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HOLONOMIC q-DIFFERENCE SYSTEM ASSOCIATED WITH THE BASIC HYPERGEOMETRIC SERIES ${}_{n+1}\varphi_n$

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Abstract. The holonomic q-difference system of the first order associated with the basic hypergeometric series is derived. The Wronskian of this system is also calculated.

In spite of widespread interests in holonomic q-difference systems (e.g. [2], [3], [5]), very few systems have been calculated in an explicit manner. Thus it seems very fundamental and important to construct such systems explicitly.

In this paper we first derive the holonomic q-difference system with respect to the variable z, which is associated with the Jackson integral

$$\int \prod_{i=1}^{n+1} t_i^{\alpha_i} \frac{(q^{\beta'_i} t_i/t_{i-1})_\infty}{(q^{\beta_i} t_i/t_{i-1})_\infty} d_q t_1 \wedge \cdots \wedge d_q t_n$$

with $t_0=1$ and $t_{n+1}=z$. This Jackson integral is one of the integral representations of the basic hypergeometric series ${}_{n+1}\varphi_n$. The special case $n=1$ corresponds to that of Heine's ${}_2\varphi_1$, and the corresponding system is known. Although the $n=2$ case corresponds to the one treated in Section 6 of [1], the equation for it there is very complicated. To avoid such complexities we modify the integrand as above, of which the last factor is changed. This permits us to derive the equation for general n.

Our second result is the determinant formula. The calculation of the determinant of the coefficient matrix A of our system leads to the q-difference equation for the Wronskian. By virtue of this equation, we get an expression for the Wronskian near the origin $z=0$ and the infinity $z=\infty$, respectively. The connection coefficient between them is also given. For works related with the determinant formulas we refer the reader to [6] and [7]. It is also noted that the condition of "n-balanced" makes our determinant constant. It seems very important to know whether the condition for a well-poised series and other conditions appearing in the theory of hypergeometric series [4] is related to the determinant of the holonomic systems or not.

Throughout this paper the number q is fixed as $0<q<1$. The symbol $(a)_\infty$ stands for the q-shifted factorial $\prod_{i=0}^{\infty}(1-aq^i)$.

1. Holonomic q-difference system. Define a meromorphic function Φ on the

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algebraic torus $(t_1, \dots, t_{n+1}) \in (C^*)^{n+1}$ by

$$\Phi = \prod_{i=1}^{n+1} t_i^{\alpha_i} \frac{(q^{\beta'_i} t_i / t_{i-1})_\infty}{(q^{\beta_i} t_i / t_{i-1})_\infty}$$

with $t_{n+1} = z$ and $t_0 = 1$. Associated with this function Φ, the q-twisted de Rham cohomology $H^n_\Phi(V, d_q)$ is defined. Under some conditions, which we do not use in this paper, the dimension $\dim H^n_\Phi(V, d_q)$ of the cohomology group is $n+1$. See [1] for more details. By means of the basis of this cohomology, defined by

$$\varphi_i = \frac{1}{\prod_{\substack{1 \le k \le n+1 \\ k \neq i}} (t_{k-1} - q^{\beta'_k} t_k)}$$

for each $1 \le i \le n+1$, we construct a holonomic q-difference system with respect to the q-shift operator T_z defined by $(T_z f)(z) = f(qz)$. For brevity, we denote the Jackson integral of the rational function φ by

$$\langle \varphi \rangle = \int \Phi \varphi \, d\tau$$

and $d\tau = d_q t_1 \wedge \cdots \wedge d_q t_n$ in what follows. The Jackson integral is taken over a suitable cycle. In this notation we have the following:

THEOREM 1. *Under the conditions $\alpha_k \neq 0$, $\beta'_k - \beta_k > 0$ for each $1 \le k \le n+1$, we have a holonomic q-difference system of rank $n+1$ such that*

$$T_z(\langle \varphi_1 \rangle, \dots, \langle \varphi_{n+1} \rangle) = (\langle \varphi_1 \rangle, \dots, \langle \varphi_{n+1} \rangle) A \,. \tag{1}$$

Here $A = (a_{ij})_{1 \le i, j \le n+1}$ is the $(n+1) \times (n+1)$-matrix with entries defined below.

Case (i) $1 \le i \le j-1$,

$$a_{ij} = q^{\sum_{k=j}^{n+1} \alpha_k + \sum_{k=1}^{i-1} \beta'_k + \sum_{k=j+1}^{n+1} \beta'_k} \frac{(q^{\beta'_j} - q^{\beta_j}) z}{1 - q^{\sum_{k=1}^{n+1} \beta'_k} z} \,.$$

Case (ii) $i = j$,

$$a_{ii} = q^{\sum_{k=i}^{n+1} \alpha_k} \frac{1 - q^{\beta_j + \sum_{1 \le k \le n+1, k \neq j} \beta'_k} z}{1 - q^{\sum_{k=1}^{n+1} \beta'_k} z} \,.$$

Case (iii) $j+1 \le i \le n$,

$$a_{ij} = q^{\sum_{k=j}^{n+1} \alpha_k + \sum_{k=j+1}^{i-1} \beta_k} \frac{(q^{\beta'_j} - q^{\beta_j}) z}{1 - q^{\sum_{k=1}^{n+1} \beta'_k} z} \,.$$

PROOF. Associated with the basis φ_i $(1 \le i \le n+1)$, we introduce another basis

$$\varphi_i' = \frac{t_{i-1} - q^{\beta_i} t_i}{\prod_{k=1}^{n+1} (t_{k-1} - q^{\beta_k'} t_k)} \qquad (1 \le i \le n+1).$$

Then we have

$$T_z \langle \varphi_i \rangle = q^{\sum_{k=i}^{n+1} \alpha_k} \langle \varphi_i' \rangle \tag{2}$$

for each i with $1 \le i \le n+1$. Namely,

$$T_z(\langle \varphi_1 \rangle, \ldots, \langle \varphi_{n+1} \rangle) = (\langle \varphi_1' \rangle, \ldots, \langle \varphi_{n+1}' \rangle) C$$

with a diagonal matrix $C = \mathrm{diag}(q^{\alpha_1 + \cdots + \alpha_{n+1}}, q^{\alpha_2 + \cdots + \alpha_{n+1}}, \ldots, q^{\alpha_{n+1}})$. To see this we first take

$$\begin{aligned} T_z \langle \varphi_i \rangle &= T_z \int \prod_{k=1}^{n+1} t_k^{\alpha_k - 1 + \delta_{ki} + \delta_{k,n+1}} \frac{(q^{\beta_k' + 1 - \delta_{ki}} t_k / t_{k-1})_\infty}{(q^{\beta_k} t_k / t_{k-1})_\infty} d\tau \\ &= q^{\alpha_{n+1}} \int \prod_{k=1}^{n+1} t_k^{\alpha_k - 1 + \delta_{ki} + \delta_{k,n+1}} \frac{(q^{\beta_k' + 1 - \delta_{ki} + \delta_{k,n+1}} t_k / t_{k-1})_\infty}{(q^{\beta_k + \delta_{k,n+1}} t_k / t_{k-1})_\infty} d\tau . \end{aligned}$$

At this step we change the variables t_k in the integral to qt_k for each k such that $i \le k \le n$. Then we have

$$T_z \langle \varphi_i \rangle = q^{\sum_{k=i}^{n+1} \alpha_k} \int \prod_{k=1}^{n+1} t_k^{\alpha_k - 1 + \delta_{ki} + \delta_{k,n+1}} \frac{(q^{\beta_k' + 1} t_k / t_{k-1})_\infty}{(q^{\beta_k + \delta_{ki}} t_k / t_{k-1})_\infty} d\tau ,$$

which is the equality (2).

Under the condition $\prod_{k=1}^{n+1} (t_{k-1} - q^{\beta_k'} t_k) \neq 0$, the equalities

$$\varphi_j' = \sum_{i=1}^{n+1} \varphi_i b_{ij} \qquad (1 \le j \le n+1)$$

are equivalent to

$$t_{j-1} - q^{\beta_j} t_j = \sum_{i=1}^{n+1} (t_{i-1} - q^{\beta_i'} t_i) b_{ij} \qquad (1 \le j \le n+1). \tag{3}$$

By comparing the constant terms and the coefficients of each t_i for i such that $1 \le i \le n$ on both sides of (3), we have a system of linear equations of rank $n+1$. From this system we can easily get an expression for b_{ij}.

Therefore, by the product of two matrices $A = BC$, we have the desired expression for a_{ij}.

Moreover, if we set z to be zero, the coefficient matrix A becomes a lower triangular matrix with the diagonal elements $a_{ii} = q^{\sum_{k=i}^{n+1} \alpha_k}$ for $1 \le i \le n+1$. Thus the rank of the system is $n+1$. ∎

2. Determinant formula. The calculation of the determinant of A is important, because it gives the equation satisfied by the Wronskian

$$\det\left(\int_{\mathscr{C}_i}\varphi_j\Phi d\tau\right).$$

Here the symbols $\mathscr{C}_i$ $(1\le i\le n+1)$ stand for suitable cycles, which are not mentioned explicitly in what follows.

PROPOSITION 2. *We have*

$$\det A=q^{\sum_{k=1}^{n+1}k\alpha_k}\frac{1-q^{\beta_1+\beta_2+\cdots+\beta_{n+1}}z}{1-q^{\beta'_1+\beta'_2+\cdots+\beta'_{n+1}}z}. \tag{4}$$

PROOF. Define $(n+1)\times(n+1)$-matrices $C_k^+=(c_{k;ij}^+)_{1\le i,j\le n+1}$ and $C_k^-=(c_{k;ij}^-)_{1\le i,j\le n+1}$ for $1\le k\le n$ as follows:

$$c_{k;ij}^+=\begin{cases} q^{\beta_k} & (i,j)=(k+1,k)\\ \delta_{ij} & \text{otherwise}\end{cases}$$

and

$$c_{k;ij}^-=\begin{cases} -q^{\beta_k} & (i,j)=(k+1,k)\\ \delta_{ij} & \text{otherwise.}\end{cases}$$

By using these matrices, the relation for the matrices

$$(C_1^-C_2^-\cdots C_n^-B)C_n^+C_{n-1}^+\cdots C_1^+=B'$$

is given. Here the elements b'_{ij} $(1\le i,j\le n+1)$ of B' are expressed as

$$b'_{ij}=\begin{cases}\dfrac{1-q^{\beta_1+\cdots+\beta_{n+1}}z}{1-q^{\beta'_1+\cdots+\beta'_{n+1}}z} & i=j=1\,,\\[2ex] \dfrac{(q^{\beta'_j+\beta'_{j+1}+\cdots+\beta'_{n+1}}-q^{\beta_j+\beta_{j+1}+\cdots+\beta_{n+1}})z}{1-q^{\beta'_1+\cdots+\beta'_{n+1}}z} & i=1,\quad j\ge 2\,,\\[2ex] \delta_{ij} & \text{otherwise}\,.\end{cases}$$

Therefore we get the desired result (4). ■

Proposition 2 leads to the q-difference equation for our Wronskian $W(z)$:

$$(T_zW)(z)=q^{\sum_{k=1}^{n+1}k\alpha_k}\frac{1-q^{\sum_{k=1}^{n+1}\beta_k}z}{1-q^{\sum_{k=1}^{n+1}\beta'_k}z}W(z)\,. \tag{5}$$

By virtue of (5), we can easily find expressions for $W^{(0)}(z)$ and $W^{(\infty)}(z)$ of our Wronskian which are characterized by the asymptotics

(6) $$\lim_{z\to 0} W^{(0)}(z)=z^{\sum_{k=1}^{n+1}k\alpha_k}(1+c_1^{(0)}z+\cdots)$$

and

(7) $$\lim_{z\to\infty} W^{(\infty)}(z)=z^{\sum_{k=1}^{n+1}k\alpha_k-\sum_{k=1}^{n+1}\beta'_k+\sum_{k=1}^{n+1}\beta_k}\left(1+\frac{c_1^{(\infty)}}{z}+\cdots\right)$$

respectively, and find the connection coefficient between them.

THEOREM 3. (i) *The expressions for the Wronskian defined above are given by*

$$W^{(0)}(z)=z^{\sum_{k=1}^{n+1}k\alpha_k}\frac{(q^{\sum_{k=1}^{n+1}\beta'_k}z)_\infty}{(q^{\sum_{k=1}^{n+1}\beta_k}z)_\infty}$$

and

$$W^{(\infty)}(z)=z^{\sum_{k=1}^{n+1}k\alpha_k-\sum_{k=1}^{n+1}\beta'_k+\sum_{k=1}^{n+1}\beta_k}\frac{(q^{1-\sum_{k=1}^{n+1}\beta_k}/z)_\infty}{(q^{1-\sum_{k=1}^{n+1}\beta'_k}/z)_\infty}.$$

(ii) *The connection coefficient C between $W^{(0)}(z)$ and $W^{(\infty)}(z)$ such that $W^{(\infty)}(z)=CW^{(0)}(z)$ is given by*

$$C=z^{\sum_{k=1}^{n+1}\beta_k-\sum_{k=1}^{n+1}\beta'_k}\frac{\theta(q^{\sum_{k=1}^{n+1}\beta_k}z)}{\theta(q^{\sum_{k=1}^{n+1}\beta'_k}z)},$$

where $\theta(z)=(z)_\infty(q/z)_\infty(q)_\infty$ is the Jacobi elliptic theta function.

REMARK. The coefficient C is not a constant with respect to the variable z but is a pseudoconstant, that is, $(T_zC)(z)=C(z)$.

It is also noted that if $\sum_{k=1}^{n+1}\beta_k=\sum_{k=1}^{n+1}\beta'_k$, then the determinant of A is constant. This condition is closely related to the "n-balanced" basic hypergeometric series ${}_{n+1}\varphi_n$. To see this, recall the integral representation of the basic hypergeometric series

(8) $$ {}_{n+1}\varphi_n\left(\begin{matrix}a_1,\ldots,a_{n+1}\\ b_1,\ldots,b_n\end{matrix};q,x\right)=\sum_{k\ge 0}\frac{(a_1)_k\cdots(a_{n+1})_k}{(b_1)_k\cdots(b_n)_k(q)_k}x^k,$$

where $(a)_k=(a)_\infty/(aq^k)_\infty$. Namely,

$$\prod_{i=1}^n\frac{\Gamma_q(\beta_i)}{\Gamma_q(\alpha_i)\Gamma_q(\beta_i-\alpha_i)}\int_{1\ge t_n\ge t_{n-1}\ge\cdots\ge t_1\ge 0}t_1^{\alpha_1-1}\cdots t_n^{\alpha_n-1}\frac{(qt_1)_\infty}{(q^{\beta_1-\alpha_1}t_1)_\infty}$$
$$\cdot\frac{(qt_2/t_1)_\infty\cdots(qt_n/t_{n-1})_\infty(q^{\alpha_{n+1}}t_nx)_\infty}{(q^{\beta_2-\alpha_2}t_2/t_1)_\infty\cdots(q^{\beta_n-\alpha_n}t_n/t_{n-1})_\infty(t_nx)_\infty}d\tau$$
$$=\prod_{i=1}^n\frac{\Gamma_q(\beta_i)}{\Gamma_q(\alpha_i)\Gamma_q(\beta_i-\alpha_i)}\cdot\frac{\theta(q^{\alpha_{n+1}}x)}{\theta(x)}\int_{1\ge t_n\ge t_{n-1}\ge\cdots\ge t_1\ge 0}t_1^{\alpha_1-1}\cdots t_{n-1}^{\alpha_{n-1}-1}t_n^{\alpha_n-\alpha_{n+1}-1}$$

$$\cdot \frac{(qt_1)_\infty(qt_2/t_1)_\infty \cdots (qt_n/t_{n-1})_\infty(qx^{-1}/t_n)_\infty}{(q^{\beta_1-\alpha_1}t_1)_\infty(q^{\beta_2-\alpha_2}t_2/t_1)_\infty \cdots (q^{\beta_n-\alpha_n}t_n/t_{n-1})_\infty(q^{1-\alpha_{n+1}}x^{-1}/t_n)_\infty} d\tau ,$$

where $a_k = q^{\alpha_k}$ $(1 \le k \le n+1)$, $b_k = q^{\beta_k}$ $(1 \le k \le n)$, and Γ_q stands for the q-gamma function $\Gamma_q(\alpha) = (q)_\infty (q^\alpha)_\infty^{-1}(1-q)^{1-\alpha}$. The last equality in the above integral representations is given by using the pseudoconstant

$$t_n^{\alpha_{n+1}} \frac{\theta(q^{\alpha_{n+1}}xt_n)}{\theta(xt_n)}$$

with respect to the variable t_n. The condition $\sum_{k=1}^{n+1} \beta_k = \sum_{k=1}^{n+1} \beta'_k$ for our integral is equivalent to the one $\sum_{k=1}^{n+1} \alpha_k + n = \sum_{k=1}^{n} \beta_k$ for the basic hypergeometric series (8), in which case, (8) is said to be "n-balanced" if $x = q$.

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