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AN INDEX FORMULA FOR THE DE RHAM COMPLEX

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Abstract. The purpose of this note is to give an *analytic* (and direct) proof of an index formula for the *relative* de Rham cohomology groups, which may be considered as a generalization of the celebrated Atiyah-Singer index theorem for the absolute de Rham cohomology groups. The crucial point is how to find an operator D for which an index formula holds. In deriving our index formula, the theory of harmonic forms satisfying an *interior boundary condition* plays a fundamental role. We remark that the operator D is no longer a local (differential) operator.

Introduction and results. Let X be an n-dimensional smooth manifold, and let $\Omega(X)$ be the space of smooth differential forms on X:

$$\Omega(X)=\bigoplus_{k=0}^{n} \Omega^k(X),$$

where $\Omega^k(X)$ is the space of smooth k-forms.

Let $d: \Omega(X)\to\Omega(X)$ be the exterior derivative on X. A smooth k-form α on X is said to be *closed* if $d\alpha=0$. It is said to be *exact* if $\alpha=d\beta$ for some smooth $(k-1)$-form β on X.

We let

$$Z^k(X)=\text{the space of closed } k\text{-forms on } X,$$

$$B^k(X)=\text{the space of exact } k\text{-forms on } X,$$

and

$$H^k(X)=Z^k(X)/B^k(X).$$

The quotient space $H^k(X)$ is called the *k-th de Rham cohomology group* of X. These groups come from a sequence of maps (the de Rham complex)

$$\Omega^{k-1}(X)\xrightarrow{d^{k-1}}\Omega^k(X)\xrightarrow{d^k}\Omega^{k+1}(X),$$

and

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$$H^k(X) = \mathrm{Ker}\, d^k / \mathrm{Im}\, d^{k-1}\,.$$

The celebrated de Rham theorem states that the de Rham cohomology groups $H^k(X)$ are isomorphic to the simplicial cohomology groups $H^k(X, \boldsymbol{R})$ defined in algebraic topology:

$$H^k(X) \cong H^k(X, \boldsymbol{R})\,.$$

We recall that the *Euler-Poincaré characteristic* $\chi(X)$ is defined by the formula:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, \boldsymbol{R})\,.$$

Now let X be a compact, oriented smooth Riemannian manifold *without* boundary. The Riemannian structure on X gives rise to a strictly positive smooth measure on X, and to an inner product $(\cdot, \cdot)$ on each $\Omega^k(X)$.

Let δ be the adjoint operator of the exterior derivative d with respect to the inner product $(\cdot, \cdot)$:

$$(\delta\alpha, \beta) = (\alpha, d\beta)\,, \quad \alpha \in \Omega^{k+1}(X)\,, \quad \beta \in \Omega^k(X)\,.$$

We "roll up" the de Rham complex, and define an operator

$$(d+\delta)_e : \Omega^e(X) \to \Omega^o(X)$$
$$\alpha \mapsto (d+\delta)\alpha\,,$$

where:

$\Omega^e(X) = \bigoplus_{i=0}^{[n/2]} \Omega^{2i}(X)$, the space of differential forms of *even* degree,
$\Omega^o(X) = \bigoplus_{i=0}^{[n/2]} \Omega^{2i+1}(X)$, the space of differential forms of *odd* degree.

We recall that the *analytical index* $\mathrm{ind}(d+\delta)_e$ of the operator $(d+\delta)_e$ is defined by the formula:

$$\mathrm{ind}(d+\delta)_e = \dim \mathrm{Ker}(d+\delta)_e - \dim \mathrm{Ker}(d+\delta)_e^*\,,$$

where $(d+\delta)_e^*$ is the adjoint operator of $(d+\delta)_e$.

Then we obtain the following index formula which is a special case of the *Atiyah-Singer index theorem* (cf. [CP], [G], [P]):

THEOREM 1. $\mathrm{ind}(d+\delta)_e = \chi(X)$.

The purpose of this note is to prove an index formula for the cohomology groups $H^{\cdot}(X, Y)$ of X *relative* to an $(n-1)$-dimensional, compact oriented submanifold Y of X. The crucial point is how to find an operator D, a generalization of $(d+\delta)_e$, for which such an index formula as in Theorem 1 holds.

We let

$$\Omega^p(X)=\text{the space of smooth } p\text{-forms on } X\,,$$

$$\Omega^p(Y)=\text{the space of smooth } p\text{-forms on } Y\,,$$

and

$$\Omega^p(X, Y)=\{\theta\in\Omega^p(X);\ \iota^*(\theta)=0\}\,,$$

where $\iota\colon Y\to X$ is the natural inclusion map. Then the exterior derivative d maps $\Omega^p(X, Y)$ into $\Omega^{p+1}(X, Y)$. Indeed, it suffices to note that $\iota^*d=d'\iota^*$ where d' is the exterior derivative on Y. Thus we have the following sequence of maps

$$\Omega^{p-1}(X, Y)\xrightarrow{d^{p-1}}\Omega^p(X, Y)\xrightarrow{d^p}\Omega^{p+1}(X, Y)\,.$$

We let

$$H^p(X, Y)=\operatorname{Ker} d^p/\operatorname{Im} d^{p-1}\,.$$

The quotient space $H^p(X, Y)$ is called the *p-th de Rham cohomology group of X relative to Y*. In other words, the relative cohomology group $H^{\cdot}(X, Y)$ is the cohomology group of the complex $\Omega^{\cdot}(X, Y)$ defined by the exact sequence of complexes

$$0\longrightarrow\Omega^{\cdot}(X, Y)\longrightarrow\Omega^{\cdot}(X)\xrightarrow{\iota^*}\Omega^{\cdot}(Y)\longrightarrow 0\,.$$

The de Rham theorem extends to this case, that is, the cohomology groups $H^p(X, Y)$ are isomorphic to the relative cohomology groups $H^p(X, Y, \boldsymbol{R})$ defined in algebraic topology:

$$H^p(X, Y)\cong H^p(X, Y, \boldsymbol{R})\,.$$

We define the *Euler-Poincaré characteristic* $\chi(X, Y)$ by the following formula:

$$\chi(X, Y)=\sum_{i=0}^{n}(-1)^i\dim H^i(X, Y, \boldsymbol{R})\,.$$

We let

$\Omega^p(X\setminus Y)=$ the space of p-currents on X which are smooth in $X\setminus Y$ and may have *jump* discontinuities at Y,

and

$$\Omega^e(X\setminus Y)=\bigoplus_i\Omega^{2i}(X\setminus Y)\,,\quad \Omega^o(X\setminus Y)=\bigoplus_i\Omega^{2i+1}(X\setminus Y)\,;$$

$$\Omega^e(Y)=\bigoplus_i\Omega^{2i}(Y)\,,\quad \Omega^o(Y)=\bigoplus_i\Omega^{2i+1}(Y)\,.$$

If T is a p-current on Y, we define a p-current $T\otimes\delta_Y$ on X by the formula:

$$\int_X \alpha\wedge *(T\otimes\delta_Y)=\int_Y \iota^*\alpha\wedge *'T\,,\quad \alpha\in\Omega^p(X)\,.$$

Here $*$ and $*'$ are the Hodge star operators on X and on Y, respectively.

We introduce a linear operator

$$D=\begin{pmatrix}(d+\delta) & -(\cdot\otimes\delta_Y)\\ \iota^* & 0\end{pmatrix}:\ \begin{matrix}\Omega^e(X\setminus Y)\\ \oplus\\ \Omega^o(Y)\end{matrix}\ \longrightarrow\ \begin{matrix}\Omega^o(X\setminus Y)\\ \oplus\\ \Omega^e(Y)\end{matrix}$$

as follows:

(1) The domain $\mathscr{D}(D)$ of D is the space

$$\mathscr{D}(D)=\left\{\begin{pmatrix}\alpha\\ S\end{pmatrix};\ \alpha\in\Omega^e(X\setminus Y),\ S\in\Omega^o(Y),\ d\alpha\in\Omega^o(X\setminus Y),\ \delta\alpha-(S\otimes\delta_Y)\in\Omega^o(X\setminus Y)\right\}.$$

(2)
$$D\begin{pmatrix}\alpha\\ S\end{pmatrix}=\begin{pmatrix}(d+\delta)\alpha-(S\otimes\delta_Y)\\ \iota^*\alpha\end{pmatrix},\quad \begin{pmatrix}\alpha\\ S\end{pmatrix}\in\mathscr{D}(D)\,.$$

Here $d\alpha$ and $\delta\alpha$ are taken in the sense of currents. Now we can state our index formula:

THEOREM 2. $\operatorname{ind} D=\chi(X, Y)=\chi(X)-\chi(Y)$.

The rest of this note is organized as follows: In Sections 1 and 2, we present a brief description of the basic definitions and results about differential operators and function spaces in differential geometry and partial differential equations. In Section 3, we consider the exterior derivative d restricted to the space $\Omega^p(X, Y)$ in the space $W_0^p(X)$ of square integrable p-currents on X, and then characterize its minimal closed extension $\bar{d}$ and the adjoint operator $\bar{d}^*$. In Section 4, via the Hilbert-Schmidt theory, we formulate the celebrated Hodge-Kodaira decomposition theorem for the Laplacian $\Delta=d\delta+\delta d$ in the framework of the Hilbert spaces $W_0^p(X)$. In particular, we have the following:

$$\operatorname{Ker}^p\Delta=\operatorname{Ker}^p(d+\delta)\cong H^p(X)\cong H^p(X, \boldsymbol{R})\,.$$

In Section 5, we study the operator D and its adjoint D^*, and characterize the kernels $\operatorname{Ker} D$ and $\operatorname{Ker} D^*$ componentwise. The characterizations of the operators $\bar{d}$ and $\bar{d}^*$ in Section 3 play an important role in the proof. Sections 6 and 7 are devoted to the proof of Theorem 2. First we consider an elliptic pseudo-differential operator P of order -1 on Y which is associated with the *interior boundary value problem* for the Laplacian $\Delta=d\delta+\delta d$:

$$\begin{cases}\Delta T=0 & \text{in}\quad X\setminus Y\,,\\ T|_Y=\varphi & \text{on}\quad Y\,.\end{cases}$$

Next, by using the operator P, we introduce a generalized Laplacian L' on Y by the formula:

$$L' = d'\delta'_1 + \delta'_1 d' ,$$

where $\delta'_1 = P\delta' P^{-1}$. It is easy to see that the Hodge-Kodaira theory extends to the operators d', δ'_1 and L':

$$\mathrm{Ker}^p L' = \mathrm{Ker}^p(d' + \delta'_1) \cong H^p(Y) \cong H^p(Y, \boldsymbol{R}) .$$

Finally we construct explicitly six mappings ρ_e, ρ'_e, ρ''_e, ρ_o, ρ'_o and ρ''_o so that the following sequence of homomorphisms forms a complex, and is *exact*:

$$\xrightarrow{\rho''_o} \mathrm{Ker}^{2i} D \xrightarrow{\rho_e} \mathrm{Ker}^{2i}(d+\delta) \xrightarrow{\rho'_e} \mathrm{Ker}^{2i}(d' + \delta'_1)$$

$$\xrightarrow{\rho''_e} \mathrm{Ker}^{2i+1} D^* \xrightarrow{\rho_o} \mathrm{Ker}^{2i+1}(d+\delta) \xrightarrow{\rho'_o} \mathrm{Ker}^{2i+1}(d' + \delta'_1) .$$

Therefore, Theorem 2 follows from an application of the well-known five lemma.

Our index formula is inspired by the work of Fujiwara [F]. The author would like to thank Professor Daisuke Fujiwara for valuable discussions.

1. Differential operators. Let X be an n-dimensional smooth manifold, and let $\Omega(X)$ be the space of smooth differential forms on X. The space $\Omega(X)$ is graded by the degrees of forms:

$$\Omega(X) = \bigoplus_{k=0}^{n} \Omega^k(X) ,$$

where $\Omega^k(X)$ is the space of smooth k-forms. There exists a unique linear map

$$d \colon \Omega(X) \to \Omega(X) ,$$

called the *exterior derivative*, such that:

(a) $d \colon \Omega^k(X) \to \Omega^{k+1}(X)$.

(b) df equals the ordinary differential df if $f \in C^\infty(X)$.

(c) If $\mu \in \Omega^k(X)$ and $\tau \in \Omega(X)$, then we have

$$d(\mu \wedge \tau) = d\mu \wedge \tau + (-1)^k \mu \wedge d\tau .$$

(d) $d^2 = 0$.

The operator d is a first-order differential operator.

Now let X be a compact, oriented smooth Riemannian manifold *without* boundary. The Riemannian structure on X gives rise to a strictly positive smooth measure μ on X, and to an inner product $(\cdot, \cdot)$ on each $\Omega^k(X)$.

Let δ be the adjoint operator of the exterior derivative d with respect to the inner product $(\cdot, \cdot)$:

$$(\delta\alpha, \beta) = (\alpha, d\beta) , \quad \alpha \in \Omega^{k+1}(X) , \quad \beta \in \Omega^k(X) .$$

The operator δ is a first-order differential operator, and is called the *codifferential operator*.

There is an isomorphism

$$*: \Omega^k(X) \to \Omega^{n-k}(X),$$

called the *Hodge star operator*, such that:

(i) $(\alpha, \beta) = \int_X \alpha \wedge *\beta$, $\alpha, \beta \in \Omega^k(X)$.

(ii) $*1 = \mu$, $*\mu = 1$.

(iii) $**\alpha = (-1)^{k(n-k)}\alpha$, $\alpha \in \Omega^k(X)$.

(iv) $(*\alpha, *\beta) = (\alpha, \beta)$, $\alpha, \beta \in \Omega^k(X)$.

We remark that the operator δ can be expressed in terms of the operator $*$ as follows:

$$\delta\alpha = (-1)^{n(k+1)+1} * d * \alpha, \quad \alpha \in \Omega^k(X).$$

We define the *Laplace-Beltrami operator* Δ on X by the formula:

$$\Delta = (d + \delta)^2 = d\delta + \delta d.$$

The operator Δ maps $\Omega^k(X)$ into itself, since d is of degree $+1$ while δ is of degree -1. It is known that Δ is a second-order *elliptic* differential operator.

2. Function spaces. First we recall the basic definitions and facts about the Fourier transform.

If $f \in L^1(\boldsymbol{R}^n)$, we define its (direct) Fourier transform $\mathscr{F}f$ by the formula

$$\mathscr{F}f(\xi) = \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \xi = (\xi_1, \ldots, \xi_n),$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$. We also denote $\mathscr{F}f$ by $\hat{f}$. Similarly, if $g \in L^1(\boldsymbol{R}^n)$, we define

$$\mathscr{F}^* g(x) = \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} e^{ix\cdot\xi} g(\xi)d\xi.$$

The function $\mathscr{F}^* g$ is called the inverse Fourier transform of g.

We introduce a subspace of $L^1(\boldsymbol{R}^n)$ which is invariant under the Fourier transform. We let

$\mathscr{S}(\boldsymbol{R}^n)$ = the space of C^∞-functions φ on $\boldsymbol{R}^n$ such that we have for any non-negative integer j

$$p_j(\varphi) = \sup_{\substack{x \in \boldsymbol{R}^n \\ |\alpha| \le j}} \{(1 + |x|^2)^{j/2} |\partial^\alpha \varphi(x)|\} < \infty.$$

The space $\mathscr{S}(\boldsymbol{R}^n)$ is called the space of C^∞-functions on $\boldsymbol{R}^n$ rapidly decreasing at infinity. We equip the space $\mathscr{S}(\boldsymbol{R}^n)$ with the topology defined by the countable family $\{p_j\}$ of seminorms. The space $\mathcal{S}(\boldsymbol{R}^n)$ is complete.

We list some basic properties of the Fourier transform:

(1) The transforms $\mathcal{F}$ and $\mathcal{F}^*$ map $\mathcal{S}(\boldsymbol{R}^n)$ continuously into itself.

(2) The transforms $\mathcal{F}$ and $\mathcal{F}^*$ are isomorphisms of $\mathcal{S}(\boldsymbol{R}^n)$ onto itself; more precisely, we have

$$\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = I \qquad \text{on} \quad \mathcal{S}(\boldsymbol{R}^n).$$

The elements of the dual space $\mathcal{S}'(\boldsymbol{R}^n)$ are called tempered distributions on $\boldsymbol{R}^n$. The direct and inverse Fourier transforms can be extended to the space $\mathcal{S}'(\boldsymbol{R}^n)$ respectively by the following formulas:

$$\langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle, \quad \varphi \in \mathcal{S}(\boldsymbol{R}^n).$$

$$\langle \mathcal{F}^*u, \varphi \rangle = \langle u, \mathcal{F}^*\varphi \rangle, \quad \varphi \in \mathcal{S}(\boldsymbol{R}^n).$$

Here $\langle \cdot, \cdot \rangle$ is the pairing between the spaces $\mathcal{S}'(\boldsymbol{R}^n)$ and $\mathcal{S}(\boldsymbol{R}^n)$. Once again, the transforms $\mathcal{F}$ and $\mathcal{F}^*$ map $\mathcal{S}'(\boldsymbol{R}^n)$ continuously into itself, and $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = I$ on $\mathcal{S}'(\boldsymbol{R}^n)$.

The function spaces we shall treat are the following (cf. [CP], [H1], [T]): If $a \in \boldsymbol{R}$, we let

$W_a(\boldsymbol{R}^n)$ = the space of distributions $u \in \mathcal{S}'(\boldsymbol{R}^n)$ such that $\hat{u} = \mathcal{F}u$ is a locally integrable function on $\boldsymbol{R}^n$ and that

$$\int_{\boldsymbol{R}^n} (1 + |\xi|^2)^a |\hat{u}(\xi)|^2 d\xi < \infty .$$

We equip the space $W_a(\boldsymbol{R}^n)$ with the inner product

$$(u, v)_a = \int_{\boldsymbol{R}^n} (1 + |\xi|^2)^a \hat{u}(\xi)\overline{\hat{v}(\xi)} d\xi ,$$

and with the associated norm

$$\| u \|_a = \left(\int_{\boldsymbol{R}^n} (1 + |\xi|^2)^a |\hat{u}(\xi)|^2 d\xi \right)^{1/2} .$$

The space $W_a(\boldsymbol{R}^n)$ is complete.

We list some basic topological properties of the spaces $W_a(\boldsymbol{R}^n)$:

(1) The space $\mathcal{S}(\boldsymbol{R}^n)$ is dense in each $W_a(\boldsymbol{R}^n)$.

(2) If $a' \leq a$, we have inclusions

$$\mathcal{S}(\boldsymbol{R}^n) \subset W_a(\boldsymbol{R}^n) \subset W_{a'}(\boldsymbol{R}^n) \subset \mathcal{S}'(\boldsymbol{R}^n),$$

with continuous injections.

(3) The spaces $W_a(\boldsymbol{R}^n)$ and $W_{-a}(\boldsymbol{R}^n)$ are dual to each other with respect to the

bilinear form:

$$\langle u, v\rangle = \int_{\boldsymbol{R}^n} \hat{u}(\xi)\hat{v}(\xi)d\xi\,, \quad u \in W_a(\boldsymbol{R}^n)\,, \quad v \in W_{-a}(\boldsymbol{R}^n)\,.$$

We let $\delta_{\boldsymbol{R}^{n-1}}(x)$ be a distribution on $\boldsymbol{R}^n$ defined by the following formula:

$$\langle \delta_{\boldsymbol{R}^{n-1}}, \varphi\rangle = \int_{\boldsymbol{R}^{n-1}} \varphi(x', 0)dx'\,, \quad \varphi \in C_0^\infty(\boldsymbol{R}^n)\,.$$

We remark that

$$\delta_{\boldsymbol{R}^{n-1}}(x', x_n) = 1 \otimes \delta(x_n)\,.$$

The next result characterizes the restrictions of elements in $W_a(\boldsymbol{R}^n)$ to the hyperplane $\{x_n = 0\}$ which enter naturally in connection with interior boundary value problems:

THEOREM 2.1. *If $a > 1/2$, then the restriction map*

$$\rho : \mathscr{S}(\boldsymbol{R}^n) \to \mathscr{S}(\boldsymbol{R}^{n-1})\,, \quad \varphi(x', x_n) \mapsto \varphi(x', 0)$$

can be extended in one and only one way to a continuous mapping ρ of $W_a(\boldsymbol{R}^n)$ onto $W_{a-1/2}(\boldsymbol{R}^{n-1})$.

If X is an n-dimensional, compact smooth manifold without boundary, then the space $W_a^p(X)$ of p-currents on X is defined to be locally the space $W_a(\boldsymbol{R}^n)$, upon using local coordinate systems $(x^1, \dots, x^n)$ flattening out X, together with a partition of unity. That is, we let

$W_a^p(X)$ = the space of p-currents α on X such that in local coordinates

$$\alpha = \sum_{1 \le i_1 < \cdots < i_p \le n} \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\,,$$

where the coefficients $\alpha_{i_1 \dots i_p}$ belong locally to the space $W_a(\boldsymbol{R}^n)$.

Then we have the following topological properties of the spaces $W_a^p(X)$ (cf. [F, Proposition 3.2]):

(1) If $a' \le a$, then we have an inclusion

$$W_a^p(X) \subset W_{a'}^p(X)\,,$$

with continuous injection.

(2) (Rellich) If $a' < a$, then the injection

$$W_a^p(X) \to W_{a'}^p(X)$$

is completely continuous (or compact).

(3) If Y is an $(n-1)$-dimensional, compact submanifold of X, then the restriction map

$$\rho: W_a^p(X) \to W_{a-1/2}^p(Y), \qquad u \mapsto u|_Y$$

is well-defined for all $a > 1/2$, and surjective.

3. The exterior derivative and the codifferential operator. We denote by d and δ the exterior derivative and the codifferential operator in the sense of currents, respectively. If T is a p-current on Y, we define a p-current $T \otimes \delta_Y$ on X by the formula:

$$\int_X \alpha \wedge *(T \otimes \delta_Y) = \int_Y \iota^* \alpha \wedge *' T, \quad \alpha \in \Omega^p(X).$$

Here $*$ and $*'$ are the Hodge star operators on X and on Y, respectively.

Then it is easy to see the following:

LEMMA 3.1. *We have for any p-current T on Y*

$$\delta(T \otimes \delta_Y) = \delta' T \otimes \delta_Y,$$

where δ' is the codifferential operator on Y.

We recall that

$$W_0^p(X) = \text{the space of square integrable } p\text{-currents on } X.$$

This is a Hilbert space with respect to the inner product

$$(\alpha, \beta) = \int_X \alpha \wedge *\beta, \quad \alpha, \beta \in W_0^p(X).$$

We let

$\bar{d}$ = the minimal closed extension in $W_0^p(X)$ of the operator d restricted to the space $\Omega^p(X, Y) = \{\alpha \in \Omega^p(X); \iota^*\alpha = 0\}$,

and

$$\bar{d}^* = \text{the adjoint of the operator } \bar{d}: W_0^p(X) \to W_0^{p+1}(X).$$

The next theorem gives a characterization of the operator $\bar{d}$ (cf. [F, Theorem 5.11]):

THEOREM 3.2. *If $\alpha \in W_0^p(X)$, $d\alpha \in W_0^p(X)$ and $\alpha|_Y = 0$, then we have*

$$\begin{cases} \alpha \in \mathscr{D}(\bar{d}), \\ \bar{d}\alpha = d\alpha. \end{cases}$$

The next theorem gives a characterization of the operator $\bar{d}^*$ (cf. [F, Theorem 5.1]):

THEOREM 3.3. *An element $\alpha \in W_0^{p+1}(X)$ belongs to the domain $\mathscr{D}(\bar{d}^*)$ of $\bar{d}^*$ if and only if there exist $\gamma \in W_0^p(X)$ and $T \in W_{-1/2}^p(Y)$ such that*

$$\delta\alpha=\gamma+(T\otimes\delta_Y).$$

In this case, we have

$$\bar{d}^*\alpha=\gamma=\delta\alpha-(T\otimes\delta_Y),$$

and

$$\delta' T\in W^{p-1}_{-1/2}(Y).$$

4. The Hodge-Kodaira decomposition theorem. Let d be the exterior derivative with domain

$$\mathscr{D}(d)=\{T\in W^p_0(X);\, dT\in W^{p+1}_0(X)\},$$

and δ the codifferential operator with domain

$$\mathscr{D}(\delta)=\{S\in W^{p+1}_0(X);\, \delta S\in W^p_0(X)\}.$$

We remark that the operators d and δ are adjoint to each other with respect to the L^2-inner product of the spaces $W^p_0(X)$:

$$(dT, S)=(T, \delta S), \quad T\in\mathscr{D}(d), \quad S\in\mathscr{D}(\delta).$$

We introduce the Laplace-Beltrami operator Δ on X by the formula:

$$\Delta=d\delta+\delta d.$$

It is easy to see that the operator Δ is a non-negative, self-adjoint operator in the Hilbert space $W^p_0(X)$. Hence we find that the resolvent $(\Delta-\lambda I)^{-1}$ exists on the space $W^p_0(X)$ for all $\lambda<0$, and that the following commutative relations hold:

(i) $\Delta d=d\Delta$ on $\mathscr{D}(d)$; $\delta\Delta=\Delta\delta$ on $\mathscr{D}(\delta)$.

(ii) $(\Delta-\lambda I)^{-1}d\subset d(\Delta-\lambda I)^{-1}$ on $\mathscr{D}(d)$; $(\Delta-\lambda I)^{-1}\delta\subset\delta(\Delta-\lambda I)^{-1}$ on $\mathscr{D}(\delta)$.

Furthermore, by virtue of Rellich's theorem, it follows that the resolvent $(\Delta-\lambda I)^{-1}$ is completely continuous on the space $W^p_0(X)$, since the domain $\mathscr{D}(\Delta)$ is contained in the space $W^p_2(X)$. Therefore, the Hilbert-Schmidt theory tells us the following:

(iii) The eigenvalues of Δ form a countable set accumulating only at $+\infty$.

We can define the harmonic operator H and the Green operator G for Δ respectively by the following formulas:

$$H=\frac{1}{2\pi i}\int_{|\lambda|=\varepsilon}(\lambda I-\Delta)^{-1}d\lambda. \tag{4.1}$$

$$G=\frac{1}{2\pi i}\int_{\Gamma}\lambda^{-1}(\lambda I-\Delta)^{-1}d\lambda. \tag{4.2}$$

Here $\varepsilon>0$ is so small that all positive eigenvalues of Δ lie outside of the circle $|\lambda|=\varepsilon$ in the complex plane, and Γ is a contour which encloses all positive eigenvalues of Δ

in the complex plane. Then we have the following:

(iv) The operator H is the orthogonal projection onto the kernel $\mathrm{Ker}^p \Delta$ of Δ, and G is a bounded operator on $W_0^p(X)$.

(v) $GH = HG = 0$ on $W_0^p(X)$; $G\Delta \subset \Delta G$ on $\mathscr{D}(\Delta)$.

Furthermore we have the following Hodge-Kodaira decomposition theorem (cf. [CP], [D], [K]):

THEOREM 4.1 (Hodge-Kodaira). *$\Delta G + H = d\delta G + \delta d G + H = I$ on $W_0^p(X)$.*

REMARK 4.2. By the elliptic regularity theorem, we find that

$$\begin{aligned}\mathrm{Ker}^p \Delta &\equiv \{T \in W_0^p(X);\ \Delta T = 0 \text{ in } X\}\\ &= \{T \in \Omega^p(X);\ \Delta T = 0 \text{ in } X\}\\ &= \{T \in \Omega^p(X);\ dT = 0,\ \delta T = 0 \text{ in } X\}\\ &= \mathrm{Ker}^p(d+\delta)\,.\end{aligned}$$

5. The operator D. We let

$\Omega^p(X \setminus Y)$ = the space of p-currents on X which are smooth in $X \setminus Y$ and may have *jump* discontinuities at Y,

and

$$\Omega^e(X \setminus Y) = \bigoplus_i \Omega^{2i}(X \setminus Y)\,, \quad \Omega^o(X \setminus Y) = \bigoplus_i \Omega^{2i+1}(X \setminus Y)\,,$$

$$\Omega^e(Y) = \bigoplus_i \Omega^{2i}(Y)\,, \quad \Omega^o(Y) = \bigoplus_i \Omega^{2i+1}(Y)\,.$$

Now we can introduce a linear operator

$$D = \begin{pmatrix} (d+\delta) & -(\cdot \otimes \delta_Y) \\ \iota^* & 0 \end{pmatrix} : \begin{matrix} \Omega^e(X \setminus Y) \\ \oplus \\ \Omega^o(Y) \end{matrix} \longrightarrow \begin{matrix} \Omega^o(X \setminus Y) \\ \oplus \\ \Omega^e(Y) \end{matrix}$$

as follows:

(a) The domain $\mathscr{D}(D)$ of D is the space

$$\mathscr{D}(D) = \left\{ \begin{pmatrix} \alpha \\ S \end{pmatrix};\ \alpha \in \Omega^e(X \setminus Y),\ S \in \Omega^o(Y),\ d\alpha \in \Omega^o(X \setminus Y),\ \delta\alpha - (S \otimes \delta_Y) \in \Omega^o(X \setminus Y) \right\}.$$

(b) $$D\begin{pmatrix} \alpha \\ S \end{pmatrix} = \begin{pmatrix} (d+\delta)\alpha - (S \otimes \delta_Y) \\ \iota^* \alpha \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ S \end{pmatrix} \in \mathscr{D}(D)\,.$$

Here $d\alpha$ and $\delta\alpha$ are taken in the sense of currents.

Near Y, we introduce coordinates (x', a) such that $x' = (x^1, \dots, x^{n-1})$ give local coordinates for Y and that $Y=\{(x', a);\ a=0\}$. We further normalize the coordinates by assuming the curves $x(a)=(x'_0, a)$, $x'_0\in Y$, are unit speed geodesics perpendicular to Y for $|a|$ sufficiently small.

If $\alpha\in\Omega^p(X)$, then we can write, near Y,

$$\alpha=\sum_{1\le i_1<\cdots<i_p\le n-1}\alpha_{i_1\ldots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

$$+\sum_{1\le i_1<\cdots<i_{p-1}\le n-1}\alpha_{i_1\ldots i_{p-1}n}dx^{i_1}\wedge\cdots\wedge dx^{i_{p-1}}\wedge da=\alpha'+\alpha''\wedge da\,,$$

where

$$\alpha'\in\Omega^p(Y)\,,\quad \alpha''\in\Omega^{p-1}(Y)\,.$$

We call α' (resp. α'') the tangential part (resp. the normal part) of α.

If $\alpha=\alpha'+\alpha''\wedge da\in\Omega^{\cdot}(X\setminus Y)$, then we have

$$d\alpha=d\alpha'+d'\alpha''\wedge da\,.$$

It is easy to see that:

(5.1) $\quad d\alpha\in\Omega^{\cdot}(X\setminus Y)\Leftrightarrow d\alpha'\in\Omega^{\cdot}(X\setminus Y)$

$\Leftrightarrow$ The tangential part α' of α does not have any jump discontinuity at Y.

Thus we can define the pull-back $\iota^*\alpha=\iota^*\alpha'$ as an element of $\Omega^{\cdot}(Y)$, that is,

$$\iota^*\alpha=\iota^*\alpha'\in\Omega^{\cdot}(Y)\qquad\text{if}\quad d\alpha\in\Omega^{\cdot}(X\setminus Y)\,.$$

We remark that

$$\delta\alpha'\in\Omega^{\cdot}(X\setminus Y)\,,$$

while the term $\delta(\alpha''\wedge da)$ may be equal to "delta functions", since we have in local coordinates

$$\delta(\alpha''\wedge da)=-\sum g^{ml}\frac{\partial}{\partial x^m}(\alpha_{li_1\ldots i_{p-2}n})dx^{i_1}\wedge\cdots\wedge dx^{i_{p-2}}\wedge da\,.$$

Hence the condition that

$$\delta\alpha-(S\otimes\delta_Y)\in\Omega^{\cdot}(X\setminus Y)$$

makes sense.

The next proposition characterizes the adjoint operator D^* of the operator D:

PROPOSITION 5.1. *The adjoint D^* of D is the operator*

$$D^* = \begin{pmatrix} (d+\delta) & (\cdot \otimes \delta_Y) \\ -\iota^* & 0 \end{pmatrix} : \begin{matrix} \Omega^o(X \setminus Y) \\ \oplus \\ \Omega^e(Y) \end{matrix} \longrightarrow \begin{matrix} \Omega^e(X \setminus Y) \\ \oplus \\ \Omega^o(Y) \end{matrix}$$

given by the following:

(c) *The domain* $\mathscr{D}(D^*)$ *of* D^* *is the space*

$$\mathscr{D}(D^*) = \left\{ \begin{pmatrix} \beta \\ T \end{pmatrix} ; \beta \in \Omega^o(X \setminus Y), T \in \Omega^e(Y), d\beta \in \Omega^e(X \setminus Y), \delta\beta + (T \otimes \delta_Y) \in \Omega^e(X \setminus Y) \right\}.$$

(d) $$D^* \begin{pmatrix} \beta \\ T \end{pmatrix} = \begin{pmatrix} (d+\delta)\beta + (T \otimes \delta_Y) \\ -\iota^*\beta \end{pmatrix}, \quad \begin{pmatrix} \beta \\ T \end{pmatrix} \in \mathscr{D}(D^*).$$

PROOF. (i) If $\beta \in \Omega^o(X \setminus Y)$ and $T \in \Omega^e(Y)$ such that

$$\begin{cases} d\beta \in \Omega^e(X \setminus Y), \\ \delta\beta + (T \otimes \delta_Y) \in \Omega^e(X \setminus Y), \end{cases}$$

then we have for all $\begin{pmatrix} \alpha \\ S \end{pmatrix} \in \mathscr{D}(D)$

$$\begin{aligned} \left\langle D\begin{pmatrix} \alpha \\ S \end{pmatrix}, \begin{pmatrix} \beta \\ T \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} d\alpha + \delta\alpha - (S \otimes \delta_Y) \\ \iota^*\alpha \end{pmatrix}, \begin{pmatrix} \beta \\ T \end{pmatrix} \right\rangle \\ &= (d\alpha + \delta\alpha - (S \otimes \delta_Y), \beta) + (\iota^*\alpha, T) \\ &= (d\alpha + \delta\alpha, \beta) - (S, \iota^*\beta) + (\iota^*\alpha, T) \\ &= (\alpha, \delta\beta + d\beta) + (\alpha, T \otimes \delta_Y) - (S, \iota^*\beta) \\ &= \left\langle \begin{pmatrix} \alpha \\ S \end{pmatrix}, \begin{pmatrix} d\beta + \delta\beta + (T \otimes \delta_Y) \\ -\iota^*\beta \end{pmatrix} \right\rangle. \end{aligned}$$

This proves that

$$\begin{pmatrix} \beta \\ T \end{pmatrix} \in \mathscr{D}(D^*),$$

and that

$$D^* \begin{pmatrix} \beta \\ T \end{pmatrix} = \begin{pmatrix} (d+\delta)\beta + (T \otimes \delta_Y) \\ -\iota^*\beta \end{pmatrix}.$$

(ii) Conversely, assume that $\beta \in \Omega^o(X \setminus Y)$ and $T \in \Omega^e(Y)$ belong to the domain $\mathscr{D}(D^*)$, that is,

there exist $\gamma \in \Omega^e(X \setminus Y)$ and $\eta \in \Omega^o(Y)$ such that for all $\begin{pmatrix} \alpha \\ S \end{pmatrix} \in \mathscr{D}(D)$ we have

$$\left\langle D\begin{pmatrix}\alpha\\ S\end{pmatrix}, \begin{pmatrix}\beta\\ T\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}\alpha\\ S\end{pmatrix}, \begin{pmatrix}\gamma\\ \eta\end{pmatrix}\right\rangle,$$

or equivalently,

$$(d\alpha+\delta\alpha, \beta)-(S\otimes\delta_Y, \beta)+(\imath^*\alpha, T)=(\alpha, \gamma)+(S, \eta).$$

Then, taking

$$\begin{cases} S=0, \\ \alpha\in\Omega^e(X), \end{cases}$$

we have for all $\alpha\in\Omega^e(X)$

$$(\alpha, \gamma)=(d\alpha+\delta\alpha, \beta)+(\imath^*\alpha, T)=(\alpha, \delta\beta+d\beta)+(\alpha, T\otimes\delta_Y),$$

so that

$$d\beta+\delta\beta+(T\otimes\delta_Y)=\gamma\in\Omega^e(X\setminus Y).$$

This gives that for all $S\in\Omega^o(Y)$

$$\begin{aligned}(S\otimes\delta_Y, \beta)&+(\alpha, (d+\delta)\beta+(T\otimes\delta_Y))=(S\otimes\delta_Y, \beta)+(\alpha, \gamma)\\ &=((d+\delta)\alpha, \beta)+(\imath^*\alpha, T)-(S, \eta)=(\alpha, (d+\delta)\beta+(T\otimes\delta_Y))-(S, \eta),\end{aligned}$$

so that

$$(S\otimes\delta_Y, \beta)=-(S, \eta).$$

This proves that

$$\imath^*\beta=-\eta\in\Omega^o(Y).$$

In other words, the tangential part β' of β does *not* have any jump discontinuity at Y. In view of assertion (5.1), it follows that

$$d\beta\in\Omega^e(X\setminus Y).$$

Therefore, we find that

$$\delta\beta+(T\otimes\delta_Y)=\gamma-d\beta\in\Omega^e(X\setminus Y).$$

This completes the proof of Proposition 5.1. ■

The next proposition characterizes the kernel $\mathrm{Ker}\,D$ of the operator D componentwise:

PROPOSITION 5.2. *An element*

$$\begin{pmatrix} \alpha \\ S \end{pmatrix} \in \begin{matrix} \Omega^e(X \setminus Y) \\ \oplus \\ \Omega^o(Y) \end{matrix}$$

belongs to the kernel of the operator D if and only if it satisfies the following conditions:

$$d\alpha_{2i}=0\,, \quad \alpha_{2i}|_Y=0\,, \quad 0 \leq i \leq [n/2]\,,$$

$$\delta\alpha_{2j+2}-(S_{2j+1}\otimes\delta_Y)=0\,, \quad 0\leq j\leq [n/2]\,.$$

Here

$$\alpha=\begin{pmatrix} \alpha_0 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{2k-2} \\ \alpha_{2k} \end{pmatrix}, \quad S=\begin{pmatrix} S_1 \\ S_3 \\ \cdot \\ \cdot \\ \cdot \\ S_{2k-1} \\ S_{2k+1} \end{pmatrix}, \quad k=\left[\frac{n}{2}\right].$$

PROOF. (i) The "only if" part: First we remark that

$$D\begin{pmatrix} \alpha \\ S \end{pmatrix}=0 \Leftrightarrow \begin{cases} \alpha_0|_Y=0, \dots, \alpha_{2k}|_Y=0\,, \\ d\alpha_0+\delta\alpha_2-(S_1\otimes\delta_Y)=0\,, \\ \cdot \\ \cdot \\ \cdot \\ d\alpha_{2k-2}+\delta\alpha_{2k}-(S_{2k-1}\otimes\delta_Y)=0\,, \\ d\alpha_{2k}-(S_{2k+1}\otimes\delta_Y)=0\,. \end{cases}$$

Hence we have

$$d\alpha_{2i}|_Y=0\,,$$
$$d\alpha_{2j}+\delta\alpha_{2j+2}-(S_{2j+1}\otimes\delta_Y)=0\,,$$
$$d\alpha_{2i}\in\Omega^{2i+1}(X\setminus Y)\subset W_0^{2i+1}(X)\,,$$
$$\alpha_{2j+2}\in\Omega^{2j+2}(X\setminus Y)\subset W_0^{2j+2}(X)\,,$$
$$S_{2j+1}\in\Omega^{2j+1}(Y)\,.$$

In view of Theorem 3.3, this implies that $\alpha_{2j+2}\in\mathscr{D}(\bar{d}^*)$, and

$$\bar{d}^*\alpha_{2j+2}=\delta\alpha_{2j+2}-(S_{2j+1}\otimes\delta_Y)=-d\alpha_{2j}\,. \tag{5.2}$$

Furthermore, by virtue of Theorem 3.2, it follows that

$$\begin{cases} d\alpha_{2j} \in \mathscr{D}(\bar{d}), \\ \bar{d}(d\alpha_{2j}) = d(d\alpha_{2j}) = 0, \end{cases}$$

since $d\alpha_{2j}|_Y = d'(\alpha_{2j}|_Y) = 0$. Therefore, we find that

$$\bar{d}(\bar{d}^* \alpha_{2j+2}) = -\bar{d}(d\alpha_{2j}) = 0 .$$

This implies that

$$(\bar{d}^* \alpha_{2j+2}, \bar{d}^* \alpha_{2j+2}) = (\alpha_{2j+2}, \bar{d}\bar{d}^* \alpha_{2j+2}) = 0 ,$$

so that $\bar{d}^* \alpha_{2j+2} = 0$. Hence we have by Formula (5.2)

$$\delta \alpha_{2j+2} - (S_{2j+1} \otimes \delta_Y) = 0 ,$$

and also $d\alpha_{2j} = 0$.

(ii) The "if" part is trivial. ■

The next theorem is an immediate consequence of Proposition 5.2:

THEOREM 5.3. $\operatorname{Ker} D = \bigoplus_{i=0}^{[n/2]} \operatorname{Ker}^{2i} D$, *where*

$$\operatorname{Ker}^{2i} D = \left\{ \begin{pmatrix} \alpha \\ S \end{pmatrix} ; \alpha \in \Omega^{2i}(X \setminus Y), S \in \Omega^{2i-1}(Y), d\alpha = 0, \alpha|_Y = 0, \delta\alpha - (S \otimes \delta_Y) = 0 \right\} .$$

Similarly, by Proposition 5.1, we can characterize the kernel $\operatorname{Ker} D^*$ of the operator D^* componentwise:

THEOREM 5.4. $\operatorname{Ker} D^* = \bigoplus_{i=0}^{[n/2]} \operatorname{Ker}^{2i+1} D$, *where*

$$\operatorname{Ker}^{2i+1} D^* = \left\{ \begin{pmatrix} \beta \\ T \end{pmatrix} ; \beta \in \Omega^{2i+1}(X \setminus Y), T \in \Omega^{2i}(Y), d\beta = 0, \beta|_Y = 0, \delta\beta + (T \otimes \delta_Y) = 0 \right\} .$$

6. The long exact sequence and the operator D. We let

$$P\varphi = G(\varphi \otimes \delta_Y)|_Y , \quad \varphi \in \Omega^p(Y) , \tag{6.1}$$

where G is the Green operator for the Laplacian Δ defined by Formula (4.2). It is known (cf. [H2], [S1], [T]) that G is an *elliptic* pseudo-differential operator of order -2 on X. Then we have the following (cf. [F, Proposition 7.6]):

THEOREM 6.1. *The operator P is an elliptic pseudo-differential operator of order -1 on Y, and it extends to an isomorphism*

$$P : W_0^p(Y) \to W_1^p(Y) .$$

PROOF. Let x_0 be an arbitrary point of Y. We remark that

$$T_{x_0}^*(X) = T_{x_0}^*(Y) \oplus N_{x_0}^*(Y) .$$

Thus we can decompose each covector $(x_0, \xi) \in T^*_{x_0}(X)$ as follows:

$$(x_0, \xi) = (x_0, \xi') \oplus (x_0, \eta) .$$

Then the principal symbol of G is equal to:

$$(|\xi'|^2 + \eta^2)^{-1} .$$

Hence we find (cf. [H2], [S1], [T]) that the principal symbol of P is given by the following:

$$-\frac{1}{2\pi}\int_{\boldsymbol{R}} \frac{d\eta}{|\xi'|^2+\eta^2} = \left(-\frac{1}{2\pi}\int_{\boldsymbol{R}} \frac{d\zeta}{1+\zeta^2}\right)\cdot |\xi'|^{-1} = \frac{1}{2}|\xi'|^{-1} .$$

This proves that P is an elliptic pseudo-differential operator of order -1 on Y.

We prove that $P\colon W^p_0(Y) \to W^p_1(Y)$ is an isomorphism. To do so, since the principal symbol of P is *real*, it suffices to show (cf. [P, Chapter XI, Theorem 12]) that P is injective, that is,

$$\varphi \in \Omega^p(Y) \quad \text{and} \quad P\varphi = 0 \Rightarrow \varphi = 0 .$$

We let

$$\Phi = G^{1/2}(\varphi \otimes \delta_Y) ,$$

where (cf. Formula (4.2))

$$G^{1/2} = \frac{1}{2\pi i}\int_{\Gamma} \lambda^{-1/2}(\lambda I - \Delta)^{-1} d\lambda .$$

We know (cf. [S2], [T]) that the operator $G^{1/2}$ is an elliptic pseudo-differential operator of order -1 on X. Then we have

$$\begin{aligned}(6.2)\quad \int_Y P\varphi \wedge *'\varphi &= \int_Y (G(\varphi\otimes\delta_Y))|_Y \wedge *'\varphi = \int_X G(\varphi\otimes\delta_Y)\wedge *(\varphi\otimes\delta_Y) \\ &= \int_X G^{1/2}(\varphi\otimes\delta_Y)\wedge G^{1/2}*(\varphi\otimes\delta_Y) = \int_X G^{1/2}(\varphi\otimes\delta_Y)\wedge *G^{1/2}(\varphi\otimes\delta_Y) \\ &= \int_X \Phi \wedge *\Phi ,\end{aligned}$$

since $*\Delta = \Delta *$ and so $*G^{1/2} = G^{1/2}*$. Therefore, it follows from Formula (6.2) that

$$\begin{aligned} P\varphi = 0 &\Rightarrow \Phi = G^{1/2}(\varphi\otimes\delta_Y) = 0 \\ &\Rightarrow G(\varphi\otimes\delta_Y) = G^{1/2}\Phi = 0 .\end{aligned}$$

Hence we have by Theorem 4.1 and Remark 4.2

$$\varphi\otimes\delta_Y = H(\varphi\otimes\delta_Y) + \Delta G(\varphi\otimes\delta_Y) = H(\varphi\otimes\delta_Y) \in \Omega^p(X) .$$

However, this happens only when $\varphi = 0$. The proof of Theorem 6.1 is complete. ∎

Since the inverse P^{-1} is a positive, elliptic pseudo-differential operator of order 1 on Y, it follows (cf. [S2], [T]) that the operator $P^{-1/2}$ is an elliptic pseudo-differential operator of order 1/2 on Y.

We equip the space $W^p_{1/2}(Y)$ with the inner product

$$\langle \varphi, \psi \rangle = (P^{-1/2}\varphi, P^{-1/2}\psi) = \int_Y P^{-1/2}\varphi \wedge *'(P^{-1/2}\psi) .$$

By Theorem 6.1, it is easy to see that the space $W^p_{1/2}(Y)$ is a Hilbert space with respect to this inner product $\langle \cdot, \cdot \rangle$. We let

$d'_1 =$ the minimal closed extension in $W^p_{1/2}(Y)$ of the operator d' restricted to the space $\Omega^p(Y)$,

and

$$\delta'_1 = \text{the adjoint of the operator } d'_1 : W^p_{1/2}(Y) \to W^{p+1}_{1/2}(Y) .$$

Then we have the following relationship between the adjoint δ' of d' and the adjoint δ'_1 of d'_1 (cf. [F], Proposition 8.1):

LEMMA 6.2. $\delta'_1 = P\delta' P^{-1}$.

We introduce a generalized Laplacian L' on Y by the formula:

$$L' = d'_1\delta'_1 + \delta'_1 d'_1 .$$

Then the operator L' is a non-negative, self-adjoint operator in the Hilbert space $W^p_{1/2}(Y)$. It is easy to see that the Hodge-Kodaira theory extends to the operators d'_1, δ'_1 and L'. More precisely, we have the following:

(i) The eigenvalues of L' form a countable set accumulating only at $+\infty$.

(ii) We can define the harmonic operator H' and the Green operator G' for L' respectively by the following formulas:

$$H' = \frac{1}{2\pi i}\int_{|\lambda| = \varepsilon} (\lambda I - L')^{-1} d\lambda .$$

$$G' = \frac{1}{2\pi i}\int_{\Gamma} \lambda^{-1}(\lambda I - L')^{-1} d\lambda .$$

Here $\varepsilon > 0$ is so small that all positive eigenvalues of L' lie outside of the circle $|\lambda| = \varepsilon$ in the complex plane, and Γ is a contour which encloses all positive eigenvalues of L' in the complex plane.

We have the following (cf. [F, Theorem 8.4]):

(ii-a) The operator H' is the orthogonal projection onto the kernel $\mathrm{Ker}^p L'$ of L', where (cf. Remark 4.2)

$$\begin{aligned}\mathrm{Ker}^p L' &\equiv \{S \in W^p_{1/2}(Y);\ L'S = 0 \text{ in } Y\}\\ &= \{S \in \Omega^p(Y);\ L'S = 0 \text{ in } Y\}\\ &= \{S \in \Omega^p(Y);\ d'S = 0,\ \delta'_1 S = 0 \text{ in } Y\}\\ &= \mathrm{Ker}^p(d' + \delta'_1),\end{aligned}$$

and the operator G' is a bounded operator on $W^p_{1/2}(Y)$.

(ii-b) $G'H' = H'G' = 0$ on $W^p_{1/2}(Y)$; $G'L' \subset L'G'$ on $\mathscr{D}(L')$.

(ii-c) $L'G' + H' = d'_1\delta'_1 G' + \delta'_1 d'_1 G' + H' = I$ on $W^p_{1/2}(Y)$.

Now we can introduce six mappings ρ_e, ρ'_e, ρ''_e, ρ_o, ρ'_o and ρ''_o as follows:

(I) $$\rho_e\colon \mathrm{Ker}^{2i} D \to \mathrm{Ker}^{2i}(d+\delta), \quad \begin{pmatrix}\alpha\\ S\end{pmatrix} \mapsto H\alpha .$$

Here H is the orthogonal projection on the space $\mathrm{Ker}^{2i}\Delta = \mathrm{Ker}^{2i}(d+\delta)$.

(II) $$\rho'_e\colon \mathrm{Ker}^{2i}(d+\delta) \to \mathrm{Ker}^{2i}(d'+\delta'_1), \quad \alpha \mapsto H'(\alpha|_Y) .$$

Here $\delta'_1 = P\delta' P^{-1}$ and H' is the orthogonal projection on the space $\mathrm{Ker}^{2i} L' = \mathrm{Ker}^{2i}(d'+\delta'_1)$.

(III) $$\rho''_e\colon \mathrm{Ker}^{2i}(d'+\delta'_1) \to \mathrm{Ker}^{2i+1} D^*, \quad T \mapsto \begin{pmatrix} dG(P^{-1}J_e T \otimes \delta_Y)\\ -P^{-1}J_e T\end{pmatrix}.$$

Here J_e is the orthogonal projection onto the orthogonal complement $(\mathrm{Im}\,\rho')_e^{\perp}$ of $\mathrm{Im}\,\rho'_e$ in the space $\mathrm{Ker}^{2i}(d'+\delta'_1)$.

(IV) $$\rho_o\colon \mathrm{Ker}^{2i+1} D^* \to \mathrm{Ker}^{2i+1}(d+\delta), \quad \begin{pmatrix}\beta\\ T\end{pmatrix} \mapsto H\beta .$$

Here H is the orthogonal projection on the space $\mathrm{Ker}^{2i+1}\Delta = \mathrm{Ker}^{2i+1}(d+\delta)$.

(V) $$\rho'_o\colon \mathrm{Ker}^{2i+1}(d+\delta) \to \mathrm{Ker}^{2i+1}(d'+\delta'_1), \quad \beta \mapsto H'(\beta|_Y) .$$

Here H' is the orthogonal projection on the space $\mathrm{Ker}^{2i+1} L' = \mathrm{Ker}^{2i+1}(d'+\delta'_1)$.

(VI) $$\rho''_o\colon \mathrm{Ker}^{2i+1}(d'+\delta'_1) \to \mathrm{Ker}^{2i+2} D, \quad T \mapsto \begin{pmatrix} dG(P^{-1}J_o T \otimes \delta_Y)\\ P^{-1}J_o T\end{pmatrix}.$$

Here J_o is the orthogonal projection onto the orthogonal complement $(\mathrm{Im}\,\rho')_o^{\perp}$ of $\mathrm{Im}\,\rho'_o$ in the space $\mathrm{Ker}^{2i+1}(d'+\delta'_1)$.

The next theorem is the essential step in the proof of Theorem 2 (cf. [F, Theorem 8.6]):

THEOREM 6.3. *The following sequence of homomorphisms forms a complex, and is exact.*

$$
(*)\quad
\begin{array}{llllll}
0 \longrightarrow & \operatorname{Ker}^0 D & \xrightarrow{\rho_e} & \operatorname{Ker}^0(d+\delta) & \xrightarrow{\rho'_e} & \operatorname{Ker}^0(d'+\delta'_1) \\
\xrightarrow{\rho''_e} & \operatorname{Ker}^1 D^* & \xrightarrow{\rho_o} & \operatorname{Ker}^1(d+\delta) & \xrightarrow{\rho'_o} & \operatorname{Ker}^1(d'+\delta'_1) \\
\xrightarrow{\rho''_o} & \operatorname{Ker}^2 D & \xrightarrow{\rho_e} & \operatorname{Ker}^2(d+\delta) & \xrightarrow{\rho'_e} & \operatorname{Ker}^2(d'+\delta'_1) \\
 & \vdots & & \vdots & & \vdots \\
\xrightarrow{\rho''_o} & \operatorname{Ker}^{2i} D & \xrightarrow{\rho_e} & \operatorname{Ker}^{2i}(d+\delta) & \xrightarrow{\rho'_e} & \operatorname{Ker}^{2i}(d'+\delta'_1) \\
\xrightarrow{\rho''_e} & \operatorname{Ker}^{2i+1} D^* & \xrightarrow{\rho_o} & \operatorname{Ker}^{2i+1}(d+\delta) & \xrightarrow{\rho'_o} & \operatorname{Ker}^{2i+1}(d'+\delta'_1) \\
 & \vdots & & \vdots & & \vdots
\end{array}
$$

Assuming this theorem for the moment, we shall prove Theorem 2. It follows from an application of the Hodge-Kodaira theorem that

$$
\begin{aligned}
&\operatorname{Ker}^j(d+\delta) \cong H^j(X) \cong H^j(X, \boldsymbol{R}), \\
&\operatorname{Ker}^j(d'+\delta'_1) \cong H^j(Y) \cong H^j(Y, \boldsymbol{R}).
\end{aligned}
$$

Therefore, by virtue of the five lemma, the long exact sequence $(*)$ implies that

$$
\operatorname{Ker}^{2i} D \cong H^{2i}(X, Y, \boldsymbol{R}), \quad \operatorname{Ker}^{2i+1} D^* \cong H^{2i+1}(X, Y, \boldsymbol{R}).
$$

Hence we have by Theorems 5.3 and 5.4

$$
\begin{aligned}
\operatorname{ind} D &= \dim \operatorname{Ker} D - \dim \operatorname{Ker} D^* \\
&= \sum_{i=0}^{[n/2]} \dim \operatorname{Ker}^{2i} D - \sum_{i=0}^{[n/2]} \dim \operatorname{Ker}^{2i+1} D^* \\
&= \sum_{i=0}^{[n/2]} \dim H^{2i}(X, Y, \boldsymbol{R}) - \sum_{i=0}^{[n/2]} \dim H^{2i+1}(X, Y, \boldsymbol{R}) \\
&= \sum_{i=0}^{n} (-1)^i \dim H^i(X, Y, \boldsymbol{R}) \\
&= \chi(X, Y) \\
&= \chi(X) - \chi(Y).
\end{aligned}
$$

7. Proof of Theorem 6.3. (I) Now we define a mapping

$$\rho : \operatorname{Ker} D \to \operatorname{Ker}(d+\delta), \quad \begin{pmatrix} \alpha \\ S \end{pmatrix} \mapsto H\alpha ,$$

and a mapping

$$\rho' : \operatorname{Ker}(d+\delta) \to \operatorname{Ker}(d'+\delta'_1), \quad \alpha \mapsto H'(\alpha|_Y) .$$

Throughout this section we drop the $2i$, $2i+1$ and use $\operatorname{Ker} D$, $\operatorname{Ker}(d+\delta)$ and $\operatorname{Ker}(d+\delta'_1)$, respectively. Then we have the following:

LEMMA 7.1. $\operatorname{Im} \rho = \operatorname{Ker} \rho'$.

PROOF. (1) Let $\begin{pmatrix} \alpha \\ S \end{pmatrix}$ be an arbitrary element of the space $\operatorname{Ker} D$, that is,

$$\begin{cases} d\alpha = 0 , \\ \alpha|_Y = \imath^* \alpha = 0 , \\ \delta\alpha - (S \otimes \delta_Y) = 0 . \end{cases}$$

Then we have

$$\alpha = H\alpha + G\Delta\alpha = H\alpha + G(d\delta\alpha + \delta d\alpha) = H\alpha + Gd(S \otimes \delta_Y) = H\alpha + dG(S \otimes \delta_Y) .$$

This gives that

$$H\alpha|_Y = (\alpha - dG(S \otimes \delta_Y))\,|_Y = -d'PS .$$

Hence we have

$$\rho'\left(\rho\begin{pmatrix} \alpha \\ S \end{pmatrix}\right) = H'(H\alpha|_Y) = -H'd'PS = 0 ,$$

since $H'd' = 0$. This proves that $\operatorname{Im} \rho \subset \operatorname{Ker} \rho'$.

(2) Conversely, assume that $\alpha \in \operatorname{Ker} \rho'$, that is,

$$\begin{cases} d\alpha = 0 , \\ \delta\alpha = 0 , \\ H'(\alpha|_Y) = 0 . \end{cases}$$

We recall that

$$d'\delta'_1 G' + \delta'_1 d' G' + H' = I .$$

Then it follows that

$$\begin{aligned} (7.1) \qquad \alpha|_Y &= d'\delta'_1 G'(\alpha|_Y) + \delta'_1 d' G'(\alpha|_Y) \\ &= d'\delta'_1 G'(\alpha|_Y) + \delta'_1 G' d'(\alpha|_Y) = d'\delta'_1 G'(\alpha|_Y) , \end{aligned}$$

since $d'(\alpha|_Y)=d\alpha|_Y=0$. If we let

$$\begin{cases} S=-P^{-1}\delta'_1 G'(\alpha|_Y)=-\delta' P^{-1}G'(\alpha|_Y)\,, \\ \beta=\alpha+dG(S\otimes\delta_Y)\,, \end{cases} \tag{7.2}$$

then we have by Formula (7.1)

$$\begin{cases} d\beta=d\alpha=0\,, \\ \beta|_Y=\alpha|_Y+d'PS=\alpha|_Y-d'\delta'_1 G'(\alpha|_Y)=0\,. \end{cases}$$

Furthermore, since we have

$$\delta' S=-\delta'\delta' P^{-1}G'(\alpha|_Y)=0\,,$$

it follows that

$$\begin{aligned} \delta\beta &= \delta dG(S\otimes\delta_Y)=(\Delta-d\delta)G(S\otimes\delta_Y) \\ &=(I-H)(S\otimes\delta_Y)-d\delta G(S\otimes\delta_Y) \\ &=(S\otimes\delta_Y)-H(S\otimes\delta_Y)-dG(\delta' S\otimes\delta_Y) \\ &=(S\otimes\delta_Y)-H(S\otimes\delta_Y)\,. \end{aligned}$$

By Theorem 3.3, this implies that

$$\begin{cases} \beta\in\mathcal{D}(\bar{d}^*)\,, \\ \bar{d}^*\beta=\delta\beta-(S\otimes\delta_Y)=-H(S\otimes\delta_Y)\,. \end{cases}$$

However, we have the following:

CLAIM 1. $H(S\otimes\delta_Y)=0$, *or equivalently*, $\delta\beta-(S\otimes\delta_Y)=0$.

PROOF. If $\{h_1,\dots,h_N\}$ is an orthonormal basis of the space $\mathrm{Ker}(d+\delta)$, then we have by Formula (7.2)

$$\begin{aligned} H(S\otimes\delta_Y)|_Y &= \sum_{j=1}^N \left(\int_X h_j\wedge *(S\otimes\delta_Y)\right)h_j|_Y \\ &= \sum_{j=1}^N \left(\int_Y h_j|_Y\wedge *'S\right)h_j|_Y \\ &= -\sum_{j=1}^N \left(\int_Y h_j|_Y\wedge *'(P^{-1}\delta'_1 G'(\alpha|_Y))\right)h_j|_Y \\ &= -\sum_{j=1}^N \langle h_j|_Y,\, \delta'_1 G'(\alpha|_Y)\rangle h_j|_Y = -\sum_{j=1}^N \langle d'(h_j|_Y),\, G'(\alpha|_Y)\rangle h_j|_Y \\ &= -\sum_{j=1}^N \langle dh_j|_Y,\, G'(\alpha|_Y)\rangle h_j|_Y=0\,, \end{aligned}$$

since $dh_j=0$. By Theorem 3.2, it follows that

$$\begin{cases} \bar{d}^*\beta=-H(S\otimes\delta_Y)\in\mathscr{D}(\bar{d}), \\ \bar{d}\bar{d}^*\beta=-dH(S\otimes\delta_Y)=0. \end{cases}$$

Hence we have

$$(H(S\otimes\delta_Y), H(S\otimes\delta_Y))=(\bar{d}^*\beta, \bar{d}^*\beta)=(\bar{d}\bar{d}^*\beta, \beta)=0.$$

This proves Claim 1. ■

Summing up, we have proved that

$$\begin{cases} d\beta=0, \\ \beta|_Y=0, \\ \delta\beta-(S\otimes\delta_Y)=0, \end{cases}$$

that is,

$$\begin{pmatrix} \beta \\ S \end{pmatrix}\in\operatorname{Ker} D,$$

and

$$\alpha=H\alpha=H\beta=\rho\begin{pmatrix} \beta \\ S \end{pmatrix}\in\operatorname{Im}\rho.$$

The proof of Lemma 7.1 is complete. ■

(II) We define

$$QS=H(S\otimes\delta_Y)|_Y,$$

and let

$$\pi=QP^{-1}.$$

Then we have the following characterization of $\operatorname{Im}\rho'$:

CLAIM 2. $\operatorname{Im}\rho'=\operatorname{Im}H'\circ\pi$.

PROOF. (i) $\operatorname{Im}H'\circ\pi\subset\operatorname{Im}\rho'$: This is trivial.

(ii) $\operatorname{Im}\rho'\subset\operatorname{Im}H'\circ\pi$: Let T be an arbitrary element of $\operatorname{Im}\rho'$, and assume that $T=\rho'(\alpha)$, $\alpha\in\operatorname{Ker}(d+\delta)$, that is,

$$T=H'(\alpha|_Y).$$

If $\{h_1,\dots,h_N\}$ is an orthonormal basis of the space $\operatorname{Ker}(d+\delta)$, then we have

$$H(S\otimes\delta_Y)=\sum_{j=1}^{N}\left(\int_X h_j\wedge *(S\otimes\delta_Y)\right)h_j=\sum_{j=1}^{N}\left(\int_Y h_j|_Y\wedge *'S\right)h_j,$$

so that

$$QS = H(S \otimes \delta_Y)|_Y = \sum_{j=1}^{N} \left(\int_Y h_j|_Y \wedge *'S \right) h_j|_Y .$$

This gives that

$$\pi S = QP^{-1}S = \sum_{j=1}^{N} \left(\int_Y h_j|_Y \wedge *'P^{-1}S \right) h_j|_Y = \sum_{j=1}^{N} \langle h_j|_Y, S \rangle h_j|_Y , \tag{7.3}$$

so that

$$H'(\pi S) = \sum_{j=1}^{N} \langle h_j|_Y, S \rangle H'(h_j|_Y) . \tag{7.4}$$

On the other hand, since we have

$$\alpha = H\alpha = \sum_{j=1}^{N} \left(\int_X h_j \wedge *\alpha \right) h_j ,$$

it follows that

$$\rho'(\alpha) = H'(\alpha|_Y) = \sum_{j=1}^{N} \left(\int_X h_j \wedge *\alpha \right) H'(h_j|_Y) .$$

However, we can find an element S_0 such that

$$\langle h_j|_Y, S_0 \rangle = \int_X h_j \wedge *\alpha , \quad 1 \leq j \leq N .$$

Hence we have

$$\rho'(\alpha) = \sum_{j=1}^{N} \langle h_j|_Y, S_0 \rangle H'(h_j|_Y) .$$

Therefore, combining this formula with Formula (7.4), we obtain that

$$T = \rho'(\alpha) = H'(\pi S_0) \in \operatorname{Im} H' \circ \pi .$$

■

REMARK 7.2. The operator π is *symmetric*, that is, we have

$$\langle \pi S, T \rangle = \langle S, \pi T \rangle .$$

Indeed, it follows from Formula (7.3) that

$$\langle \pi S, T \rangle = \sum_{j=1}^{N} \langle h_j|_Y, S \rangle \langle h_j|_Y, T \rangle = \langle S, \pi T \rangle .$$

(III) Now we define a linear mapping

$$\rho'' : \operatorname{Ker}(d'+\delta_1') \to \operatorname{Ker} D\,, \quad T \mapsto \begin{pmatrix} dG(P^{-1}JT \otimes \delta_Y) \\ P^{-1}JT \end{pmatrix}.$$

Here J is the orthogonal projection onto the orthogonal complement $(\operatorname{Im}\rho')^{\perp}$ of $\operatorname{Im}\rho'$ in the space $\operatorname{Ker}(d'+\delta_1')$.

(III-a) First we check the *well-definedness* of the mapping ρ'': If we let

$$\begin{cases} \alpha = dG(P^{-1}JT \otimes \delta_Y)\,, \\ S = P^{-1}JT\,, \end{cases}$$

then we have

$$\begin{cases} d\alpha = 0\,, \\ \alpha|_Y = d'P(P^{-1}JT) = d'JT = 0\,, \end{cases}$$

since $JT \in \operatorname{Ker}(d'+\delta_1')$. Further it follows that

$$\begin{aligned} \delta\alpha &= \delta dG(S\otimes\delta_Y) = (\Delta - d\delta)G(S\otimes\delta_Y) = (I - H - d\delta G)(S\otimes\delta_Y) \\ &= (S\otimes\delta_Y) - H(S\otimes\delta_Y) - d\delta G(S\otimes\delta_Y)\,. \end{aligned} \tag{7.5}$$

However, we have the following:

CLAIM 3. $H(S\otimes\delta_Y)=0$, $d\delta G(S\otimes\delta_Y)=0$.

PROOF. First we have

$$d\delta G(S\otimes\delta_Y)|_Y = dG\delta(S\otimes\delta_Y)|_Y = d'P\delta'S = d'(P\delta'P^{-1})JT = d'\delta_1'JT = 0\,, \tag{7.6}$$

since $JT \in \operatorname{Ker}(d'+\delta_1')$.

If $T = T_1 + T_2$ with $T_1 \in \operatorname{Im}\rho'$ and $T_2 \in (\operatorname{Im}\rho')^{\perp}$, then we have

$$H(S\otimes\delta_Y)|_Y = QS = QP^{-1}JT = QP^{-1}JT_2 = QP^{-1}T_2 = \pi T_2\,,$$

since $JT_1 = 0$ and $JT_2 = T_2$.

However, if $\{h_1, \dots, h_N\}$ is an orthonormal basis of the space $\operatorname{Ker}(d+\delta)$, then it follows from Formula (7.3) that

$$\pi T_2 = \sum_{j=1}^{N} \langle h_j|_Y, T_2\rangle h_j|_Y = \sum_{j=1}^{N} \langle h_j|_Y, H'(T_2)\rangle h_j|_Y = \sum_{j=1}^{N} \langle H'(h_j|_Y), T_2\rangle h_j|_Y = 0\,,$$

since $T_2 \in (\operatorname{Im}\rho')^{\perp} \subset \operatorname{Ker}(d'+\delta_1')$ and $H'(h_j|_Y) = \rho'(h_j) \in \operatorname{Im}\rho'$. Hence we have

$$H(S\otimes\delta_Y)|_Y = \pi T_2 = 0\,. \tag{7.7}$$

Thus, in view of Theorem 3.2, it follows from Assertions (7.6) and (7.7) that

$$H(S\otimes\delta_Y) + d\delta G(S\otimes\delta_Y) \in \mathscr{D}(\bar{d})\,.$$

Therefore, since we have by Formula (7.5)

$$\bar{d}^*\alpha=\delta\alpha-(S\otimes\delta_Y)=-H(S\otimes\delta_Y)-d\delta G(S\otimes\delta_Y)\in\mathscr{D}(\bar{d});$$

it follows that

$$(\bar{d}^*\alpha,\bar{d}^*\alpha)=(\bar{d}\bar{d}^*\alpha,\alpha)=0,$$

so that

$$0=\bar{d}^*\alpha=-H(S\otimes\delta_Y)-d\delta G(S\otimes\delta_Y).$$

This proves Claim 3, since $Hd=0$. ∎

By Claim 3, it follows from Formula (7.5) that $\delta\alpha-(S\otimes\delta_Y)=0$. Summing up, we have proved that

$$\begin{pmatrix}\alpha\\ S\end{pmatrix}\in\operatorname{Ker}D.$$

(III-b) Next we show the following:

LEMMA 7.3. $\operatorname{Im}\rho'=\operatorname{Ker}\rho''$.

PROOF. (1) $\operatorname{Ker}\rho''\subset\operatorname{Im}\rho'$: If $T\in\operatorname{Ker}(d'+\delta'_1)$ and

$$\rho''(T)=\begin{pmatrix}dG(P^{-1}JT\otimes\delta_Y)\\ P^{-1}JT\end{pmatrix}=0,$$

then we have $T\in\operatorname{Im}\rho'$, since $JT=0$.

(2) $\operatorname{Im}\rho'\subset\operatorname{Ker}\rho''$: This is trivial. ∎

(IV) Finally it remains to show the following:

LEMMA 7.4. $\operatorname{Im}\rho''=\operatorname{Ker}\rho$.

PROOF. (1) $\operatorname{Im}\rho''\subset\operatorname{Ker}\rho$: This is trivial, since $Hd=0$.

(2) $\operatorname{Ker}\rho\subset\operatorname{Im}\rho''$: If $\begin{pmatrix}\alpha\\ S\end{pmatrix}\in\operatorname{Ker}D$ and $\rho\begin{pmatrix}\alpha\\ S\end{pmatrix}=0$, then we have

$$\begin{cases}d\alpha=0,\\ \alpha|_Y=0,\\ \delta\alpha-(S\otimes\delta_Y)=0,\\ H\alpha=0.\end{cases}$$

Thus α can be written in the following form:

$$\alpha=G\Delta\alpha=Gd\delta\alpha=Gd(S\otimes\delta_Y)=dG(S\otimes\delta_Y).$$

If we let

$$T = PS\,,$$

then it follows that

$$d'T = dG(S \otimes \delta_Y)\big|_Y = \alpha\big|_Y = 0\,,$$

and from Lemmas 6.2 and 3.1 and also Formula (6.1) that

$$\delta_1' T = P\delta' S = G(\delta' S \otimes \delta_Y)\big|_Y = G\delta(S \otimes \delta_Y)\big|_Y = G\delta(\delta\alpha)\big|_Y = 0\,.$$

Hence we have $T \in \operatorname{Ker}(d' + \delta_1')$. However, we have $JT = T$, that is,

(7.8) $$T \in (\operatorname{Im} \rho')^{\perp}\,.$$

Indeed, since we have

$$\pi T = \pi PS = QS = H(S \otimes \delta_Y)\big|_Y = H(\delta\alpha)\big|_Y = 0\,,$$

we find from Remark 7.2 that for all $\varphi \in \Omega^{\bullet}(Y)$

$$\langle T, H'\pi\varphi \rangle = \langle H'T, \pi\varphi \rangle = \langle T, \pi\varphi \rangle = \langle \pi T, \varphi \rangle = 0\,,$$

so that by Claim 2

$$T \perp \operatorname{Im} H' \circ \pi = \operatorname{Im} \rho'\,.$$

This proves assertion (7.8).

In view of assertion (7.8), it follows that

$$P^{-1}JT = P^{-1}T = S\,.$$

Hence we have

$$\begin{pmatrix} \alpha \\ S \end{pmatrix} = \begin{pmatrix} dG(S \otimes \delta_Y) \\ S \end{pmatrix} = \begin{pmatrix} dG(P^{-1}JT \otimes \delta_Y) \\ P^{-1}JT \end{pmatrix} = \rho''(T) \in \operatorname{Im} \rho''\,.$$

This completes the proof of Lemma 7.4. ■

Now the proof of Theorem 6.3 and hence that of Theorem 2 is complete.

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