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GENERATORS FOR THE IDEAL OF A PROJECTIVELY EMBEDDED TORIC SURFACE

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Abstract. We show that the ideal of a projectively embedded toric surface is generated by polynomials of degrees 2 and 3.

1. Introduction. Let X be a toric surface. It is well known (see [Da]) that X is determined by a fan Δ in $\boldsymbol{Z}^2$. We will use the notation used in the book of Oda [Od] and denote $X = T_{\mathrm{emb}}(\Delta)$. An ample line bundle $\mathscr{L}$ on X is determined by a certain integral convex polygon P and the cohomology group $H^0(X, \mathscr{L})$ corresponds in a natural way to P (see [Od, Paragraph 2.2]). Since we are in dimension 2, an ample line bundle $\mathscr{L}$ is also a very ample line bundle (see [Ko, Lemma 1.6.3]), hence $\mathscr{L}$ gives an embedding in some projective space.

It is an interesting problem to determine equations for this embedded surface. Especially how many equations should one determine? The answer to this problem is given in this article: one has to determine the equations of degrees 2 and 3. The basic idea is that we will rewrite every monomial, which appears in a defining equation, in some kind of standard monomial. This rewriting uses the equations of degrees 2 and 3.

In this article we start with an integral convex polygon P and we consider the toric surface X_P (see [Da, 5.8]). Let $\mathscr{L}_P$ be the line bundle on X_P corresponding to P and let Δ_P be the fan such that $X_P = T_{\mathrm{emb}}(\Delta_P)$.

2. The generators of the ideal. Let P be an integral convex polygon in $\boldsymbol{R}^2$ and let $X = T_{\mathrm{emb}}(\Delta_P)$. Then $\mathscr{L}_P$ gives an embedding $\phi: X \to \boldsymbol{P}^{n-1}$, where $n = h^0(X, \mathscr{L}_P)$. Let $\{x_1, \dots, x_n\}$ be a basis for $H^0(X, \mathscr{L}_P)$, let $I \subset \boldsymbol{C}[x_1, \dots, x_n]$ be the ideal of X and let $I_d = I \cap \boldsymbol{C}[x_1, \dots, x_n]_d$ be the homogeneous part of I of degree d. Then, we have the following exact sequence

$$0 \longrightarrow I_d \longrightarrow \mathrm{Sym}^d(H^0(X, \mathscr{L}_P)) \xrightarrow{\phi_d} H^0(X, \mathscr{L}_P^{\otimes d}) \longrightarrow 0 .$$

DEFINITION 2.1. Let P be an integral convex polygon in $\boldsymbol{R}^2$. We define dP as the convex polygon which we get by multiplying P by d.

The line bundle $\mathscr{L}_P^{\otimes d}$ corresponds to the polygon dP. Let P contain the points

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$m_1, \ldots, m_n$ with $m_i \in \mathbf{Z}^2$ for $i=1, \ldots, n$. A point m_i corresponds to the section x_i. By abuse of notation we also use x_i if we mean the point m_i. A monomial $x^d \in \mathrm{Sym}^d(H^0(X, \mathscr{L}_P))$ is a monomial in the variables $x_1, \ldots, x_n$.

DEFINITION 2.2. Let $Q \in dP$. A *path of length d* to Q is a set of d points $\langle y_1, \ldots, y_d \rangle$ (not necessarily distinct) such that $y_i \in P$, with $1 \leq i \leq d$ and $\sum_{i=1}^{d} y_i = Q$. Each y_i is called a *step*.

Let us remark that a path is just a set of steps, hence the order of the steps is not determined. A monomial m of degree d is a path of length d to $\phi_d(m) \in dP$ and conversely, every path to an element of dP is a monomial of degree d in the variables $\{x_1, \ldots, x_n\}$.

LEMMA 2.3. *Let P be the triangle given by $x_0=(0, 0)$, $x_1=(1, 0)$, $x_2=(1, 1)$. Let $Q \in dP$. The there exists a unique path to Q.*

PROOF.

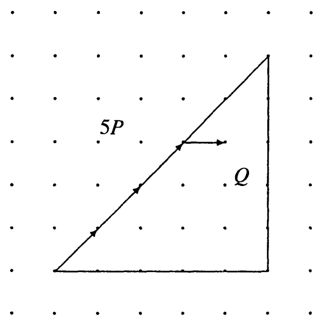

FIGURE 1.

Let $Q=(a, b) \in dP$. Take

$$S=\langle \underbrace{x_1, \ldots, x_1}_{a-b}, \underbrace{x_2, \ldots, x_2}_{b}, \underbrace{x_0, \ldots, x_0}_{d-a} \rangle .$$

Then S is a path to Q. This is a well defined path because $d \geq a \geq b$ and $Q \in dP$. It is unique because $\{x_1, x_2\}$ is a basis for $\mathbf{Z}^2$. □

DEFINITION 2.4 (*height function*). Let $L \subset \mathbf{R}^2$ be a line through zero such that there exists a point $R=(r_0, r_1) \in \mathbf{Z}^2$ on L. Take R in such a way that $r_1 \geq 0$ and $\gcd(r_0, r_1)=1$. If $r_1=0$ then take $r_0=1$. Let $h(x, L)=\det(R, x)$, which is also called the lattice distance from x to L.

The height function is additive, hence $h(x+y, L)=h(x, L)+h(y, L)$ for all $x, y \in \mathbf{Z}^2$.

DEFINITION 2.5. An n-triangulation V_n of a convex polygon P is a set of triangles $V_n=\{P_i\}$ such that

1. $\mathrm{Area}(P_i)=n^2/2$ for all i.
2. $P=\bigcup_i P_i$.
3. $P_i \cap P_j \subset \partial P_i$, $i \neq j$.

LEMMA 2.6. *Let P be a convex polygon and $Q \in dP$. Then there exists a path of length d to Q.*

PROOF.

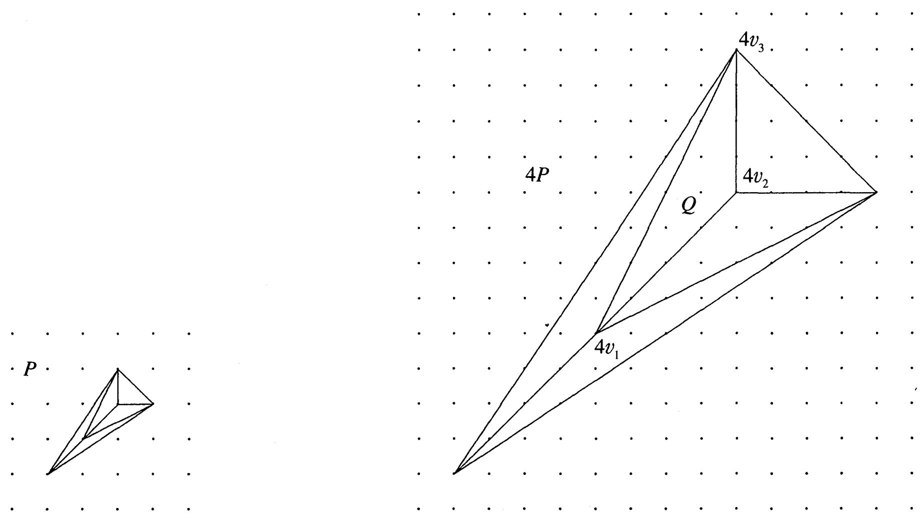

FIGURE 2.

Let $V_1=\{P_i\}$ be a 1-triangulation of P. Then $V_d=\{dP_i\}$ is a d-triangulation of dP. Hence, $Q \in dP_i$ for a certain i. Let v_1, v_2, v_3 be the vertices of P_i. Then, it follows from Lemma 2.3 that there exist unique $a, b, c \in \boldsymbol{N}$ such that $a(v_2-v_1)+b(v_3-v_1)+c\cdot 0=Q-dv_1$ with $a+b+c=d$. Hence, $av_2+bv_3+cv_1=Q$. □

From this lemma, it follows that ϕ_d is surjective.

THEOREM 2.7. *The ideal I is generated by polynomials of degrees 2 and 3.*

The next lemmas will serve to prove this theorem. From the way that we look at the problem, we see that I_d is generated by polynomials of the form $\boldsymbol{x}^d-\boldsymbol{y}^d$ such that the monomials $\boldsymbol{x}^d, \boldsymbol{y}^d \in \mathrm{Sym}^d(H^0(X, \mathscr{L}_P))$ are mapped by ϕ_d to the same image.

DEFINITION 2.8. Let P be a convex polygon. An *operation of degree n* on a path $S=\langle x_1, \dots, x_d\rangle$ to $Q \in dP$ is the substitution of a subset $S'=\langle y_1, \dots, y_n\rangle \subset S$ by a subset $S''=\langle u_1, \dots, u_n\rangle$, $u_i \in P$, such that

$$\sum_{x \in S} x = \sum_{x \in (S \setminus S') \cup S''} x = Q .$$

LEMMA 2.9. *Let P be a convex polygon. Let $v_1, \ldots, v_n$ be its vertices arranged clockwise in this order. Let $v_1=(0,0)$, let B_i, $i=1,\ldots,n-2$ be the triangle with vertices v_1, v_{i+1}, v_{i+2} which we get by drawing the lines L_i from $(0,0)$ to the vertices $v_3, \ldots, v_{n-1}$ (see Figure 3). Thus $B_1, \ldots, B_{n-2}$ give a triangulation of P. Suppose that we have a path $S=\langle x_1, \ldots, x_d\rangle$ to $Q\in dP$. Then, by operations of degree 2, we can change S into a path $S'=\langle x'_1, \ldots, x'_d\rangle$ to Q so that $x'_i\in B_{i_0}$ for all i and a certain i_0.*

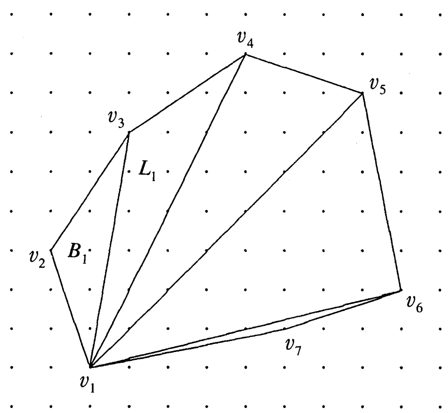

FIGURE 3.

PROOF. Let $T=\langle x_i\in S\,|\,x_i\in B_1, x_i\notin B_j \text{ if } j\neq 1\rangle$. Denote $h:=\sum_{x\in T} h(x, L_1)$ which is a nonnegative integer. We may suppose that there is a $y\in S$ and $y\notin B_1$, because if such a y does not exist, then all x_i belong to B_1 and hence nothing is left to prove.

Choose and fix any $x\in T$ and denote $R=y+x$. Then $R\in 2P$, hence $R\in 2B_j$ for a certain j. Thus, by Lemma 2.6 there exist $y', x'\in B_j$ such that $R=y'+x'$. Now replace in S the steps x by x' and y by y'. Then we get a new path S' to Q. Let $T'=\langle x'_i\in S'\,|\,x'_i\in B_1, x'_i\notin B_j \text{ if } j\neq 1\rangle$. We obtain the set T' from the set T in the following way:

Case 1. $y+x\in 2B_1$.

- If $h(x', L_1)>0$, then replace in T the step x by x', or else remove x from T.
- If $h(y', L_1)>0$, add the step y' to T.

Case 2. $y+x\notin 2B_1$. Then remove x from T.

Denote $h':=\sum_{x\in T'} h(x, L_1)$. In Case 1, we see that $h(x', L_1)+h(y', L_1)=h(x, L_1)+h(y, L_1)<h(x, L_1)$ because $h(y, L_1)<0$. In Case 2, we removed a point x from T with $h(x, L_1)>0$. The conclusion is that $h'<h$. Therefore, if we continue this process, two things are possible. Either h becomes 0 or all the points are in B_1. If h becomes 0, then we can start all over with B_2, etc. We see that at the end, all steps are in one triangle. The replacements in S are all operations of degree 2.

LEMMA 2.10. *Let P be a triangle. Let $Q\in 3P$. Then there exists a path*

$S=\langle x_1, x_2, x_3\rangle$ to Q, $x_i \in P$, such that one of the x_i is a vertex.

PROOF.

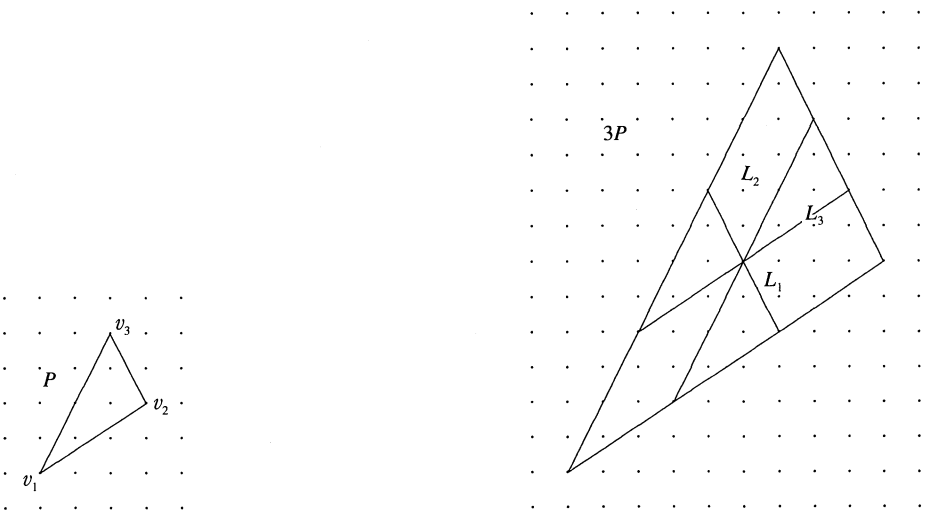

FIGURE 4.

Let v_1, v_2, v_3 be the vertices of P. Without loss of generality, we may assume $v_1=(0, 0)$. Let $2P_i := v_i + 2P$ for $i = 1, 2, 3$. Thus $2P_1$ (resp. $2P_2$, resp. $2P_3$) is a triangle with vertices $\mathbf{0}$, $2v_2$, $2v_3$ (resp. $3v_2$, v_2, v_2+2v_3, resp. $3v_3$, v_3, $2v_2+v_3$).

Let L_i be the edge of $2P_i$ that goes through v_2+v_3. It is clear that every point $Q \in 3P$ is in $2P_{i_0}$ for a certain i_0. Hence, from Lemma 2.6, it follows that there is a path (starting from v_{i_0}) to Q of length 2. If we also use v_{i_0} as a step, then we have a path from $\mathbf{0}$ of length 3 to Q. □

LEMMA 2.11. *Let P be a triangle. Let $S=\langle x_1, \dots, x_d\rangle$ be a path to $Q \in dP$. Then by operations of degree* 3, *we can change S in such a way that at most two steps of S are not vertices.*

PROOF. Take any three steps. Change them by an operation of degree 3 into three steps that contain a vertex. This is possible because of Lemma 2.10. Continue this process until there are no three steps left which are not vertices. □

LEMMA 2.12. *Let P be a triangle with vertices v_1, v_2, v_3. Let*

$$S=\langle \underbrace{v_1, \dots, v_1}_{a}, \underbrace{v_2, \dots, v_2}_{b}, \underbrace{v_3, \dots, v_3}_{c}, k_1, k_2\rangle$$

be a path of length $d \geq 4$ to a point Q. Then, there exists no other path of length d

$$S' = \langle \underbrace{v_1, \ldots, v_1}_{a'}, \underbrace{v_2, \ldots, v_2}_{b'}, \underbrace{v_3, \ldots, v_3}_{c'}, k'_1, k'_2 \rangle$$

to Q such that $S \cap S' = \varnothing$.

PROOF.

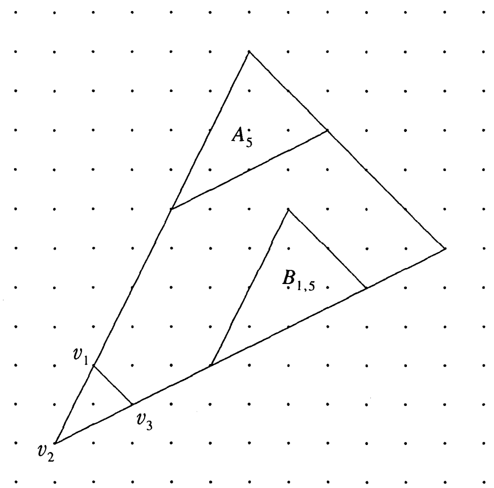

FIGURE 5.

Let the vertices of P be v_1, v_2, v_3 numbered as in Figure 5. Without loss of generality we may assume that $v_2 = (0, 0)$. Let

$$S' = \langle \underbrace{v_1, \ldots, v_1}_{a'}, \underbrace{v_2, \ldots, v_2}_{b'}, \underbrace{v_3, \ldots, v_3}_{c'}, k'_1, k'_2 \rangle$$

be any path of length d such that $S' \cap S = \varnothing$. Let S' be a path to Q'. Now we have to prove that Q' cannot be equal to Q.

Without loss of generality we may assume that $(a, b, c) = (d-2, 0, 0)$ and $(a', b', c') = (0, k, d-2-k)$ with $0 \le k \le d-2-k$. Then Q lies in the triangle A_d which has vertices $(d-2)v_1$, dv_1, $(d-2)v_1 + 2v_3$, and Q' lies in the triangle $B_{k,d}$ which has vertices $2v_1 + (d-2-k)v_3$, $(d-2-k)v_3$, $(d-k)v_3$ (see Figure 5).

If $d \ge 5$ then the triangle A_d and the triangle $B_{k,d}$ have no points in common, hence the lemma is true. If $d = 4$ then the two triangles have exactly one point in common namely $2v_1 + (2-k)v_3$. Hence Q and Q' can only be equal if $k'_1 = k'_2 = v_1$. Hence S and S' have a step in common. □

PROOF OF THE THEOREM. Suppose that we have a relation $x_1^d = x_2^d$. Hence, we have two different paths to $Q = \sum_{i=1}^d x_{1,i} = \sum_{i=1}^d x_{2,i}$. If we triangulate P as in Lemma 2.9, we can change both paths into paths which contain only steps of a certain triangle, by using only operations of degree 2. Hence, we get a relation $\sum_{i=1}^{d} x'_{1,i} = \sum_{i=1}^{d} x'_{2,i}$ with $x'_{1,i}, x'_{2,i} \in B_{i_0}$. By using relations of degree 3, we can even get in the situation that $x'_{1,i}$ (and also $x'_{2,i}$) are all vertices of B_{i_0} except two of them (Lemma 2.11).

Now we prove the theorem by induction. For $d=3$, the theorem is true. Suppose that $d>3$. From Lemma 2.12, it follows that $S_1 = \langle x_{1,i} \rangle$ and $S_2 = \langle x_{2,i} \rangle$ have a step in common. Hence, if we divide the relation by this variable, we get a relation of lower degree. But, by induction, this relation was in the ideal generated by I_2 and I_3 and therefore, the original relation was also in this ideal. □

Lemma 2.12 proves that to $Q \in dP$ there exists a kind of standard path consisting of the vertices of the triangle B of the polygon, in which Q lies, and of two steps which are allowed to be in the interior of B.

In higher dimension the natural generalization fails. This is shown in the following example.

EXAMPLE 2.13. Let P be the convex hull of the points $v_1 = (0, 0, 0)$, $v_2 = (0, 0, 3)$, $v_3 = (1, 2, 0)$, $v_4 = (2, 1, 0)$ (see Figure 6). With the criterion of Oda [Od, Theorem 2.13] one can check that $\mathscr{L}_P$ is a very ample line bundle on X_P. Let x_i be the variable corresponding to v_i, $i = 1, \ldots, 4$. Name the other points of $P \cap \mathbf{Z}^3$ as follows: $x_5 = (0, 0, 1)$, $x_6 = (0, 0, 2)$, $x_7 = (1, 1, 1)$ and $x_8 = (1, 1, 0)$.

Let $Q \in 5P$, $Q = (3, 3, 3)$. The natural generalization would be that a standard path consists of two vertices and three internal points. However, if we take the paths S_1 and S_2 to Q, where $S_1 = \langle x_3, x_4, x_5, x_5, x_5 \rangle$ and $S_2 = \langle x_1, x_2, x_8, x_8, x_8 \rangle$, then we notice that $S_1 \cap S_2 = \varnothing$.

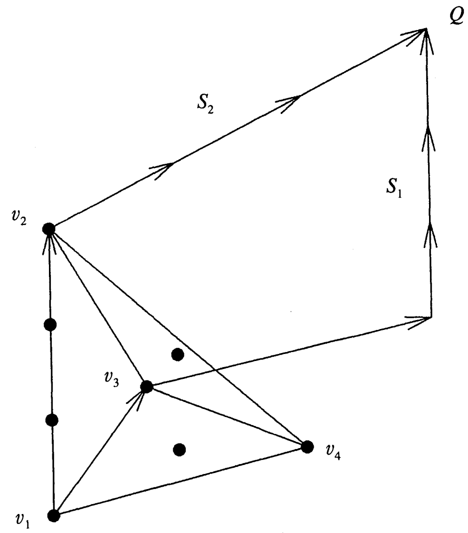

FIGURE 6.

Hence a better notion of standard path should be found for higher dimension. Although this notion of standard path fails, it is still likely that relations up to the degree $n+1$, where $n+1$ is the number of vertices of the standard simplex in dimension n, will suffice.

In the above example we have the relations $x_1x_2=x_5x_6$, $x_5^2=x_1x_6$ and $x_8^3=x_1x_3x_4$, hence the polynomial $x_1x_2x_8^3-x_3x_4x_5^3$ is in the ideal generated by the relations of degrees 2, 3 and 4 because we have

$$x_1x_2(x_8^3-x_1x_3x_4)-x_3x_4x_5(x_5^2-x_1x_6)+x_1x_3x_4(x_1x_2-x_5x_6)=x_1x_2x_8^3-x_3x_4x_5^3 .$$

Therefore I will make the following:

CONJECTURE 2.14. Let P be an integral convex polytope in $\boldsymbol{R}^n$ such that X_P is a toric variety of dimension n and that $\mathscr{L}_P$ is a very ample line bundle on X_P. Then the ideal I of X embedded in a projective space by $\mathscr{L}_P$, is generated by polynomials of degrees at most $n+1$.

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