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# CONJUGATE EXPANSIONS FOR ULTRASPHERICAL FUNCTIONS

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**Abstract.** We define and investigate the Hilbert transform for expansions with respect to the system of ultraspherical functions. Using deep estimates done by Muckenhoupt and Stein in the polynomial expansion case we prove the existence of boundary values of the conjugate Poisson integrals of integrable functions. The limit function then satisfies usual $L^p$ and weak type (1, 1) estimates.

**1. Introduction.** Thirty years ago in their classical paper [6] Muckenhoupt and Stein investigated the ultraspherical expansions from the harmonic analysis point of view. One of the main results they proved was that the conjugacy mapping $f \mapsto \tilde{f}$ is a bounded operator on $L^p$, $1<p<\infty$. Here the definition of conjugacy is insightfully introduced in such a way that the Poisson integral and the conjugate Poisson integral corresponding to $f$ are related by suitable Cauchy-Riemann equations. More specifically, if $f \in L^1((0,\pi),(\sin\theta)^{2\lambda}d\theta)$, $\lambda>0$, has the expansion $\sum a_n P_n^\lambda(\cos\theta)$ then its conjugate $\tilde{f}$ is formally defined by

$$\sum \frac{2\lambda}{n+2\lambda} a_n \sin\theta \cdot P_{n-1}^{\lambda+1}(\cos\theta) .$$

Then the Poisson integral

$$f(r,\theta)=\sum a_n r^n P_n^\lambda(\cos\theta)$$

and the conjugate Poisson integral

$$\tilde{f}(r,\theta)=\sum \frac{2\lambda}{n+2\lambda} a_n r^n \sin\theta \cdot P_{n-1}^{\lambda+1}(\cos\theta)$$

satisfy

$$\frac{\partial}{\partial r}((r\sin\theta)^{2\lambda}\tilde{f})=-r^{2\lambda-1}(\sin\theta)^{2\lambda}\frac{\partial f}{\partial\theta},$$

$$\frac{\partial}{\partial\theta}((r\sin\theta)^{2\lambda}\tilde{f})=r^{2\lambda+1}(\sin\theta)^{2\lambda}\frac{\partial f}{\partial r}.$$

In [4], [5] Muckenhoupt proved $L^p$ conjugate function theorems for Hermite and

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Laguerre expansions and the definition of conjugacy this time was done according to a general principle given by Stein [7]. In all these cases only polynomial expansions were discussed. Parallelly to such expansions it is natural to consider expansions with respect to the system of corresponding orthonormal functions that arise by multiplying polynomials by the square root of appropriate weight function. This approach was undertaken by Gosselin and Stempak in [3] where they discussed conjugacy for expansions with respect to Hermite functions. The main objective of this paper is to develop conjugacy theory for ultraspherical functions. By these we mean

$$\varphi_n^\lambda(\theta)=d(\lambda, n)P_n^\lambda(\cos\theta)(\sin\theta)^\lambda, \qquad n=0, 1, 2, \ldots$$

where

$$d(\lambda, n)=2^{\lambda-1/2}\Gamma(\lambda)\pi^{-1/2}((n+\lambda)n!/\Gamma(n+2\lambda))^{1/2}$$

and $P_n^\lambda(t)$ is the $n$-th ultraspherical polynomial of type $\lambda>0$. The system $\{\varphi_n^\lambda(\theta)\}$ is a complete orthonormal system in $L^2((0,\pi), d\theta)$ and was discussed in the literature for instance by Askey and Wainger [1], [2].

In [7] Stein suggests the following definition of the Hilbert transform associated with a general Sturm-Liouville operator

$$L=\frac{d^2}{dx^2}+a(x)\frac{d}{dx} \tag{1.1}$$

acting on an interval $(\alpha_1, \alpha_2)$: if $\{\varphi_n\}$, $n\geq 0$, is a complete orthonormal set of eigenfunctions of $L$ with eigenvalues $\{-\lambda_n^2\}$ then a formal integration by parts shows that the functions $\{\lambda_n^{-1}d\varphi_n/dx\}$ form an orthonormal set again and thus the proposed Hilbert transform would provide the mapping $\varphi_n\mapsto\lambda_n^{-1}d\varphi_n/dx$. In the situation we consider the functions $\varphi_n^\lambda(\theta)$ are eigenfunctions of the differential operator

$$L=\frac{d^2}{d\theta^2}-\lambda(\lambda-1)\frac{1}{\sin^2\theta} \tag{1.2}$$

with eigenvalues $\{-\mu_n^2\}$, $\mu_n=n+\lambda$. The operator (1.2) is no longer of the form (1.1). However the following result motivates our definition of Hilbert transform that follows.

PROPOSITION 1.1. *Let $\{\varphi_n\}$, $n\geq 0$, be an orthonormal set of functions in $L^2((0,\pi), d\theta)$ consisting of eigenfunctions of the differential operator* (1.2) *with eigenvalues $-\mu_n^2$ (precise values of $\mu_n$'s are not important here). Suppose also that $\varphi_n$'s are of sufficient decay at* 0 *and* $\pi$: $\lim_{\theta\to 0,\pi}\varphi_n(\theta)\varphi_m(\theta)\cot\theta=0$ *and* $\lim_{\theta\to 0,\pi}\varphi_n'(\theta)\varphi_m(\theta)=0$, *say. Then the set* $\{\psi_n\}$, $n\geq 1$, *where*

$$\psi_n(\theta)=(\mu_n^2-\lambda^2)^{-1/2}(\varphi_n'(\theta)-\lambda\cot\theta\,\varphi_n(\theta))$$

*is also an orthonormal system in* $L^2((0,\pi), d\theta)$.

PROOF. We have from the decay at 0 and $\pi$

$$0=\int_0^\pi (\varphi_n\varphi_m \cot\theta)' d\theta=\int_0^\pi (\varphi'_n\varphi_m+\varphi_n\varphi'_m)\cot\theta\, d\theta-\int_0^\pi \varphi_n\varphi_m \frac{1}{\sin^2\theta}\, d\theta .$$

Therefore

$$\begin{aligned}((\mu_n^2-\lambda^2)(\mu_m^2-\lambda^2))^{1/2}&\int_0^\pi \psi_n\psi_m d\theta\\ &=\int_0^\pi (\varphi'_n-\lambda\cot\theta\,\varphi_n)(\varphi'_m-\lambda\cot\theta\,\varphi_m)d\theta\\ &=\int_0^\pi (\varphi'_n\varphi'_m+\lambda^2\cot^2\theta\,\varphi_n\varphi_m)d\theta-\lambda\int_0^\pi(\varphi'_n\varphi_m+\varphi_n\varphi'_m)\cot\theta\, d\theta\\ &=\int_0^\pi (-\varphi''_n+\lambda^2\cot^2\theta\,\varphi_n)\varphi_m d\theta-\lambda\int_0^\pi \varphi_n\varphi_m\frac{1}{\sin^2\theta}\, d\theta\\ &=(\mu_n^2-\lambda^2)\delta_{nm} ,\end{aligned}$$

which finishes the proof of the proposition.

It may be easily verified by using differential properties of the ultraspherical polynomials that $(d/d\theta-\lambda\cot\theta)\varphi_n^\lambda(\theta)=-\sqrt{n(n+2\lambda)}\,\varphi_{n-1}^{\lambda+1}(\theta)$. This and the explicit form of $\mu_n$'s suggests that the Hilbert transform for the ultraspherical functions can be defined by the mapping

$$\varphi_n^\lambda(\theta)\mapsto\varphi_{n-1}^{\lambda+1}(\theta) .$$

However, the same argument we used defining conjugate functions for the Hermite function expansions [3] forces us to define the Hilbert transform for ultraspherical functions by

$$\varphi_n^\lambda(\theta)\mapsto \frac{\sqrt{n(n+2\lambda)}}{n+\lambda}\,\varphi_{n-1}^{\lambda+1}(\theta) . \tag{1.3}$$

Then, if $f(\theta)\in L^1$ has the expansion $\sum b_n\varphi_n^\lambda(\theta)$, defining the Poisson integral of $f$ by

$$f(t,\theta)=\sum_{n=0}^{\infty} e^{-t(n+\lambda)}b_n\varphi_n^\lambda(\theta) \tag{1.4}$$

and its conjugate Poisson integral by

$$\tilde f(t,\theta)=\sum_{n=1}^{\infty} e^{-t(n+\lambda)}\frac{\sqrt{n(n+2\lambda)}}{n+\lambda}\,b_n\varphi_{n-1}^{\lambda+1}(\theta) , \tag{1.5}$$

one can easily verify that they are related by the differential equation

$$\frac{\partial \tilde{f}}{\partial t} = \left(\frac{\partial}{\partial \theta} - \lambda \cot \theta\right) f . \tag{1.6}$$

Note that (1.3) maps an orthonormal set into only an orthogonal set but due to the factor in front of $\varphi_{n-1}^{\lambda+1}(\theta)$ we have (1.6) that will be crucial in what follows.

The main goal of the paper is to prove the existence of boundary values of the conjugate Poisson integrals of integrable functions. In §2 we discuss the Poisson integrals and prove their convergence at the boundary both almost everywhere and in $L^p$ norms, $1<p<\infty$. §3 is devoted to the study of conjugate Poisson integrals. When studying the Poisson and conjugate Poisson integrals the main problem is of course a good estimate for corresponding kernels or their derivatives that arise. It occurs, fortunately, that we can take a profit from deep estimates done by Muckenhoupt and Stein in the polynomial case. While it is quite straightforward when dealing with the Poisson kernel, a modification must be done in the conjugate case. The fact that unweighted case is considered explains that the estimates and arguments we use are somehow simpler than in the (weighted) polynomial case. As usual the letter $C$ will denote a constant varying from line to line.

**2. Poisson integrals.** Rather than working with the Poisson integral of $f$ given by (1.4) we will use the following more convenient version of it. Given $f(\theta) \in L^1$ with the expansion $\sum b_n \varphi_n^\lambda(\theta)$ the Poisson integral of $f$ is defined by

$$f(r, \theta) = \sum_{n=0}^{\infty} r^n b_n \varphi_n^\lambda(\theta) , \qquad 0<r<1 . \tag{2.1}$$

Then the Poisson kernel

$$P(r, \theta, \eta) = \sum_{n=0}^{\infty} r^n \varphi_n^\lambda(\theta) \varphi_n^\lambda(\eta) \tag{2.2}$$

differs from the Poisson kernel associated to polynomial ultraspherical expansions only by the factor $(\sin\theta \sin\eta)^\lambda$. Therefore, due to the fundamental estimate given by Muckenhoupt and Stein, [6, Lemma 1, p. 27], we have the following.

LEMMA 2.1. *The Poisson kernel* $P(r, \theta, \eta)$ *satisfies*

$$P(r, \theta, \eta) \le C \frac{1-r}{(1-r)^2 + (\theta-\eta)^2} , \tag{2.3}$$

*where* $C$ *is a constant independent of* $0<r<1$ *and* $0<\theta, \eta<\pi$.

Note that the estimate (4.1) in [6] is essentially given for $1/2<r<1$ but the inequality

$$|\varphi_n^\lambda(\theta)| \le C n^{2\lambda-1} \tag{2.4}$$

shows that $P(r, \theta, \eta)$ is uniformly bounded for $0<r<1/2$. Also positivity of the Poisson kernel follows from a well known formula

(2.5)
$$P(r, \theta, \eta)=\frac{\lambda}{\pi}(1-r^2)(\sin\theta\sin\eta)^\lambda\int_0^\pi\frac{(\sin\tau)^{2\lambda-1}}{(1-2r(\cos\theta\cos\eta+\sin\theta\sin\eta\cos\tau)+r^2)^{\lambda+1}}d\tau .$$

The inequality (2.4) allows us to write

$$f(r, \theta)=\int_0^\pi P(r, \theta, \eta)f(\eta)d\eta$$

as well as to differentiate (2.1) term by term to check that $f(r, \theta)$ is a $C^\infty$-function. In the polynomial expansion case the Poisson kernel has the integrals equal to one. In the case we consider we do not know the exact values of the integrals $\int_0^\pi P(r, \theta, \eta)d\eta$ but all we need to know is the uniform estimate

$$\int_0^\pi P(r, \theta, \eta)d\eta\le C \tag{2.6}$$

that follows from (2.3).

In what follows by $f^*$ we denote the usual Hardy-Littlewood maximal function of $f$, a locally integrable function on $(0, \pi)$, and by $\|f\|_p$ we mean the $L^p$-norm of $f$ with respect to the Lebesgue measure. The estimate (2.3) leads to the following result.

THEOREM 2.2. *Let $f\in L^1((0, \pi), d\theta)$ and let $f(r, \theta)$ denote the Poisson integral of $f$ given by* (2.1). *Then*

(a) $\sup_{r<1}|f(r, \theta)|\le Cf^*(\theta)$,

(b) $f(r, \theta)\to f(\theta)$ *a.e. as* $r\to 1$.

*Moreover, if $f\in L^p$, $1\le p\le\infty$, then*

(c) $\|f(r, \theta)\|_p\le C\|f\|_p$,

(d) $\|f(r, \theta)-f(\theta)\|_p\to 0$ *as* $r\to 1$, $p\ne\infty$.

PROOF. Almost everywhere convergence of Poisson integrals is clearly a consequence of the estimate (a) and the fact that the linear space spanned by the functions $\varphi_n^\lambda$, $n=0, 1, 2, \ldots$, is dense in $L^p((0, \pi), d\theta)$, $1\le p<\infty$. The inequality in (a) follows easily from (2.3). The estimate in (c) is implied by (2.6) and then (d) also follows.

Note that almost everywhere convergence in (b) may be obtained from the corresponding result for the polynomial expansions (but this is not the case of norm convergence in (d) or the weak-type (1, 1) estimate for the maximal function in (a)). More precisely, if $f\in L^1(d\theta)$ then $f(\theta)(\sin\theta)^{-\lambda}\in L^1((\sin\theta)^{2\lambda}d\theta)$ and

$$\sum_0^\infty r^n d(n, \lambda)^2\langle f(\theta)(\sin\theta)^{-\lambda}, P_n^\lambda(\cos\theta)\rangle_{L^2((\sin\theta)^{2\lambda}d\theta)}P_n^\lambda(\cos\theta)\to f(\theta)(\sin\theta)^{-\lambda}$$

almost everywhere as $r \to 1$ by [6, Theorem 2, p. 31]. Hence

$$\sum_0^\infty r^n \langle f, \varphi_n^\lambda \rangle_{L^2(d\theta)} \varphi_n^\lambda(\theta) \to f(\theta)$$

a.e. as $r \to 1$.

**3. Conjugate Poisson integrals.** Since for technical reasons we decided to work with the Poisson integral given by (2.1) rather than by (1.4) we now define $\tilde{f}(r, \theta)$, the conjugate Poisson integral of $f(\theta) \sim \sum b_n \varphi_n^\lambda(\theta)$ by

$$\tilde{f}(r, \theta) = \sum_{n=1}^\infty r^n \frac{\sqrt{n(n+2\lambda)}}{n+\lambda} b_n \varphi_{n-1}^{\lambda+1}(\theta) . \tag{3.1}$$

Thus (1.6), satisfied by $f(t, \theta)$ and $\tilde{f}(t, \theta)$, transforms to the differential equation

$$\frac{\partial}{\partial r}(r^\lambda \tilde{f}) = -r^{\lambda-1}\left(\frac{\partial}{\partial \theta} - \lambda \cot \theta\right) f \tag{3.2}$$

with $f(r, \theta)$ and $\tilde{f}(r, \theta)$ involved. Let

$$Q(r, \theta, \eta) = \sum_{n=1}^\infty r^n \frac{\sqrt{n(n+2\lambda)}}{n+\lambda} \varphi_{n-1}^{\lambda+1}(\theta) \varphi_n^\lambda(\eta) \tag{3.3}$$

so that

$$\tilde{f}(r, \theta) = \int_0^\pi Q(r, \theta, \eta) f(\eta) d\eta .$$

Then (3.2) gives

$$Q(r, \theta, \eta) = -r^{-\lambda} \int_0^r t^{\lambda-1}\left(\frac{\partial}{\partial \theta} - \lambda \cot \theta\right) P(t, \theta, \eta) dt , \tag{3.4}$$

where $P(t, \theta, \eta)$ is the Poisson kernel (2.2). To obtain $L^p$ theorems for conjugate functions we need good estimates for the kernel $Q$. In the case of polynomial expansions Muckenhoupt and Stein doing a careful analysis received a sophisticated estimate of conjugate kernel (cf. [6, Lemma 4 of §7]). Even if we cannot use this estimate directly (conjugate kernels does not differ by the factor $(\sin \theta \sin \eta)^\lambda$ as it was in the case of Poisson kernels), it occurs that we do rely on their work after a reduction is made. The estimate of $Q$ is the following.

LEMMA 3.1. *Let* $0 \le \theta \le \pi$, $0 \le \eta \le \pi/2$. *Then*

$$Q(r, \theta, \eta) = \begin{cases} O((\sin \eta)^{-1}) & \textit{if } \ 2\theta < \eta , \\ O((\sin \theta)^{-1}) & \textit{if } \ \theta > 2\eta . \end{cases}$$

*If* $\theta/2 < \eta < 2\theta$ *then*

$$Q(r, \theta, \eta)=c_\lambda r^\lambda \frac{\sin(\theta-\eta)}{1-2r\cos(\theta-\eta)+r^2}+O\left((\sin\theta)^{-1}\cdot\left(1+\log^+\left(\frac{\sin\theta\sin\eta}{1-\cos(\theta-\eta)}\right)\right)\right).$$

PROOF. Differentiating (2.5) shows

$$\left(\frac{\partial}{\partial\theta}-\lambda\cot\theta\right)P(t, \theta, \eta)=-\frac{\lambda(\lambda+1)}{\pi}t(1-t^2)(\sin\theta\sin\eta)^\lambda\int_0^\pi \frac{b(\sin\tau)^{2\lambda-1}}{(1-2at+t^2)^{\lambda+2}}d\tau,$$

where

$$a=\cos\theta\cos\eta+\sin\theta\sin\eta\cos\tau$$

and

$$b=2(\sin\theta\cos\eta-\cos\theta\sin\eta\cos\tau).$$

Thus, (3.4) and the elementary identity

$$\int_0^r \frac{t^\lambda(1-t^2)}{(1-2at+t^2)^{\lambda+2}}dt=\frac{1}{\lambda+1}\cdot\frac{r^{\lambda+1}}{(1-2ar+r^2)^{\lambda+1}}$$

give

$$Q(r, \theta, \eta)=\frac{\lambda}{\pi}(\sin\theta\sin\eta)^\lambda\int_0^\pi \frac{D_\theta}{D^{\lambda+1}}(\sin\tau)^{2\lambda-1}d\tau, \tag{3.5}$$

where

$$D=1-2r(\cos\theta\cos\eta+\sin\theta\sin\eta\cos\tau)+r^2.$$

This is the point where we simply take the profit from hard estimates of Muckenhoupt and Stein. Denoting

$$R(r, \theta, \eta)=\int_0^\pi D_\theta D^{-\lambda-1}(\sin\tau)^{2\lambda-1}d\tau$$

we see that $R(r, \theta, \eta)$, up to an ignorable factor, is the kernel considered by the forementioned authors in (7.4) of [6] for which the following estimates hold:

$$R(r, \theta, \eta)=O((\sin\eta)^{-2\lambda-1}) \quad \text{if} \quad 2\theta<\eta, \tag{3.6}$$

$$R(r, \theta, \eta)=O((\sin\theta)^{-2\lambda-1}) \quad \text{if} \quad \theta/2>\eta. \tag{3.7}$$

Therefore, in the case $2\theta<\eta$, multiplying (3.6) by $(\sin\theta\sin\eta)^\lambda$ and using $1/\sin\eta\leq C(1/\sin\theta)$ we get

$$Q(r, \theta, \eta)=O((\sin\eta)^{-1}).$$

Consider now the second case when $\theta>2\eta$. If $0<\theta<\pi/2$ then $\sin\eta<\sin\theta$ and multiplying (3.7) by $(\sin\theta\sin\eta)^\lambda$ gives

$$Q(r,\theta,\eta)=O((\sin\theta)^{-1}).$$

If $\pi/2\le\theta\le 3\pi/4$ then $|D_\theta|\le C\sin\eta$ and

$$D^{\lambda+1}\ge C(1-\cos(\theta-\eta))^{\lambda+1}=C2^{\lambda+1}\left(\sin^2\frac{\theta-\eta}{2}\right)^{\lambda+1}\ge C(\sin\theta\sin\eta)^{\lambda+1},$$

hence $R(r,\theta,\eta)=O(\sin\eta/(\sin\theta\sin\eta)^{\lambda+1})$ which gives the required estimate for $Q$. The case $3\pi/4\le\theta\le\pi$ is trivial since then $D^{\lambda+1}\ge C$. In the most critical range, $\theta/2\le\eta\le 2\theta$ we only note that the error term $O(|\sin(\theta-\eta)|(r\sin\theta\sin\eta)^{-\lambda-1})$ on the second line on p. 38 of [6] may be changed to $O(\sin\eta/(\sin\theta\sin\eta)^{\lambda+1})$ and the error term on the seventh line on p. 38 is exactly what we need to get our estimate for $R(r,\theta,\eta)$ that implies the required estimate for $Q$. This concludes the proof of the lemma.

The estimates of Lemma 3.1 are crucial to prove the main results of this section.

THEOREM 3.2. *The conjugate maximal operator is of weak-type* (1, 1) *which means that*

$$\int_{E_s} d\theta\le\frac{C}{s}\|f\|_1,$$

*where*

$$E_s=\left\{\theta\in(0,\pi):\sup_{0<r<1}|\tilde f(r,\theta)|>s\right\}.$$

PROOF. It suffices to follow line by line the proof of Theorem 4 in [6, p. 38] using the estimates of Lemma 3.1 instead of those from Lemma 4 in [6, p. 35]. Clearly we also use the classical weak-type result for the conjugate maximal operator for trigonometric series.

COROLLARY 3.3. *Let $f(\theta)\in L^1((0,\pi),d\theta)$ and let $\sum_{n=0}^{\infty}b_n\varphi_n^\lambda(\theta)$ denote the expansion of $f$ with respect to the ultraspherical functions $\varphi_n^\lambda$, $n=0,1,\ldots$. Then the limit $\lim_{r\to1}\tilde f(r,\theta)$, denoted by $\tilde f(\theta)$, exists almost everywhere and*

$$\int_{\{|\tilde f|>s\}} d\theta\le\frac{C}{s}\|f\|_1.$$

*Moreover, if $f(\theta)\in L^p$, $1<p<\infty$, then $\|\tilde f\|_p\le C_p\|f\|_p$, $\tilde f(r,\theta)$ converges in $L^p$ norm to $\tilde f(\theta)$ as $r\to1$ and $\tilde f(\theta)$ has the expansion*

$$\tilde f(\theta)\sim\sum_{n=1}^{\infty}\frac{\sqrt{n(n+2\lambda)}}{n+\lambda}b_n\varphi_{n-1}^{\lambda+1}(\theta).$$

PROOF. The existence of the boundary value $\tilde f(\theta)$ follows from the estimate for the conjugate maximal operator and the fact that the linear space generated by the functions $\varphi_n^\lambda$ is dense in $L^p$, $1 \leq p < \infty$.

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