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SINGULAR VARIATION OF THE GROUND STATE EIGENVALUE FOR A SEMILINEAR ELLIPTIC EQUATION

Dedicated to Professor Takeshi Kotake on his sixtieth birthday

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Abstract. A study is made of the asymptotic behaviour for the ground state eigenvalue concerning certain semi-linear elliptic operators under singular variation of domains.

1. Introduction. Let M be a bounded domain in $\boldsymbol{R}^3$ with smooth boundary ∂M. Let w be a fixed point in M. We remove from M an open ball $B(\varepsilon; w)$ of radius ε with the center w and write $M_\varepsilon = M \setminus \overline{B(\varepsilon; w)}$.

In the present note, we consider the minimizing problem:

$$(1.1)_\varepsilon \qquad \lambda(\varepsilon) = \inf_{X_\varepsilon} \int_{M_\varepsilon} |\nabla u|^2 dx\,,$$

where $X_\varepsilon = \{u;\, u \in H_0^1(M_\varepsilon),\ \|u\|_{L^{p+1}(M_\varepsilon)} = 1,\ u \geq 0\}$, and investigate the asymptotic behaviour of $\lambda(\varepsilon)$ when $\varepsilon \to 0$.

It is easy to see that when $p \in (1, 5)$ there exists at least one positive solution u_ε which attains $(1.1)_\varepsilon$ and which satisfies

$$(1.2) \qquad \begin{aligned} -\Delta u_\varepsilon &= \lambda(\varepsilon) u_\varepsilon^p \quad &&\text{in} \quad M_\varepsilon\,, \\ u_\varepsilon &= 0 &&\text{on} \quad \partial M_\varepsilon\,. \end{aligned}$$

Let A denote the operator $v \mapsto \Delta v$ from $H^2(M_\varepsilon) \cap H_0^1(M_\varepsilon)$ to $L^2(M_\varepsilon)$ associated with the boundary condition (1.2).

Along with $(1.1)_\varepsilon$, we consider the minimizing problem:

$$(1.3) \qquad \lambda(0) = \inf_{X} \int_{M} |\nabla u|^2 dx\,,$$

where $X = \{u;\, u \in H_0^1(M),\ u_{|\partial M} = 0,\ \|u\|_{L^{p+1}(M)} = 1,\ u \geq 0\}$.

THEOREM. *Assume that the positive solution of $-\Delta \tilde{u} = \tilde{u}^p$ in M under the Dirichlet condition on ∂M is unique. Assume also that for any small $0 < \varepsilon \ll 1$ the ground state*

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solution u_ε *of* $(1.1)_\varepsilon$ *is unique and* $\mathrm{Ker}(A+\lambda(\varepsilon)pu_\varepsilon^{p-1})=\{0\}$. *Here* u_ε *is the positive minimizer of* $(1.1)_\varepsilon$. *Then,*

$$\lambda(\varepsilon)-\lambda(0)=4\pi\varepsilon u(w)^2+o(\varepsilon) \tag{1.4}$$

holds for $p\in(1, 5)$. *Here* u *is the minimizer of* (1.3).

REMARK. The domain (such that the positive solution of $-\Delta \underset{\sim}{u}=\underset{\sim}{u}^p$ in M under the Dirichlet condition on ∂M is unique) is given by Gidas-Ni-Nirenberg [4] and Dancer [3].

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2. Preliminary lemmas. In the following we assume $\dim M=3$.

LEMMA 2.1. *Assume that* u *satisfies*

$$\Delta u(x)=0 \qquad x\in M\setminus\overline{B(\varepsilon; w)} \tag{2.1}$$

$$u(x)=0 \qquad x\in\partial M \tag{2.2}$$

$$u(x)=L(\theta) \qquad x=w+\varepsilon\theta\,,\quad \theta=(\theta_1, \theta_2, \theta_3)\in S^2\,. \tag{2.3}$$

Then,

$$\int_{S^2}\left(\left(\frac{\partial u}{\partial \nu}\right)(x)\right)^2_{\big|\partial S_\varepsilon}\varepsilon^2 d\theta\le C\left(\max_\theta L(\theta)^2+W\right),$$

where

$$W=(\max L(\theta)^2)^{\sigma/(1+\sigma)}(\|L\|^2_{H^1(S^2)}+\|L\|^2_{C^{1+\sigma'}(S^2)})^{1/(1+\sigma)}$$

for $\sigma'>\sigma>0$. *Here* $B_\varepsilon=B(\varepsilon; w)$.

PROOF. Let $-\Delta_{S^2}$ be the Laplace-Beltrami operator on S^2 with canonical metric. It has the eigenvalue series $0=\mu_0<\mu_1\le\mu_2\le\dots$. Let $\{\varphi_j(x)\}_{j=0}^\infty$ be a complete orthonormal basis of $L^2(S^2)$ consisting of the eigenfunctions of Δ_{S^2}. It is well known that $\mu_j\sim Cj$ for $j\to\infty$. Let

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_{S^2}$$

be the Laplacian on $R^3\setminus\{0\}$. We put $\Delta(r^{q_j}\varphi_j(\theta))=0$. Then, $(q_j(q_j-1)+2q_j-\mu_j)r^{q_j-2}=0$. When $\mu_j=0$, then $q_j=0, -1$. When $\mu_j\neq 0$, then $q_j=-(1/2)-(\mu_j+(1/4))^{1/2}$ is a candidate so that $r^{q_j}\to 0$ when $r\to\infty$, and behaves like $q_j\sim -c'j^{1/2}$ as $j\to\infty$.

We put $\tilde{S}_r=\{x\in \boldsymbol{R}^3; |x|=r\}$, $\psi\in C^\infty(R^3)$, $\psi_{|S_\varepsilon}=\psi(\varepsilon\theta)$, $\theta\in S^2$. Note that $S_1=S^2$. In terms of $\{\varphi_j\}_{j=0}^\infty$ we have the following expansion:

$$\psi(\varepsilon\theta)=\sum_{j=0}^\infty b_j\varphi_j(\theta)\,.$$

CLAIM. *The solution of the boundary value problem*

$$\Delta U(x)=0 \qquad x\in \boldsymbol{R}^3\setminus \bar{B}_\varepsilon$$

$$U_{|\tilde{S}_\varepsilon}=\psi(\varepsilon\theta)$$

$$\lim_{|x|\to\infty} U(x)=0$$

is given by

$$U(x)=\sum_{j=0}^{\infty} b_j(\varepsilon/r)^{-q_j}\varphi_j(\theta) \tag{2.4}$$

with $q_0=-1$. *And*

$$\|U\|_{L^2(\tilde{S}_R)}\le C(R)\|\psi\|_{L^2(\tilde{S}_\varepsilon)} \tag{2.5}$$

$$\left\|\frac{\partial U}{\partial r}\right\|_{L^2(\tilde{S}_R)}\le C'(R)\|\psi\|_{L^2(\tilde{S}_\varepsilon)} \tag{2.6}$$

hold.

PROOF OF CLAIM. It is known that the eigenvalue of the n-th spherical harmonic function is $n(n+1)$. Thus, $q_j\le -2$ for $\mu_j\neq 0$. We therefore get (2.5), (2.6). q.e.d.

CONTINUATION OF THE PROOF OF LEMMA 2.1. Assume that $w=\{0\}$ and choose R so that $\{x\in \boldsymbol{R}^3; \varepsilon<|x|<R\}\subset M$. By the Green formula for $\Delta U\cdot u-\Delta u\cdot U$ we get

$$\int_{\tilde{S}_\varepsilon}\psi\frac{\partial u}{\partial r}\Big|_{S_\varepsilon} d\tilde{S}_\varepsilon=J_1+J_2\,, \tag{2.7}$$

where

$$J_1=\int_{\tilde{S}_\varepsilon} L\frac{\partial U}{\partial r}\Big|_{\tilde{S}_\varepsilon} d\tilde{S}_\varepsilon$$

$$J_2=\int_{\tilde{S}_R}\left(U\frac{\partial u}{\partial r}-u\frac{\partial U}{\partial r}\right)\Big|_{\tilde{S}_R} d\tilde{S}_R\,.$$

We here have

$$\|u\|_{L^\infty(M)}\le \|L\|_{L^\infty(\tilde{S}_1)} \tag{2.8}$$

$$\left\|\frac{\partial u}{\partial r}\right\|_{L^2(\tilde{S}_R)}\le C\|u\|_{L^2(M)}\le C'\|L\|_{L^\infty(\tilde{S}_1)} \tag{2.9}$$

by the maximum principle and elliptic estimates.

Therefore,

$$|J_2|\le C\|L\|_{L^\infty(\tilde{S}_1)}\|\psi\|_{L^2(\tilde{S}_\varepsilon)}\,. \tag{2.10}$$

On the other hand,

$$J_1=\int_{\tilde{S}_\varepsilon} L\frac{\partial U}{\partial r}\Big|_{\tilde{S}_\varepsilon} d\tilde{S}_\varepsilon=\int_{\tilde{S}_1} L(\theta)\frac{\partial U}{\partial r}(\theta)_{|r=\varepsilon}\varepsilon^2 d\theta$$

$$=\int_{\tilde{S}^1}\left(\sum_{j=0}^{\infty} a_j\varphi_j(\theta)\right)\left(-\sum_{j=0}^{\infty} b_j q_j\varphi_j(\theta)\right)\varepsilon d\theta$$

$$=-\varepsilon\sum_{j=0}^{\infty} a_j b_j q_j .$$

Therefore,

$$J_1=\left(\varepsilon^2\sum_{j=0}^{\infty} b_j^2\right)^{1/2}\left(\sum_{j=0}^{\infty} q_j^2 a_j^2\right)^{1/2} .$$

Since

$$\varepsilon^2\sum_{j=0}^{\infty} b_j^2=\|\psi\|^2_{L^2(\tilde{S}_\varepsilon)} ,$$

we get

$$|J_1|\le\left(\sum_{j=0}^{\infty} q_j^2 a_j^2\right)^{1/2}\|\psi\|_{L^2(\tilde{S}_\varepsilon)} .$$

By (2.7) we have

$$\left\|\frac{\partial u}{\partial r}\right\|_{L^2(\tilde{S}_\varepsilon)}=\sup_{\|\psi\|_{L^2(\tilde{S}_\varepsilon)}=1}\left|\int_{\tilde{S}_\varepsilon}\psi\frac{\partial u}{\partial r}\Big|_{\tilde{S}_\varepsilon} d\tilde{S}_\varepsilon\right| \tag{2.11}$$

$$\le C'\left(\|L\|_{L^\infty(\tilde{S}_1)}+\left(\sum_{j=0}^{\infty} q_j^2 a_j^2\right)^{1/2}\right).$$

Since $q_j\sim -cj^{1/2}$ as $j\to\infty$, we see that the second term on the right hand side of (2.11) does not exceed

$$C'\left(\sum_{j=0}^{\infty} a_j^2 j^{1+\sigma}\right)^{1/(1+\sigma)}\left(\sum_{j=0}^{\infty} a_j^2\right)^{\sigma/(1+\sigma)} .$$

Clearly,

$$\int_{S^2} L(\theta)^2 d\theta=ca_0^2+C'\sum_{j=1}^{\infty} a_j^2\le C''\max L(\theta)^2 .$$

Thus, (2.11) is estimated by

$$C\max_\theta L(\theta)^2+(\|L\|_{H^{1+\sigma}(S^2)})^{2/(1+\sigma)}\max_\theta L(\theta)^{2\sigma/(1+\sigma)} .$$

Here we used the fact that

$$\left(a_0^2+\sum_{j=1}^{\infty} a_j^2 j^{\xi}\right)^{1/2}$$

is equivalent to the norm $\|L\|_{H^{\xi}(S^2)}$, $H^{\xi}(S^2)$ denoting the L^2-Sobolev space of fractional order.

Now it is well known that $H^{\xi}(S^2)\ni f$ has the following equivalent norm for $1<\xi<2$ (see Adams [1]):

$$\left(\|f\|_{H^1(S^2)}^2+\sum_{|\alpha|=1}\int_{S^2}\int_{S^2}|D^{\alpha}f(x)-D^{\alpha}f(y)||x-y|^{-2\xi}dxdy\right)^{1/2}.$$

Hence, $\|L\|_{H^{1+\sigma}(S^2)}\leq\|L\|_{H^1(S^2)}+\|L\|_{C^{1+\sigma'}(S^2)}$ for $\sigma'>\sigma>0$. Thus, we get Lemma 2.1.

q.e.d.

Let $G_{\varepsilon}(x, y)$ be the Green function of $-\Delta$ in M_{ε} under the Dirichlet condition on ∂M_{ε}, that is, it satisfies $-\Delta_x G_{\varepsilon}(x, y)=\delta(x-y)$, $x, y\in M_{\varepsilon}$ and $G_{\varepsilon}(x, y)=0$ for $x\in\partial M_{\varepsilon}$. Let $G(x, y)$ be the Green function of $-\Delta$ in M under the Dirichlet condition on ∂M. We introduce the following kernel function $p_{\varepsilon}(x, y)$:

$$p_{\varepsilon}(x, y)=G(x, y)-4\pi\varepsilon G(x, w)G(w, y).$$

Let $\boldsymbol{G}_{\varepsilon}$, $\boldsymbol{P}_{\varepsilon}$, $\boldsymbol{G}$ be the operators given by

$$\boldsymbol{G}_{\varepsilon}f(x)=\int_{M_{\varepsilon}} G_{\varepsilon}(x, y)f(y)dy$$

$$\boldsymbol{P}_{\varepsilon}f(x)=\int_{M_{\varepsilon}} p_{\varepsilon}(x, y)f(y)dy$$

$$\boldsymbol{G}g(x)=\int_{M} G(x, y)g(y)dy.$$

As for the regularity of operator $\boldsymbol{G}$, we refer the reader to the literature (for instance [2], [5]).

LEMMA 2.2. *There exist constants h, $C>0$ such that*

$$\int_{S^2}\left(\frac{\partial}{\partial\nu}(\boldsymbol{P}_{\varepsilon}-\boldsymbol{G}_{\varepsilon})f\right)^2_{\Big|\partial B_{\varepsilon}}\varepsilon^2 d\theta\leq C\varepsilon^h\|f\|_{L^q(M_{\varepsilon})}$$

holds for $f\in L^q(M_{\varepsilon})$ with $q>3$.

PROOF. Let $\tilde{f}$ denote the extension of f to M_{ε} defined as 0 outside M_{ε}. Let $(\boldsymbol{P}_{\varepsilon}-\boldsymbol{G}_{\varepsilon})f=v_{\varepsilon}$. Then,

$$\Delta v_{\varepsilon}(x)=0 \qquad x\in M_{\varepsilon}$$

$$v_\varepsilon(x)=0 \qquad x\in\partial M$$

$$v_\varepsilon(x)_{|\partial B_\varepsilon}=G\tilde{f}(x)-G\tilde{f}(w)-4\pi\varepsilon S(x, w)G\tilde{f}(w)\,,$$

where $S(x, y)=G(x, y)-(4\pi)^{-1}|x-y|^{-1}$.

Therefore, $v_\varepsilon(x)_{|\partial B_\varepsilon}\equiv L(\theta)$ satisfies

$$\max_\theta |L(\theta)|\le C\varepsilon^\tau\|G\tilde{f}\|_{C^\tau(M)}+O(\varepsilon)|G\tilde{f}(w)| \le C\varepsilon^\tau\|f\|_{L^q(M_\varepsilon)} \tag{2.12}$$

with $\tau>0$, provided $q>3/2$.

Furthermore,

$$\begin{aligned}\|L\|^2_{H^1(S^2)}+\|L\|^2_{C^{1+\sigma'}(M)} &\le C(\|G\tilde{f}\|^2_{H^1(S^2)}+\|G\tilde{f}\|^2_{C^{1+\sigma'}(M)}+O(\varepsilon)|G\tilde{f}(w)|^2)\\ &\le C'\|f\|^2_{L^r(M_\varepsilon)}\end{aligned} \tag{2.13}$$

for $r>3$ if we take sufficiently small $\sigma'>0$. Therefore, by Lemma 2.1 we get Lemma 2.2. q.e.d.

LEMMA 2.3. *Let $p\in(1, 5)$ and let u_ε be the solution of $(1.1)_\varepsilon$. Then, we have*

$$\sup_{0<\varepsilon\le\varepsilon_0}\ \sup_{x\in M_\varepsilon}|u_\varepsilon(x)|<C<\infty\,.$$

PROOF. We continue u_ε into M by setting 0 outside M_ε and denote by $\tilde{u}_\varepsilon$ the function thus extended. It is clear that $\{\tilde{u}_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ forms a bounded set in $H^1_0(M)$, so that by Sobolev lemma we first obtain

$$\sup_{0<\varepsilon\le\varepsilon_0}\ \|\tilde{u}_\varepsilon\|_{L^6(M)}\le C\,. \tag{2.14}$$

On the other hand, from $u_\varepsilon=\lambda(\varepsilon)G_\varepsilon u^p_\varepsilon$ and from the monotonicity of the Green function, it follows that

$$(0\le)u_\varepsilon\le\lambda(\varepsilon_0)G\tilde{u}^p_\varepsilon\,, \tag{2.15}$$

where we used the fact that $\lambda(\varepsilon)\le\lambda(\varepsilon_0)$ for $\varepsilon\le\varepsilon_0$.

Suppose now $p\in(1, 4)$. Then, in view of (2.14) we get $\sup_{0<\varepsilon\le\varepsilon_0}\|\tilde{u}^p_\varepsilon\|_{L^q(M)}\le C$ with $q>3/2$. Thus, the Lemma follows from (2.15) and the regularity of G.

Consider next the case where $p\in[4, 5)$. Using again the regularity of G, we see that there exists a constant $C'>0$, independent of ε $(\le\varepsilon_0)$, such that for $\varepsilon\le\varepsilon_0$

$$\|\tilde{u}_\varepsilon\|_{L^q(M)}\le\lambda(\varepsilon_0)\|G\tilde{u}^p_\varepsilon\|_{L^q(M)}\le C'\,,$$

where $q=6/(p-4)$ if $p\in(4, 5)$, and $q>1$ may be any positive number when $p=4$. We thus obtain a better estimate than (2.14), from which we started.

Proceeding in this manner step by step, the proof is completed. q.e.d.

The following Lemma is crucial for our study. Let Ω be a bounded domain in $\boldsymbol{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. Let ρ be a smooth function on $\partial\Omega$. We denote by $\nu(x)$ the exterior unit normal vector at $x \in \partial\Omega$. If ε is small enough, we have a new domain Ω_ε bounded by $\partial\Omega_\varepsilon = \{x + \varepsilon\rho(x)\nu(x)\,;\, x \in \partial\Omega\}$. Let p be a fixed number satisfying $1 < p < \infty$ for $N = 2$, $1 < p < (N+2)/(N-2)$ for $N \geq 3$.

We consider the minimizing problem

$$\mu(\varepsilon) = \inf_{Y\varepsilon} \int_{\Omega_\varepsilon} |\nabla u|^2 dx \,, \tag{2.16$_\varepsilon$}$$

where $Y_\varepsilon = \{u\,;\, u \in H_0^1(\Omega_\varepsilon),\ \|u\|_{L^{p+1}(\Omega_\varepsilon)} = 1,\ u_\varepsilon \geq 0\}$.

We have the following.

Lemma 2.4 (cf. Osawa [7]). Assume that the positive solution u which minimizes $(2.16)_0$ is unique. Assume also that $\operatorname{Ker}(T + \mu(0)pu^{p-1}) = \{0\}$, where T denotes the operator $v \mapsto \Delta v$ from $H^2(\Omega) \cap H_0^1(\Omega)$ to $L^2(\Omega)$. Then, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}(\mu(\varepsilon) - \mu(0)) = -\int_{\partial\Omega} (\partial u/\partial \nu_x)^2 \rho(x) d\sigma_x \,,$$

where $\partial/\partial\nu_x$ denotes the differentiation along the exterior normal.

Lemma 2.5. *$\lambda(\varepsilon)$ converges to $\lambda(0)$ as $\varepsilon \to 0$.*

Proof. From the monotonicity of the eigenvalues $\lambda(\varepsilon)$, it follows that as $\varepsilon \to 0$, $\lambda(\varepsilon) \to \lambda^* \geq \lambda(0)$. Since $\{\tilde{u}_\varepsilon\}_{0 < \varepsilon \leq \varepsilon_0}$ is bounded in $H_0^1(M)$, we may extract a subsequence $\{\tilde{u}_{\varepsilon_n}\}$ which converges to u^* weakly in $H_0^1(M)$ and strongly in $L^q(M)$ for any $q \in (1, 6)$. It is not difficult to verify here that $\operatorname{supp}(-\Delta u^* - \lambda^* u^{*p}) \in \{w\}$ and $-\Delta u^* - \lambda^* u^{*p} \in H^{-1}(M) + L^1(M)$. Therefore, $-\Delta u^* = \lambda^* u^{*p}$ in M. Suppose now $\lambda^* \gneq \lambda(0)$. Then, we shall have two positive solutions of $-\Delta \underset{\sim}{u} = \underset{\sim}{u}^p$ in M, namely $\lambda^{*1/(p-1)} u^*$, $\lambda(0)^{1/(p-1)} u$. This contradicts the assumption of the Theorem, and so the proof is complete. q.e.d.

3. Proof of the Theorem. Applying Lemma 2.4 to our situation, we obtain

$$\lambda'(t) = \int_{S^2} (\partial u_t/\partial \nu)^2|_{\partial B_t} t^2 d\theta$$

so that

$$\lambda(\varepsilon) - \lambda(0) = \int_0^\varepsilon \lambda'(t) dt \,. \tag{3.1}$$

Let $\lambda'(t) = \lambda(t)^2(K_1 + K_2 + K_3)$,
where

$$K_1=\int_{S^2}(\partial \boldsymbol{P}_t u_t^p/\partial v_x)^2 t^2 d\theta$$

$$K_2=2\int_{S^2}(\partial \boldsymbol{P}_t u_t^p/\partial v_x)(\partial(\boldsymbol{P}_t-\boldsymbol{G}_t)u_t^p/\partial v_x)t^2 d\theta$$

$$K_3=\int_{S^2}(\partial(\boldsymbol{P}_t-\boldsymbol{G}_t)u_t^p/\partial v_x)^2 t^2 d\theta$$

with $x=w+t\theta$, $\theta\in S^2$.

If we prove that

$$K_1\le C\,, \tag{3.2}$$

then by Lemma 2.2 and 2.3 we get $K_2=O(t^{\nu/2})$, $K_3=O(t^{\nu})$ for some $\nu>0$.

To this effect, write $\lambda(t)^2K_1=L_1+L_2+L_3$, where

$$L_1=\lambda(t)^2\int_{S^2}(\partial \boldsymbol{G}\tilde{u}_t^p/\partial v_x)^2 t^2 d\theta$$

$$L_2=-8\pi t\lambda(t)^2\int_{S^2}\left(\frac{\partial}{\partial v_x}\boldsymbol{G}\tilde{u}_t^p(x)\right)\left(\frac{\partial}{\partial v_x}G(x,w)\boldsymbol{G}\tilde{u}_t^p(w)\right)t^2 d\theta$$

$$L_3=16\pi^2t^2\lambda(t)^2\int_{S^2}\left(\frac{\partial}{\partial v_x}G(x,w)\right)^2(\boldsymbol{G}\tilde{u}_t^p(w))^2t^2d\theta\,.$$

By Lemma 2.3 we then have $L_1=O(t^2)$, $L_2=O(t)$ and

$$L_3=16\pi^2t^2\lambda(t)^2\int_{S^2}\left(\frac{\partial}{\partial v_x}\frac{1}{4\pi}|x-w|^{-1}\right)^2(\boldsymbol{G}\tilde{u}_t^p(w))^2t^2d\theta+O(t^4)$$

$$=O(1)\,.$$

We thus have proved (3.2).

Using now the estimates for $K_i\,(i=1,2,3)$, we obtain $\lambda(\varepsilon)-\lambda(0)=O(\varepsilon)$, together with

$$\lambda(t)^2K_1=4\pi\lambda(0)^2\boldsymbol{G}\tilde{u}_t^p(w)+O(t)\,.$$

Further, from Lemma 2.3 and the proof of Lemma 2.5, it is easy to show that as $t\to 0$, $\tilde{u}_t^p$ actually converges to u^p in $L^q(M)$ with $q>3/2$, so that $\boldsymbol{G}\tilde{u}_t^p(w)=\boldsymbol{G}u^p(w)+o(1)$.

Combining the estimate obtained above and noting that $u=\lambda(0)\boldsymbol{G}u^p$ in M, we finally obtain

$$\lambda'(t)=4\pi u(w)^2+o(1)\,,$$

and the proof of the Theorem is complete.

4. Comment. For related topics, the reader may be referred to Osawa [7], Osawa-Ozawa [8], Ozawa [9], Dancer [3], Lin [6]. The result of this paper was announced in Ozawa [10].

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Addendum (Received February 10, 1993). After this paper was accepted, it turned out, thanks to a recent result of S. Roppongi: "The Hadamard variation of the ground state value of some quasi-linear elliptic equations (preprint)", that the assumption $\mathrm{Ker}(A+\lambda(\varepsilon)pu_\varepsilon^{p-1})=\{0\}$ for $\varepsilon>0$ is unnecessary to obtain the formula

$$\lambda'(\varepsilon)=\int_{S^2}\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)^2\Big|_{\partial B_\varepsilon}\varepsilon^2 d\theta\,.$$

Thus, the conclusion of the Theorem holds as it stands without the assumption above mentioned.

Moreover, it was pointed out by Professor E. N. Dancer that his results in Math. Z. 206 (1991), 551–562, imply that if the positive solution of $-\Delta\tilde{\tilde{u}}=\lambda\tilde{\tilde{u}}^p$ in M under the Dirichlet condition on ∂M is unique, the ground state solution u_ε is then unique for $0<\varepsilon\ll 1$, provided that $\mathrm{Ker}(A+\lambda p\tilde{\tilde{u}}^{p-1})=\{0\}$, λ being the ground state value.

As the uniqueness of the positive solution of $-\Delta\tilde{u}=\tilde{u}^p$ is actually equivalent to the uniqueness of the positive solution of $-\Delta\tilde{\tilde{u}}=\lambda\tilde{\tilde{u}}^p$, the proof of the Theorem might be further simplified.

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