i)}\int_A\sum_{\{j\in[0,k]/v^{-1}(T^jx)>y\}}v(T^jx)d\mu.$$

Finally, we bound above the right hand side of (1.8):

$$\mu(A)(s(i)+1)=\int_X\sum_{j=0}^{s(i)}\chi_{T^jA}(x)d\mu\le\frac{2}{C(i)}\int_X\sum_{j=0}^{k}\chi_{T^jA}(x)d\mu=\frac{2}{C(i)}(k+1)\mu(A). \tag{1.11}$$

Now, (1.8), (1.9), (1.10), (1.11) and the fact $1/p+1/p'=1$ allow to cancell $C(i)$ and to obtain

$$\left(\int_A\sum_{j=0}^{k}u(T^jx)d\mu\right)^{1/p}\left(q'\int_0^\infty\left(\int_A\sum_{\{j\in[0,k]/v^{-1}(T^jx)>y\}}v(T^jx)d\mu\right)^{q'/p'}y^{q'-1}dy\right)^{1/q'} \tag{1.12}$$
$$\le C(k+1)\mu(A).$$

Step 3. From (1.12) we get

$$(1.13)\quad \left(\int_A \sum_{j=0}^{k} u(T^j x)d\mu\right)^{p'/p} \int_A \left(\int_0^\infty \left(\sum_{\{j\in[0,k]/v^{-1}(T^jx)>y\}} v(T^jx)\right)^{q'/p'} y^{q'-1}dy\right)^{p'/q'} d\mu$$

$$\leq C(k+1)^{p'}\mu(A)^{p'}$$

and then, since A is an arbitrary subset of $H_{n_0,n_1,\ldots,n_k}$, the union of the H's is A_α, the union of the A_α's is B_i and the union of the B_i's is X, we obtain

$$\left(\sum_{j=0}^{k} u(T^j x)\right)^{1/p} \left(\int_0^\infty q'\left(\sum_{\{j\in[0,k]/v^{-1}(T^jx)>y\}} v(T^jx)\right)^{q'/p'} y^{q'-1}dy\right)^{1/q'} \leq C(k+1)$$

a. e. $x\in X$, that is,

$$\|\chi_{[0,k]}\|_{p,q;u^x}\|\chi_{[0,k]}(v^x)^{-1}\|_{p',q';v^x}\leq C(k+1)\,.$$

Let us consider the second factor on the left hand side of (1.13) and let us dominate it by the corresponding term in (1.12):

$$\int_A \left(\int_0^\infty \left(\sum_{\{j\in[0,k]/v^{-1}(T^jx)>y\}} v(T^jx)\right)^{q'/p'} y^{q'-1}dy\right)^{p'/q'} d\mu$$

$$\leq \int_A \left(\sum_{n=-\infty}^{\infty} \int_{3^n}^{3^{n+1}} \left(\sum_{\{j\in[0,k]/v^{-1}(T^jx)>3^n\}} v(T^jx)\right)^{q'/p'} y^{q'-1}dy\right)^{p'/q'} d\mu$$

$$= C\int_A \left(\sum_{n=-\infty}^{+\infty} \int_{3^{n-1}}^{3^n} \left(\sum_{\{j\in[0,k]/v^{-1}(T^jx)>3^n\}} v(T^jx)\right)^{q'/p'} y^{q'-1}dy\right)^{p'/q'} d\mu$$

$$\leq C\int_A \left(\sum_{n=-\infty}^{+\infty} \int_{3^{n-1}}^{3^n} 2^{(a_n+1)q'/p'} y^{q'-1}dy\right)^{p'/q'} d\mu$$

$$= C\left(\sum_{n=-\infty}^{+\infty} \int_{3^{n-1}}^{3^n} \left(\int_A 2^{a_n}d\mu\right)^{q'/p'} y^{q'-1}dy\right)^{p'/q'}$$

$$\leq C\left(\sum_{n=-\infty}^{+\infty} \int_{3^{n-1}}^{3^n} \left(\int_A \sum_{\{j\in[0,k]/v^{-1}(T^jx)>3^n\}} v(T^jx)d\mu\right)^{q'/p'} y^{q'-1}dy\right)^{p'/q'}$$

$$\leq C\left(\int_0^\infty \left(\int_A \sum_{\{j\in[0,k]/v^{-1}(T^jx)>y\}} v(T^jx)d\mu\right)^{q'/p'} y^{q'-1}dy\right)^{p'/q'}.$$

Then, the left hand side of (1.13) is smaller than the left hand side of (1.12) raised to p'. Now (1.13) follows from (1.12) and the implication is proved.

(c)$\Rightarrow$(a) Let f be a positive function. Let $L\in\mathbf{N}$, $N\in\mathbf{N}$, $\lambda>0$ and $O_\lambda=\{x\in X/M_Lf(x)>\lambda\}$, where M_L is the truncated maximal operator defined by $M_Lf=\sup_{n,m\leq L}T_{n,m}|f|$.

Then,

$$(1.14)\qquad u(O_\lambda)=\int_{O_\lambda} u\,d\mu=\frac{1}{N+1}\sum_{j=0}^{N}\int_{O_\lambda} u\,d\mu=\frac{1}{N+1}\sum_{j=0}^{N}\int_{T^{-j}(O_\lambda)} u(T^j x)d\mu$$

$$=\int_X \frac{1}{N+1}\sum_{j=0}^{N}\chi_{T^{-j}(O_\lambda)}(x)u(T^j x)d\mu$$

$$=\int_X \frac{1}{N+1}\sum_{\{j\in[0,N]/M_L f(T^j x)>\lambda\}} u^x(j)d\mu$$

$$=\int_X \frac{1}{N+1}\sum_{\{j\in[0,N]/m_L f^x(j)>\lambda\}} u^x(j)d\mu$$

$$\le\int_X \frac{1}{N+1}\sum_{\{j\in N/m(f^x\chi_{[0,N+L]})(j)>\lambda\}} u^x(j)d\mu\,,$$

where m is the maximal operator in $\boldsymbol{Z}$ and m_L its truncated.

Condition $A_{p,q}(T)$ means that (u^x, v^x) verify $A_{p,q}$ in the integers uniformly in x. Theorem 2 in [1] (adapted to two weights) ensures that

$$(1.15)\qquad \sum_{\{j\in Z/m(f^x\chi_{[0,N+L]})(j)>\lambda\}} u^x(j)\le\frac{C}{\lambda^p}\|f^x\chi_{[0,N+L]}\|^p_{p,q;v^x}\,.$$

Then (1.14) together with (1.15) give:

$$(1.16)\qquad u(O_\lambda)\le\frac{C}{\lambda^p}\int_X \frac{1}{N+1}\|f^x\chi_{[0,N+L]}\|^p_{p,q;v^x}d\mu\,.$$

By the definition of the $L_{p,q}$-norm and Minkowski's integral inequality ($p/q\ge 1$) we obtain

$$(1.17)\quad u(O_\lambda)\le\frac{C}{\lambda^p}\frac{1}{N+1}\left(q\int_0^\infty\left(\int_X\left(\sum_{\{j\in[0,N+L]/f^x(j)>y\}} v^x(j)\right)d\mu\right)^{q/p} y^{q-1}dy\right)^{p/q}$$

Since T is a measure preserving transformation, the right hand side of (1.17) equals

$$(1.18)\qquad \frac{C}{\lambda^p}\frac{1}{N+1}\left(q\int_0^\infty\left((N+L+1)\int_X v(x)\chi_{\{z/f(z)>y\}}(x)d\mu\right)^{q/p} y^{q-1}dy\right)^{p/q}$$

$$=\frac{C}{\lambda^p}\frac{N+L+1}{N+1}\|f\|^p_{p,q;v}\,.$$

Therefore we have

$$(1.19)\qquad u(O_\lambda)\le\frac{C}{\lambda^p}\frac{N+L+1}{N+1}\|f\|^p_{p,q;v}\,.$$

Letting N and then L tend to infinity we obtain

$$u(\{x/Mf(x)>\lambda\})\leq\frac{C}{\lambda^p}\|f\|^p_{p,q;v}\,,\quad\text{that is}\,,$$

$$\|Mf\|_{p,\infty;u}\leq C\|f\|_{p,q;v}\,.$$

REMARK. Observe that (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) also hold with $1<p<\infty$ and $1<q<\infty$.

3. **Proof of Theorem 2.** The implications (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (b) are clear. The implication (b) $\Rightarrow$ (e) follows as in Theorem 1 (see the previous remark). To prove the equivalence (f) $\Leftrightarrow$ (e) just write the proof of Theorem 4 in [1] in the integers.

Finally, let us see simultaneously that (f) $\Rightarrow$ (a) and (f) $\Rightarrow$ (c). We will need the following well-known properties of $A_p(T)$-weights:

(i) $A_p(T)$ is contained in $A_r(T)$ if $p<r$, and

(ii) $w\in A_p(T)$ implies $w\in A_{p-\varepsilon}(T)$ for some $\varepsilon>0$ with $p-\varepsilon>1$.

These properties and Theorem 1 (or Theorem 2.7 in [6]) imply that if $w\in A_p(T)$, then there exist r_1 and r_2 with $r_2>p>r_1$ such that

$$\|Mf\|_{r_1,\infty;w}\leq C\|f\|_{r_1;w}\quad\text{and}\quad\|Mf\|_{r_2,\infty;w}\leq C\|f\|_{r_2;w}\,.$$

By Marcinkiewickz's interpolation theorem (see [4]),

$$\|Mf\|_{p,s;w}\leq C\|f\|_{p,s;w}\,,\qquad 1\leq s\leq\infty\,.$$

4. **Proof of Theorem 3.** The implications (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d), (d) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are obvious. We only have to prove (b) $\Rightarrow$ (c). Let L be a Banach limit. Let

$$\mu(E)=L\left(\left\{\int_X T_{0,n}\chi_E\,dv\right\}\right).$$

μ is well defined by (b) and by the finiteness of v. Moreover, since T is invertible, μ is an invariant measure equivalent to v, i.e., $v=wd\mu$ where w is a positive a.e., measurable function. Therefore, Theorem 2 gives (c).

Since $L_{p,q}(w)$ is reflexive, $\|T_{n,n}\|_{p,q;w}\leq C$ and $n^{-1}(f\circ T^n)$ converges to 0 in $L_{p,q}(w)$ for every $f\in L_\infty(w)$, we have that $L_{p,q}(w)=F\oplus\mathrm{cl}(N)$, where

$$F=\{h\in L_{p,q}(w)/h\circ T=h\}$$

and $\mathrm{cl}(N)$ is the clousure of the set N defined by

$$N=\{h\in L_{p,q}(w)/h=g-g\circ T,\ g\in L_\infty(w)\}$$

(see [5, pp. 71–74]).

The a.e. convergence of $\{T_{n,n}f\}$ is clear for $f=h+g-g\circ T$ with $g\in L_\infty(w)$ and $h\in F$. On the other hand, Theorem 2 (the hard part) gives us a weak inequality for M. We can apply Banach's principle to obtain the a.e. convergence of $\{T_{n,n}f\}$ for every

$f \in L_{p,q}(w)$ from the a. e. convergence in the dense class.

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