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WALSH FUNCTIONS AND UNIFORM DISTRIBUTION MOD 1

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Abstract. A Weyl criterion using Walsh functions is established and the relation between uniform distribution mod 1 and dyadic addition on the real line is investigated. Further, we develop the relationship between the modified integrals of the Walsh functions and uniform distribution mod 1: a "Walsh integrals" Weyl criterion is developed and analogues of the LeVeque inequality and the Erdös-Turán inequality are obtained. These bounds are easier to compute than the classical bounds.

1. Introduction. A major tool for the study of the uniform distribution mod 1 of sequences has been the classical Weyl criterion (see for example [1]). Uniform distribution mod 1 is defined as follows. For a real number x let $[x]$ denote the integer part of x and $I=[0, 1)$. Let $\omega=\{x_i\}$ be a sequence of real numbers, $N\in \boldsymbol{N}$ and $E\subset I$. Then, let the counting function $A(E; N, \omega)$ be the number of terms

$$0\le i\le N-1\,, \qquad x_i-[x_i]\in E\,.$$

The sequence of real numbers $\omega=\{x_i\}$, $i=0, 1, 2, \ldots$ is said to be uniformly distributed modulo 1 (abbreviated u.d. mod 1) if for every pair of real numbers with $0\le a\le b\le 1$ we have

$$\lim_{N\to\infty}\frac{A([a, b); N, \omega)}{N}=b-a\,.$$

A consequence of this definition is that $\{x_i\}$ is u.d. mod 1 if and only if, for any $f\in C^{\circ}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1} f(x_i)=\int_0^1 f(x)dx\,. \tag{1}$$

We use Walsh functions to derive a further necessary and sufficient condition (a Walsh function Weyl criterion) for a sequence to be u.d. mod 1.

Dyadic addition on the real line, $\dot{+}$, was first used by Fine [2] and has proven to be of fundamental importance when used in conjunction with Walsh function analysis. However there has been no investigation of how sequences of real numbers are uniformly distributed mod 1 when combined with this operation. One of the aims of this paper is to initiate such a study.

A comprehensive treatment of uniform distribution of sequences is given in [1].

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The representation theory of compact Abelian groups could be used following [1] to provide a proof of our Theorem 1. However the Walsh functions are not precisely the characters of the dyadic group (see [3] for a discussion of the method required to form these characters). So a proof of Theorem 1 following the classical Weyl method is more natural than the alternative (using group representation theory) mentioned above. Our theorems and Corollary all appear to be new results.

Section 2 of this paper briefly introduces the properties of the Walsh functions and dyadic addition and we introduce the use of Walsh functions in the study of uniformly distributed mod 1 sequences. In Section 3, we define the modified integrals of the Walsh functions and obtain another Weyl criterion using these Walsh integrals. Further, analogues of both LeVeque's inequality and the Erdös-Turán inequality are derived. These inequalities are of considerable interest in view of their ease of computation.

2. Walsh functions and a Weyl criterion. Since their introduction in [4], the Walsh functions have been studied extensively. Following Paley [5], [6] we define the Walsh functions in terms of the Rademacher functions $\{r_n\}_{n=0}^{\infty}$. The Rademacher functions were published in [7] and they form an incomplete (with respect to L^2) set of orthogonal functions. The Rademacher functions are defined as follows:

$$r_n: \boldsymbol{R} \to \boldsymbol{R},\ r_n(x+1) = r_n(x)\,, \qquad \text{for} \quad n \in \boldsymbol{N} \cup \{0\}$$

$$r_0(x) = \begin{cases} 1 & \text{for} \quad x \in [0, 1/2) \\ -1 & \text{for} \quad x \in [1/2, 1) \end{cases}$$

$$r_n(x) = r_0(2^n x)\,, \qquad \text{for} \quad n \geq 1\,.$$

The Walsh functions are complete with respect to L^2 and are defined as follows:

$$W_0(x) = 1\,, \qquad W_n(x) = \prod_{i=1}^{r} r_{n_i}(x)\,,$$

where

$$n = \prod_{i=1}^{r} 2^{n_i}\,, \qquad n_{i+1} > n_i$$

is the unique decomposition of n into the sum of strictly increasing powers of 2.

The Walsh functions can be considered essentially as the characters of the dyadic group, G. Let $\bar{x} \in G$ so that $\bar{x} = \{x_1, x_2, \dots\}$, $x_n \in \{0, 1\}$ with group operation addition termwise modulo 2. Fine [2] defined the function $\lambda: G \to [0, 1]$ where

$$\lambda(\bar{x}) = \sum_{n=1}^{\infty} 2^{-n} x_n\,.$$

Clearly λ does not have a single-valued inverse because for each finite expansion

$$\sum_{n=1}^{m} 2^{-n}x_n, \qquad x_n \in \{0, 1\},$$

there is a corresponding infinite expansion. In such cases the finite expansion is taken, and so for all real $x \in [0, 1)$, writing the inverse as μ,

$$\lambda(\mu(x)) = x - [x] \quad \text{and} \quad \mu(\lambda(\bar{x})) = \bar{x},$$

provided $\lambda(\bar{x})$ is a dyadic irrational.

Letting $\overset{*}{+}$ be the operation in G, that is, termwise addition modulo 2, we abbreviate

$$\lambda(\mu(x) \overset{*}{+} \mu(y)) \qquad \text{as} \quad x \dot{+} y$$

for any real x and y.

It is straightforward (following the classical Weyl method) to establish the following (which we state without proof):

THEOREM 1 (a Walsh function Weyl criterion). *$\{x_i\}$ is* u.d. mod 1 *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} W_h(x_i) = 0, \qquad \textit{for any} \quad h \geq 1.$$

This Walsh function Weyl criterion can easily be used to provide proofs (omitted here) for the following two direct analogues of well known classical results [1].

THEOREM 2. *If $\{x_i\}$ is* u.d. mod 1, *$\{a_i\} \subset I$ is such that $a_i \to 0$ as $i \to \infty$ and $x_i \dot{+} a_i$ is a dyadic irrational for each i then $\{x_i \dot{+} a_i\}$ is* u.d. mod 1.

We note that this theorem provides a new way to generate u.d. mod 1 sequences from a known u.d. mod 1 sequence $\{x_i\}$. There are two special cases of interest, namely, when the x_i are either all dyadic rationals or all dyadic irrationals. Similarly, Theorem 3 (following) can also be used to generate new u.d. mod 1 sequences.

THEOREM 3. *If $\{x_i\}_{i=0}^{\infty}$ is* u.d. mod 1, *$a_j \in \mathbf{R}$ $0 \leq j \leq N$ and $I_K = \{x_i \dot{+} \dot{\sum}_{j=0}^{K} a_j\}$ is a set of dyadic irrationals for each $0 \leq K \leq N$, then I_K is* u.d. mod 1 *where $\dot{\sum}$ is defined inductively as*

$$\dot{\sum}_{j=0}^{0} a_j = a_0 \quad \textit{and} \quad \dot{\sum}_{j=0}^{K-1} a_j \dot{+} a_K = \dot{\sum}_{j=0}^{K} a_j.$$

3. The Walsh integrals and u.d. mod 1. The integrals of the Walsh functions, defined by

$$J_k(x) = \int_0^x W_k(x)dx$$

were investigated by Fine [2]. Here, the set of "Walsh integrals", $\{J_k\}$, with a slight modification (see the Corollary below), is used to investigated u.d. mod 1.

We now establish a partial extension of the Weyl criterion for functions, possibly discontinuous, which have a Walsh series expansion subject to a moderate regularity condition. This result is used to prove our Corollary on the integrals of the Walsh functions. This Corollary is used later as part of the "Walsh integrals" Weyl criterion.

THEOREM 4. *If* $\{x_i\}$ *is* u.d. mod 1, *then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} F_K(x_i)=0$$

with

$$F_K(x)=\sum_{k=1}^{g(K)} a_k W_k(x)+\sum_{k=g(K)+1}^{\infty} b_k W_k(x)\,, \qquad |b_i|\leq C\frac{1}{2^{i+1}}\,,$$

where $C\in \boldsymbol{R}^+$ *and does not depend on i while* $g\colon \boldsymbol{N}\to\boldsymbol{N}$ *is arbitrary.*

PROOF.

$$\begin{aligned}\left|\frac{1}{N}\sum_{i=0}^{N-1} F_K(x_i)\right| &= \left|\frac{1}{N}\sum_{i=0}^{N-1}\sum_{k=1}^{g(K)} a_k W_k(x_i)+\frac{1}{N}\sum_{i=0}^{N-1}\sum_{k=g(K)+1}^{\infty} b_k W_k(x_i)\right| \\ &\leq \left|\sum_{k=1}^{g(K)} a_k \frac{1}{N}\sum_{i=0}^{N-1} W_k(x_i)+\sum_{k=g(K)+1}^{M} b_k \frac{1}{N}\sum_{i=0}^{N-1} W_k(x_i)\right|+\frac{C}{2^M} \\ &\leq \sum_{k=1}^{g(K)} |a_k| \left|\frac{1}{N}\sum_{i=0}^{N-1} W_k(x_i)\right|+\sum_{k=g(K)+1}^{M} |b_k| \left|\frac{1}{N}\sum_{i=0}^{N-1} W_k(x_i)\right|+\frac{C}{2^M} \\ &\to \frac{C}{2^M} \qquad \text{as} \quad N\to\infty\,.\end{aligned}$$

But M was arbitrary, so

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{i=0}^{N-1} F_K(x_i)\right|=0\,.$$

□

COROLLARY. *Let* $\{x_i\}$ *be* u.d. mod 1 *and* $h=2^n+l$, $0\leq l<2^n$. *Then a partial analogue of the Weyl criterion holds, namely*:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1} J_h'(x_i)=0 \qquad \textit{for any} \quad h\in \boldsymbol{N}\,,$$

where the modified Walsh integrals are

$$J_h'=J_h-\delta_{h,2^n}2^{-(n+2)}\,.$$

PROOF. From Fine [2], we have

$$J_h(x)=2^{-(n+2)}\left\{(W_l(x)-\sum_{r=1}^{\infty} 2^{-r}W_{2^{n+r}+h}(x)\right\}.$$

Hence the result follows from Theorem 4. □

We note that this result may also be obtained, via equation (1), from the symmetry (about $x=1/2$) properties of the Walsh functions and the continuity properties of the J_h.

In the following, integrals of the Walsh functions are used to devise an analogue of the LeVeque inequality. From this the modified integrals of the Walsh functions are used to obtain another Weyl criterion: the "Walsh integrals" Weyl criterion whose first part is stated as our Corollary above.

For use in the proof of the following lemma, we establish a preliminary result. From Fine [2], we note that

$$x-[x]=\frac{1}{2}-\frac{1}{4}\sum_{n=0}^{\infty} 2^{-n}W_{2^n}(x).$$

Also, the series

$$\sum_{n=0}^{\infty} 2^{-n}W_{2^n}(x)$$

is uniformly convergent, so

$$\int_0^1 xW_h(x)dx=-\frac{1}{4}\sum_{n=0}^{\infty} 2^{-n}\int_0^1 W_{2^n}W_h dx=-\sum_{n=0}^{\infty} 2^{-(n+2)}\delta_{2^n,h}.$$

Now, suppose that $x_0, \ldots, x_{N-1}$ are N points in I. For $0\le x\le 1$, led $R_N(x)=A([0, x); N)-Nx$. Then

LEMMA.

$$\int_0^1 R_N^2(x)dx=\left(\sum_{m=0}^{N-1}\left(x_m-\frac{1}{2}\right)\right)^2+\sum_{h=1}^{\infty}\left(N2^{-(n+2)}\delta_{2^n,h}-\sum_{m=0}^{N-1} J_h(x_m)\right)^2.$$

PROOF. The proof is a modification of the proof of Lemma 2.8 in [1]. Since the Walsh functions are complete in $L^2[0, 1)$, the Walsh Fourier series of $R_N(x)$ converges in L^2 to $R_N(x)$. Also, for $0\le m\le N-1$ we let $C_m(x)$ be the characteristic function of the interval $(x_m, 1]$. Then

$$A([0, x); N)=\sum_{m=0}^{N-1} C_m(x)$$

and so, letting

$$\sum_{k=0}^{\infty} a_k W_k(x)$$

be the Walsh Fourier series of $R_N(x)$, a_0 is obtained as in the classical proof, and for $h \geq 1$ we obtain

$$\begin{aligned} a_h &= \sum_{m=0}^{N-1} \int_0^1 C_m(x) W_h(x) dx - N \int_0^1 x W_h(x) dx \\ &= \sum_{m=0}^{N-1} \int_{x_m}^1 W_h(x) dx + N \sum_{m=0}^{\infty} 2^{-(m+2)} \delta_{2^m,h} \\ &= \sum_{m=0}^{N-1} (J_h(1) - J_h(x_m)) + N \sum_{m=0}^{\infty} 2^{-(m+2)} \delta_{2^m,h} \\ &= -\sum_{m=0}^{N-1} J_h(x_m) + N \sum_{m=0}^{\infty} 2^{-(m+2)} \delta_{2^m,h} \\ &= N 2^{-(n+2)} \delta_{2^n,h} - \sum_{m=0}^{N-1} J_h(x_m) . \end{aligned}$$

The result follows by Parseval's identity. □

The discrepancy D_N is a measure of the deviation of a sequence from being uniformly distributed mod 1 and is defined as

$$D_N = D_N(x_0, \ldots, x_{N-1}) = \sup_{0 \leq a < b \leq 1} \left| \frac{A([a, b); N)}{N} - (b-a) \right| .$$

THEOREM 5 (analogue of LeVeque's inequality). *The discrepancy D_N of the finite sequence $x_0, \ldots, x_{N-1}$ in I satisfies*

$$D_N \leq \left(12 \sum_{h=1}^{\infty} \left(2^{-(n+2)} \delta_{2^n,h} - \frac{1}{N} \sum_{m=0}^{N-1} J_h(x_m) \right)^2 \right)^{1/3} = \left(\frac{12}{N^2} \sum_{h=1}^{\infty} \left(\sum_{m=0}^{N-1} J_h'(x_m) \right)^2 \right)^{1/3} .$$

PROOF. The proof uses the above Lemma and follows the one given in [1]. The details are omitted here.

REMARK. The constant in the classical LeVeque inequality is the best possible. To establish a corresponding result here, consider $\{x_m\}_{m=0}^{N-1} = \{0\}$ for which

$$D_N \leq \left(12 \sum_{h=1}^{\infty} (2^{-(n+2)} \delta_{2^n,h})^2 \right)^{1/3} = \left(12 \sum_{h=1}^{\infty} 2^{-2n-4} \delta_{2^n,h} \right)^{1/3} = \left(12 \sum_{n=0}^{\infty} 2^{-2n-4} \right)^{1/3} = 1 .$$

So the constant here, 12, is also the best possible.

THEOREM 6 (the "Walsh integrals" Weyl criterion). *$\{x_i\}$ is uniformly distributed* mod 1 *if and only if* $\lim_{N \to \infty} (1/N) \sum_{i=0}^{N-1} J_h'(x_i) = 0$, *for any* $h \in \mathbf{N}$.

PROOF. The "only if" part of the theorem is established as the Corollary above. For the "if" part assume $\{x_i\}$ is not uniformly distributed mod 1, then there exists a sequence $\{N_k\}$ such that

$$\lim_{k\to\infty} D_{N_k} = D > 0 .$$

Now, for any $N \geq 1$, from the analogue of LeVeque's inequality, letting $H = 2^M + L$, $0 \leq L < 2^M$,

$$D_N^3 \leq 12 \sum_{h=1}^{\infty} \left(\frac{1}{N} \sum_{i=0}^{N-1} J_h'(x_i) \right)^2 < 12 \sum_{h=1}^{H} \left(\frac{1}{N} \sum_{i=0}^{N-1} J_h'(x_i) \right)^2 + \frac{12}{2^{M-1}} .$$

Now choose M so that

$$\frac{12}{2^{M-1}} < \frac{1}{2} D^3 ,$$

and let $\varepsilon > 0$ be given such that $N_k > N_0$ implies

$$D^3 - \varepsilon < D_{N_k}^3 < D^3 + \varepsilon .$$

Thus,

$$D^3 - \varepsilon < D_{N_k}^3 < 12 \sum_{h=1}^{H} \left(\frac{1}{N_k} \sum_{i=0}^{N_k-1} J_h'(x_i) \right)^2 + \frac{12}{2^{M-1}} .$$

So

$$\frac{1}{2} D^3 - \varepsilon < 12 \sum_{h=1}^{H} \left(\sum_{i=0}^{N_k-1} \frac{1}{N_k} J_h'(x_i) \right)^2 .$$

Now choose N_0 so that

$$\frac{1}{2} D^3 - \varepsilon > \frac{1}{4} D^3 .$$

Thus

$$0 < \frac{1}{4} D^3 < 12 \sum_{h=1}^{H} \left(\frac{1}{N_k} \sum_{i=0}^{N_k-1} J_h'(x_i) \right)^2 ,$$

which implies that

$$0 < \frac{1}{4} D^3 \leq 12 \sum_{h=1}^{H} \left(\lim_{k\to\infty} \frac{1}{N_k} \sum_{i=0}^{N_k-1} J_h'(x_i) \right)^2 .$$

So, for some $h \geq 1$

$$\left|\lim_{k\to\infty}\frac{1}{N_k}\sum_{i=0}^{N_k-1}J'_h(x_i)\right|>0\,.$$

□

We conclude with:

THEOREM 7 (an analogue of the Erdös-Turán inequality). *For any finite sequence* $x_0, \ldots, x_{N-1}$ *of real numbers and any positive* H, *we have for* $H=2^M+L$, $0\le L<2^M$

$$D_N\le\left(12\sum_{h=1}^{H-1}\left|\sum_{n=0}^{N-1}\frac{1}{N}J'_h(x_n)\right|^2+\frac{3}{2^{M-1}}\left(1-\frac{1}{2^{M+1}}\right)\right)^{1/3}.$$

PROOF.

$$\begin{aligned}D_N&\le\left(12\sum_{h=1}^{\infty}\left|\sum_{n=0}^{N-1}\frac{1}{N}J'_h(x_n)\right|^2\right)^{1/3}\\
&=\left(12\left(\sum_{h=1}^{H-1}\left|\sum_{n=0}^{N-1}\frac{1}{N}J'_h(x_n)\right|^2+\sum_{h=H}^{\infty}\left|\sum_{n=0}^{N-1}\frac{1}{N}J'_h(x_n)\right|^2\right)\right)^{1/3}\\
&\le\left(12\left(\sum_{h=1}^{H-1}\left|\sum_{n=0}^{N-1}\frac{1}{N}J'_h(x_n)\right|^2+\sum_{h=H}^{\infty}\left(\sum_{n=0}^{N-1}\frac{1}{N}|J'_h(x_n)|\right)^2\right)\right)^{1/3}\\
&\le\left(12\left(\sum_{h=1}^{H-1}\left|\sum_{n=0}^{N-1}\frac{1}{N}J'_h(x_n)\right|^2+\frac{12}{2^{M+1}}\left(1-\frac{1}{2^{M+1}}\right)\right)\right)^{1/3}\\
&=\left(12\sum_{h=1}^{H-1}\left|\sum_{n=0}^{N-1}\frac{1}{N}J'_h(x_n)\right|^2+\frac{3}{2^{M-1}}\left(1-\frac{1}{2^{M+1}}\right)\right)^{1/3}\end{aligned}$$

since, for any x

$$\begin{aligned}\sum_{h=H}^{\infty}(J'_h(x))^2&\le\sum_{i=M}^{\infty}\sum_{h=2^i}^{2^{i+1}-1}(J'_h(x))^2\le\sum_{i=M}^{\infty}2^i\left(\frac{1}{2^{i+1}}\right)^2-\sum_{i=M}^{\infty}\frac{3}{2^{2(i+2)}}\\
&=\sum_{i=M}^{\infty}\frac{1}{2^{i+2}}-\sum_{i=M}^{\infty}\frac{3}{2^{2(i+2)}}=\frac{1}{2^{M+2}}\sum_{i=0}^{\infty}\frac{1}{2^i}-\frac{3}{4}\frac{1}{4^{M+1}}\sum_{i=0}^{\infty}\frac{1}{4^i}\\
&=\frac{2}{2^{M+2}}-\frac{3}{4}\frac{1}{4^{M+1}}\frac{1}{1-1/4}=\frac{1}{2^{M+1}}\left(1-\frac{1}{2^{M+1}}\right).\end{aligned}$$

□

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