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DISCRETENESS OF FINITE DIMENSIONAL TEICHMÜLLER SPACES IN THE UNIVERSAL TEICHMÜLLER SPACE

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Abstract. The universal Teichmüller space contains all Teichmüller spaces of hyperbolic Riemann surfaces. We shall investigate how the Teichmüller spaces of punctured spheres vary in the universal Teichmüller space when the base points change.

1. Introduction. The universal Teichmüller space $T(1)$ is the quotient space of all quasiconformal automorphisms of the upper half plane $\boldsymbol{H}$ fixing 0, 1, and ∞. (We refer to [L] for definitions and fundamental results on Teichmüller spaces.) Hence for any two such quasiconformal homeomorphisms f_1 and f_2, the assignment $([f_1], [f_2]) \mapsto [f_1 \circ f_2^{-1}]$ is well-defined, where $[f_1]$ and $[f_2]$ are the equivalence classes of $T(1)$ determined by f_1 and f_2, respectively. This induces the group structure on $T(1)$. On the other hand, $T(1)$ is a topological space equipped with the Teichmüller distance and is embedded via the Bers embedding onto a bounded domain of the Banach space

$$A_\infty(L, 1) = \left\{ \varphi\text{: holomorphic on } L,\ \|\varphi\| = \sup_{z \in L} |(\operatorname{Im} z)^2 \varphi(z)| < \infty \right\},$$

where L denotes the lower half plane. However, it is well-known (cf. [L, Theorem 3.3 in Chapter III]) that $T(1)$ is not a topological group with respect to this topology. In this note, we shall observe this fact from a new point of view. Namely, we shall show that finite dimensional Teichmüller spaces of quasiconformally equivalent Riemann surfaces of certain kinds are embedded in $T(1)$ discretely.

Our results are motivated by the paper [K], which contains a subtle argument. Let Γ be a Fuchsian group acting on $\boldsymbol{H}$. For each Beltrami coefficient μ for Γ, we denote by f^μ the quasiconformal homeomorphism of $\boldsymbol{H}$ fixing 0, 1, and ∞ with Beltrami coefficient μ. The Teichmüller space $T(\Gamma)$ is the quotient space of all such quasiconformal mappings, and is naturally identified with a subset of $T(1)$. The argument in [K] would be made simpler if the assignment $[f^\mu]_\Gamma \mapsto [(f^\mu)^{-1}]$ is a continuous mapping of $T(\Gamma)$ into $T(1)$ when $\boldsymbol{H}/\Gamma$ is a Riemann surface of finite type, where $[\cdot]_\Gamma$ and $[\cdot]$ denote the equivalence classes in $T(\Gamma)$ and $T(1)$ respectively. We disprove that the assignment $[f^\mu]_\Gamma \mapsto [(f^\mu)^{-1}]$ is continuous as follows. Identify $T(1)$ with its image by the Bers

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embedding in $A_\infty(L, 1)$, and identify all Teichmüller spaces with corresponding subset in $A_\infty(L, 1)$ via the Bers embedding. Then for a Fuchsian group Γ, the Teichmüller space $T(\Gamma)$ is contained in the subspace $A_\infty(L, \Gamma)$ of $A_\infty(L, 1)$ defined by

$$A_\infty(L, \Gamma) = \{\varphi \in A_\infty(L, 1) : \varphi \circ \gamma \times (\gamma')^2 = \varphi \text{ for all } \gamma \in \Gamma\} .$$

Setting $\Gamma^\mu = f^\mu \Gamma (f^\mu)^{-1}$, we investigate how the subset $T(\Gamma^\mu) \subset A_\infty(L, \Gamma) \subset A_\infty(L, 1)$ in $T(1)$ varies as $[f^\mu]$ varies in a neighborhood of $[\text{id.}] = [f^0]$ in $T(\Gamma)$. Here, the topology of $T(\Gamma)$ is induced by the Teichmüller distance $d_{T(\Gamma)}$, which induces the same topology as that by $d_{T(1)} | T(\Gamma)$.

THEOREM 1. *Let Γ be a Fuchsian group acting on $\mathbf{H}$ such that $\mathbf{H}/\Gamma$ is a Riemann surface of $(0, m)$ type with $m > 3$. Then for each $[f^{\mu_0}]_\Gamma \in T(\Gamma)$ and for each positive number k there exists a positive number $\varepsilon = \varepsilon([\mu_0], k)$ and a neighborhood U of $[f^{\mu_0}]_\Gamma$ in $T(\Gamma)$ such that*

$$\text{dist}(T(\Gamma^{\mu_0}) \cap B_k^c,\ T(\Gamma^\mu) \cap B_k^c) \geq \varepsilon$$

for all $[f^\mu]_\Gamma \in U \setminus \{[f^{\mu_0}]\}$. Here, dist *denotes the distance in the Banach space $A_\infty(L, 1)$, and B_k is the ball of radius k centered at 0 in $A_\infty(L, 1)$.*

This theorem implies that, in investigating Teichmüller theory via quasiconformal mappings, one has to carefully treat variations with respect to the source surfaces. This is in contrast with the approach to Teichmüller theory via harmonic mappings, where variations with respect to source surfaces are studied as well as variations with respect to target surfaces (see Wolf [W], for example).

COROLLARY. *Let Γ be a Fuchsian group as above. For each $[f^{\mu_0}]_\Gamma \in T(\Gamma)$, the mapping $T(\Gamma) \to T(1)$ defined by $[f^\mu]_\Gamma \mapsto [f^{\mu_0} \circ (f^\mu)^{-1}]$ is continuous only at the point $[f^{\mu_0}]_\Gamma$. In particular, the universal Teichmüller space is not a topological group.*

REMARK. Recently, this observation was completely extended by Matsuzaki [M] to what one might expect. After the preparation of this paper the author noticed the result of Gardiner-Sullivan [GS] which includes closely related topics.

2. The proofs of Theorem 1 and the Corollary. Before proving Theorem 1, we show how to derive the Corollary from Theorem 1.

PROOF OF THE COROLLARY. It is easy to see that $\lim_{[f^\mu]_\Gamma \to [f^{\mu_0}]_\Gamma} [f^{\mu_0} \circ (f^\mu)^{-1}] = [\text{id.}]$.

Assume conversely that $[f^\mu]_\Gamma \mapsto [f^{\mu_0} \circ (f^\mu)^{-1}]$ is continuous at a point $[f^\nu]_\Gamma \in T(\Gamma)$. Let φ_ν denote the point of $A_\infty(L, 1)$ determined by $[f^{\mu_0} \circ (f^\nu)^{-1}]$. Then

$$\lim_{[f^{\mu_0} \circ (f^\mu)^{-1}] \to [f^{\mu_0} \circ (f^\nu)^{-1}]} \|\varphi_\nu - \varphi_\mu\| = 0 ,$$

since the Bers embedding $T(1) \hookrightarrow A_\infty(L, 1)$ is continuous. Hence if $\|\varphi_\nu\| > 0$, then there exists a positive number δ such that $\|\varphi_\mu\| > \|\varphi_\nu\|/2$ whenever $[f^\mu]_\Gamma \in T(\Gamma)$ satisfies $d_{T(\Gamma)}([f^\mu]_\Gamma, [f^\nu]_\Gamma) < \delta$. On the other hand, $f^{\mu_0} \circ (f^\nu)^{-1}$ and $f^{\mu_0} \circ (f^\mu)^{-1}$ determine points in $T(\Gamma^\mu)$ and $T(\Gamma^\nu)$, respectively. Hence we have

$$\begin{aligned}\|\varphi_\nu - \varphi_\mu\| &\geq \operatorname{dist}(T(\Gamma^\mu) \cap B^c_{\|\varphi_\nu\|/2}, T(\Gamma^\nu) \cap B^c_{\|\varphi_\nu\|/2}) \\ &\geq \varepsilon([f^\nu]_\Gamma, \|\varphi_\nu\|/2) > 0\end{aligned}$$

if $[f^\mu]_\Gamma$ is sufficiently close to $[f^\nu]_\Gamma$ and $[f^\mu]_\Gamma \neq [f^\nu]$ by Theorem 1. These two formulae yield a contradiction. Hence we have $\|\varphi_\nu\| = 0$, namely, $[f^\nu]_\Gamma = [f^{\mu_0}]_\Gamma$. □

Proof of Theorem 1. We shall show the claim by contradiction. If the statement of Theorem 1 is false, then there exists a point $[f^{\mu_0}]_\Gamma \in T(\Gamma)$ and a sequence $\{[f^{\mu_n}]_\Gamma\}$ with $\lim_{n\to\infty}[f^{\mu_n}]_\Gamma = [f^{\mu_0}]_\Gamma$ such that there exist $\varphi \in T(\Gamma^\mu) \subset A_\infty(L, 1)$ and $\varphi_n \in T(\Gamma^{\mu_n}) \subset A_\infty(L, 1)$ with $\|\varphi\| > k$, $\|\varphi_n\| > k$ for some $k > 0$ and $\lim_{n\to\infty}\|\varphi_n - \varphi\| = 0$. Namely,

$$\lim_{n\to\infty} \sup_{z\in L} |(\operatorname{Im} z)^2\varphi - (\operatorname{Im} z)^2\varphi_n(z)| = 0\,.$$

On the other hand, $|(\operatorname{Im} z)^2\varphi(z)|$ and $|(\operatorname{Im} z)^2\varphi_n(z)|$ are automorphic functions for Γ and Γ^{μ_n}, respectively, hence induce continuous functions on L/Γ^{μ_0} and L/Γ^{μ_n}, respectively. We may assume that $L/\Gamma^{\mu_0} = \hat{C}\setminus\{0, 1, \infty, p_1, \ldots, p_{m-3}\}$, $L/\Gamma^{\mu_n} = \hat{C}\setminus\{0, 1, \infty, p_1^{(n)}, \ldots, p_{m-3}^{(n)}\}$ and $\lim_{n\to\infty} p_j^{(n)} = p_j$, $j = 1, 2, \ldots, m-3$. Then the functions induced by $|(\operatorname{Im} z)^2\varphi(z)|$ and $|(\operatorname{Im} z)^2\varphi_n(z)|$ are of the form $|\lambda^{-2}\psi|$ and $|\lambda_n^{-2}\psi_n|$, where λ and λ_n are the Poincaré densities of $\hat{C}\setminus\{0, 1, \infty, p_1, \ldots, p_{m-3}\}$ and $\hat{C}\setminus\{0, 1, \infty, p_1^{(n)}, \ldots, p_{m-3}^{(n)}\}$, respectively, and ψ and ψ_n are integrable holomorphic quadratic differentials on $\hat{C}\setminus\{0, 1, \infty, p_1, \ldots, p_{m-3}\}$ and $\hat{C}\setminus\{0, 1, \infty, p_1^{(n)}, \ldots, p_{m-3}^{(n)}\}$, respectively.

Now we show that $\lim_{n\to\infty}\| |\lambda^{-2}\psi| - |\lambda_n^{-2}\psi_n| \|_\infty = 0$, where $\|\cdot\|_\infty$ denotes the supremum norm on $\hat{C}$. First, since ψ and ψ_n are regarded as integrable rational functions on $\hat{C}$ with at most simple poles at 0, 1, p_j and $p_j^{(n)}$, respectively ($j = 1, 2, \ldots, m-3$), they are represented as

$$\psi(z) = \sum_{j=1}^{m-3} \frac{a_j}{z(z-1)(z-p_j)}, \qquad \psi_n(z) = \sum_{j=1}^{m-3} \frac{a_j^{(n)}}{z(z-1)(z-p_j^{(n)})}\,.$$

By the assumption $\lim_{n\to\infty}\sup_{z\in L}|(\operatorname{Im} z)^2\varphi - (\operatorname{Im} z)^2\varphi_n(z)| = 0$, the sequence $\{\psi_n\}$ converges to ψ on compact sets in $\hat{C}\setminus\{0, 1, \infty, p_1, \ldots, p_{m-3}\}$. Hence we have $\lim_{n\to\infty} a_j^{(n)} = a_j$. Next we observe the behavior of λ_n^{-2} and λ^{-2} near $p_j^{(n)}$ and p_j. Fix arbitrary $j \in \{1, 2, \ldots, m-3\}$ and let ρ_n and ρ denote the Poincaré density of $\hat{C}\setminus\{0, \infty, p_j^{(n)}\}$ and $\hat{C}\setminus\{0, \infty, p_j\}$, respectively. Then we have $\rho_n^{-2} \geq \lambda_n^{-2}$ and $\rho^{-2} \geq \lambda^{-2}$. Note that $\rho_n = \gamma_n^*\rho$ with $\gamma_n(z) = (p_j/p_j^{(n)}) \times z$ and for a local parameter ζ around p_j with $\zeta(p_j) = 0$, $\rho^{-2}(\zeta) = |\zeta|^2(\log(1/|\zeta|))^2$. From this formula together with the behavior of ψ_n, we have that for each $\varepsilon > 0$ there exists a small disc D_ε^j centered at p_j such that $\sup_{D_\varepsilon^j}|\rho^{-2}\psi| < \varepsilon$ and $\sup_{D_\varepsilon^j}|\rho_n^{-2}\psi_n| < \varepsilon$ for all sufficiently large n. It follows that

$\sup_{D_\varepsilon^j}|\lambda^{-2}\psi|<\varepsilon$ and $\sup_{D_\varepsilon^j}|\lambda_n^{-2}\psi_n|<\varepsilon$ for sufficiently large n. On the other hand, the sequences $\{\lambda_n\}$ and $\{\psi_n\}$ converge uniformly on $\hat{C}\setminus\bigcup_{j=1}^{m-3}D_\varepsilon^j$ to λ and ψ respectively. Hence we have $\||\lambda_n^{-2}\psi_n|-|\lambda^{-2}\psi|\|_\infty<2\varepsilon$ for sufficiently large n.

Now, take $\delta>0$ sufficiently small so that $3\delta<\|\varphi\|$ and that the set

$$E_{3\delta}=\{z\in\hat{C}:|\lambda^{-2}(z)\psi(z)|\leq 3\delta\}$$

is contained in a disjoint union of topological discs around zeros of ψ and the punctures $\{0, 1, \infty, p_1, \dots, p_{m-3}\}$. The latter set is lifted onto the disjoint union of relatively compact topological discs and horospheres at parabolic fixed points of Γ on L via a universal covering $\pi\colon L\to\hat{C}\setminus\{0, 1, \infty, p_1, \dots, p_{m-3}\}$. By the above observation we may assume that $E_{2\delta}^n=\{z\in\hat{C};|\lambda_n^{-2}\psi|<2\delta\}\subset E_{3\delta}$ and $E_\delta\subset E_{2\delta}^n$ if n is sufficiently large. Here, note that there exists a finite set of parabolic fixed points F of Γ such that if $[f^{\mu_n}]\neq[f^{\mu_0}]$ then $f^{\mu_n}(q)\neq f^{\mu_0}(q)$ for some $q\in F$. Taking a subsequence if necessary, we may assume that there exists a parabolic fixed point q of Γ corresponding to the puncture p_1 such that $f^{\mu_n}(q)\neq f^{\mu_0}(q)$ for all n.

There exists a point $o\in L$ such that the closure of the Dirichlet fundamental region D of Γ^{μ_0} centered at o contains the parabolic fixed point $f^{\mu_0}(q)$ of Γ^{μ_0}. Let γ^{μ_0} denote the primitive parabolic element of Γ^{μ_0} with fixed point $f^{\mu_0}(q)$. There exists a smooth $\langle\gamma^{\mu_0}\rangle$ invariant Jordan curve J surrounding a Jordan domain R such that $\sup_R|(\operatorname{Im} z)^2\varphi(z)|\leq\delta$. It follows that $\sup_R|(\operatorname{Im} z)^2\varphi_n(z)|\leq 2\delta$ if n is sufficiently large. On the other hand, there exists a Dirichlet region D_n for Γ^{μ_n} with $\overline{D_n}\ni f^{\mu_n}(q)$ such that $\{D_n\}$ converges to D in Hausdorff topology. Hence $D_n\cap J$ for large n consists of a finite union of cross cuts of D_n, at least one components of which divides D_n into two components, one is projected onto a punctured disc around $p_1^{(n)}$ and the other is projected onto its complement. Choose such a component of $D_n\cap J$ and denote it by c_n. Let γ^{μ_n} denote the primitive parabolic element of Γ^{μ_n} with fixed point $f^{\mu_n}(q)$. Then $J_n=\bigcup_{h=-\infty}^{\infty}(\gamma^{\mu_n})^h c_n\cup\{f^{\mu_n}(q)\}$ is a Jordan curve surrounding a Jordan region R_n. Since $\sup_{c_n}|(\operatorname{Im} z)^2\varphi_n(z)|\leq 2\delta$ and $|(\operatorname{Im} z)^2\varphi_n(z)|$ is a Γ^{μ_n} invariant function, $\sup_{J_n}|(\operatorname{Im} z)^2\varphi_n(z)|\leq 2\delta$. Hence it follows that both of J_n and J are contained in some components of $\pi_n^{-1}(E_{2\delta}^n)$, where $\pi_n\colon L\to\hat{C}\setminus\{0, 1, \infty, p_1^{(n)}, \dots, p_{m-3}^{(n)}\}$ is a universal covering, chosen to be a Γ^{μ_n}-invariant function.

On the other hand, since $J_n\cap J\neq\varnothing$, J_n and J are contained in the same component of $\pi_n^{-1}(E_{2\delta}^n)$. This is, however, impossible, since $\pi_n^{-1}(E_{2\delta}^n)$ is contained in a disjoint union of Γ^{μ_n}, as $E_{2\delta}^n$ is contained in a disjoint union of discs around zeros of ψ_n and those around the punctures, while $J_n\cap \boldsymbol{R}=f^{\mu_n}(q)\neq f^{\mu_0}(q)=J\cap\boldsymbol{R}$. □

3. Another approach. If one needs only the fact that the mapping $T(\Gamma)\to T(1)$ defined by $[f^\mu]_\Gamma\mapsto[(f^\mu)^{-1}]$ is not continuous, there is a simpler approach. We can deal with more general Fuchsian groups and the approach is completely different from the previous section.

THEOREM 2. *Let Γ be a finitely generated Fuchsian group of the first kind acting on $\boldsymbol{H}$ such that $\boldsymbol{H}/\Gamma$ is a hyperbolic Riemann surface with at least one puncture. Then for each $[f^{\mu}]_{\Gamma} \in T(\Gamma)$ and each positive number $k>0$, there exist a positive number $\varepsilon = \varepsilon([f^{\mu}]_{\Gamma}, k)$ and a sequence $\{[f^{\mu_n}]_{\Gamma}\} \subset T(\Gamma)$ with $\lim_{n\to\infty} [f^{\mu_n}]_{\Gamma} = [f^{\mu}]_{\Gamma}$ such that*

$$\mathrm{dist}(T(\Gamma^{\mu_n}) \cap B_k^c,\ T(\Gamma^{\mu}) \cap B_k^c) \geq \varepsilon\,,$$

where dist *stands for the distance in the Banach space $A_{\infty}(L, 1)$ and B_k is the ball of $A_{\infty}(L, 1)$ with radius k centered at* 0.

PROOF. For each parabolic fixed point q of Γ, the function on $T(\Gamma)$ defined by $[f^{\nu}]_{\Gamma} \mapsto f^{\nu}(q)$ for $[f^{\nu}]_{\Gamma} \in T(\Gamma)$ is real analytic. We choose a parabolic fixed point q such that this function is non-constant. Then for each $[f^{\mu}]_{\Gamma} \in T(\Gamma)$ and each neighborhood U of $[f^{\mu}]_{\Gamma}$ in $T(\Gamma)$, the set $\{f^{\nu}(q); [f^{\nu}]_{\Gamma} \in U\}$ contains a neighborhood of $f^{\mu}(q)$ in the real axis. Let l_{ν} denote the vertical line segment terminating in $f^{\nu}(q)$ with Euclidean length 1. Then, by Myrberg's theorem, for almost all l_{ν} the image of l_{ν} under the universal covering mapping $\pi: L \to L/\Gamma$ is dense in L/Γ. Moreover, the image of any subray of l_{ν} terminating in $f^{\nu}(q)$ is dense in L/Γ for almost all l_{ν}. Choose a sequence $\{[f^{\mu_n}]_{\Gamma}\}$ such that each l_{μ_n} has this property and $\lim_{n\to\infty} [f^{\mu_n}]_{\Gamma} = [f^{\mu}]_{\Gamma}$.

Now, suppose that $\{[f^{\mu_n}]_{\Gamma}\}$ does not satisfy the claim of Theorem 2. Then we may assume that there exist $\varphi \in T(\Gamma^{\mu}) \cap B_k^c$ and $\varphi_n \in T(\Gamma^{\mu_n}) \cap B_k^c$ with $\lim_{n\to\infty} \|\varphi - \varphi_n\| = 0$. We shall draw a contradiction. For each n, l_{μ_n} terminates in the parabolic fixed point $f^{\mu_n}(q)$, hence there exists a point z_n on l_{μ_n} such that the part l'_{μ_n} of l_{μ_n} connecting z_n and $f^{\mu_n}(q)$ satisfies $\sup_{z\in l'_{\mu_n}} |(\mathrm{Im}\, z)^2 \varphi_n(z)| < k/3$. For sufficiently large n, we have $\|\varphi - \varphi_n\| < k/3$. It follows that $\sup_{z\in l'_{\mu_n}} |(\mathrm{Im}\, z)^2 \varphi(z)| < 2k/3$. On the other hand, the image $\pi(l'_{\mu_n})$ is dense in L/Γ and $|(\mathrm{Im}\, z)^2 \varphi(z)|$ induces a continuous function on L/Γ. It follows that $\sup_{z\in l'_{\mu_n}} |(\mathrm{Im}\, z)^2 \varphi(z)| = \|\varphi\| > k$. This is a contradiction. □

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