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LEFT CELLS IN AFFINE WEYL GROUPS

Dedicated to Professor R. W. Carter on his sixtieth birthday.

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Abstract. We prove a property of left cells in certain crystallographic groups W, by which we formulate an algorithm to find a representative set of left cells of W in any given two-sided cell. As an illustration, we make some applications of this algorithm to the case where W is the affine Weyl group of type $\tilde{F}_4$.

The cells of affine Weyl groups W, as defined by Kazhdan and Lusztig in [6], have been described explicitly in certain special cases: for W of rank 2, see [11]; for W of type $\tilde{A}_n$, see [16], [10]; for W of rank 3, see [1], [4]; for the cells with a-values 1, 2 and $|\Phi|/2$ in a general W, see [2], [8], [9], [18], [19], where Φ is the root system determined by W. It is known that there exists a bijection between the set of two-sided cells in an affine Weyl group W and the set of unipotent classes in a certain complex reductive group G associated with W. It is also known that the value of the a-function on a two-sided cell of W is equal to the dimension of the variety of Borel subgroups of G containing an element of the corresponding unipotent class (see [14]). Thus for an affine Weyl group W, the two-sided cells of W are relatively well understood to certain extent. But the classification of left cells in a given two-sided cell of W is not known in general, even the number of these left cells. In the present paper, we shall introduce an algorithm to find a representative set of left cells of W' in a given two-sided cell, where W' is a group belonging to a certain family of crystallographic groups including all the Weyl groups and all the affine Weyl groups. This algorithm has been used by several person to describe the left cells in the following cases: for W of types $\tilde{B}_4$, $\tilde{C}_4$, $\tilde{D}_4$ (see [20], [21], [24]), for the ones of a-value 3 in a general irreducible affine Weyl group W (see [15]), for the ones of a-values 4, 5, in W of type $\tilde{F}_4$.

The content of this paper is organized as follows. § 1 serves as the preliminaries. Some basic concepts and known results concerning the cells of certain crystallographic group W are introduced. We prove a property of left cells of W in § 2. This property is crucial for us to formulate an algorithm in § 3, which is the main purpose of this paper. The algorithm given in § 3 is to find a representative set of left cells of W in a given two-sided cell. It needs some technique in applying this algorithm. Thus a part of § 3 together with the whole of § 4 provide more concepts and results to this end.

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Finally, in § 5, we illustrate this algorithm by applying it to finding representative sets of left cells in certain two-sided cells of the affine Weyl group of type $\tilde{F}_4$.

1. Preliminaries.

1.1. Let $W=(W, S)$ be a Coxeter group with S its Coxeter generator set. Let $\leq$ be the Bruhat order on W: $y\leq w$ in W means that there exist some reduced forms $w=s_1s_2\cdots s_l$ and $y=s_{i_1}s_{i_2}\cdots s_{i_t}$ with $s_i\in S$ such that $i_1, i_2, \ldots, i_t$ is a subsequence of $1, 2, \ldots, l$. For $w\in W$, we denote by $\ell(w)$ the length of w.

1.2. Let $\mathscr{A}=\boldsymbol{Z}[u, u^{-1}]$ be the ring of all Laurent polynomials in an indeterminate u with integer coefficients. The Hecke algebra $\mathscr{H}$ of W over $\mathscr{A}$ has two sets of $\mathscr{A}$-bases $\{\tilde{T}_x\}_{x\in W}$ and $\{C_w\}_{w\in W}$ which satisfy the relations

$$(1.2.1)\qquad \begin{cases}\tilde{T}_w\tilde{T}_{w'}=\tilde{T}_{ww'}, & \text{if}\quad \ell(ww')=\ell(w)+\ell(w');\\ (\tilde{T}_s-u^{-1})(\tilde{T}_s+u)=0, & \text{if}\quad s\in S,\end{cases}$$

and

$$(1.2.2)\qquad C_w=\sum_{y\leq w}u^{\ell(w)-\ell(y)}P_{y,w}(u^{-2})\tilde{T}_y,$$

where $P_{y,w}\in\boldsymbol{Z}[u]$ satisfies that $P_{w,w}=1$, $P_{y,w}=0$ if $y\nleq w$ and $\deg P_{y,w}\leq(1/2)(\ell(w)-\ell(y)-1)$ if $y<w$. The $P_{y,w}$'s are called Kazhdan-Lusztig polynomials [6].

1.3. For $y, w\in W$ with $\ell(y)\leq\ell(w)$, we denote by $\mu(y, w)$ or $\mu(w, y)$ the coefficient of $u^{(1/2)(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$. We say that y and w are jointed, and written y—w, if $\mu(y, w)\neq 0$. To any $x\in W$, we associate two subsets of S:

$$\mathscr{L}(x)=\{s\in S\mid sx<x\}\quad\text{and}\quad \mathscr{R}(x)=\{s\in S\mid xs<x\}.$$

We have the following relations: for any $x\in W$ and $s\in S$,

$$(1.3.1)\qquad C_sC_x=\begin{cases}(u^{-1}+u)C_x, & \text{if}\quad s\in\mathscr{L}(x);\\ \sum\limits_{\substack{y—x\\ sy<y}}\mu(x, y)C_y, & \text{if}\quad s\notin\mathscr{L}(x);\end{cases}$$

and

$$(1.3.2)\qquad C_xC_s=\begin{cases}(u^{-1}+u)C_x, & \text{if}\quad s\in\mathscr{R}(x);\\ \sum\limits_{\substack{y—x\\ ys<y}}\mu(x, y)C_y, & \text{if}\quad s\notin\mathscr{R}(x);\end{cases}$$

where the numbers of elements y occurring on the right hand sides of (1.3.1) and (1.3.2) are finite. Moreover, $\{C_s\mid s\in S\}$ forms a generator set of the algebra $\mathscr{H}$ over $\mathscr{A}$.

1.4. In the present paper, we assume once and forever that W is irreducible (i.e. its Coxeter diagram is connected) and satisfies the following conditions:

(1) For any $x, y, z \in W$, we define $h_{x,y,z} \in \mathscr{A}$ by

$$C_x C_y = \sum_z h_{x,y,z} C_z \,. \tag{1.4.1}$$

Then there exists a positive integer N such that

$$u^N h_{x,y,z} \in \boldsymbol{Z}[u]\,, \qquad \text{for all} \quad x, y, z \in W; \tag{1.4.2}$$

(2) $P_{y,w}$ has non-negative coefficients for any pair $y, w \in W$.

These include all the Weyl groups and all the affine Weyl groups (cf. [6], [7]).

1.5. Let $x, y \in W$. We denote $x \leq_L y$ (resp. $x \leq_R y$), if there exists some $w \in W$ with $h_{w,y,x} \neq 0$ (resp. $h_{y,w,x} \neq 0$). We denote $x \leq_{LR} y$, if there exists some $w \in W$ with $x \leq_L w \leq_R y$, or equivalently, if there exists some $w' \in W$ with $x \leq_R w' \leq_L y$. We write $x \sim_L y$ (resp. $x \sim_R y$, resp. $x \sim_{LR} y$), if the relation $x \leq_L y \leq_L x$ (resp. $x \leq_R y \leq_R x$, resp. $x \leq_{LR} y \leq_{LR} x$) holds. These are equivalence relations on W, and the equivalence clases of W with respect to $\sim_L$ (resp. $\sim_R$, resp. $\sim_{LR}$) are called the left (resp. right, resp. two-sided) cells of W. The preorders $\leq_L$, $\leq_R$ and $\leq_{LR}$ on elements of W induce partial orders on the corresponding cells of W.

1.6. By (1.3.1) and (1.3.2), we see that for $x, y, z \in W$, $h_{x,y,z}$ has non-negative coefficients as a Laurent polynomial in u and $h_{x,y,z}(u) = h_{x,y,z}(u^{-1})$. By the assumption 1.4, (1), we can define a function $a: W \to \boldsymbol{N}$ by

$$a(z) = \max_{x,y \in W} \deg h_{x,y,z}\,, \qquad \text{for} \quad z \in W\,. \tag{1.6.1}$$

The following are known properties of the a-function:

(1) $x \leq_{LR} y \Rightarrow a(x) \geq a(y)$. In particular, $x \sim_{LR} y \Rightarrow a(x) = a(y)$. So we may define the a-value $a(\Gamma)$ on a left (resp. right, resp. two-sided) cell Γ of W by $a(x)$ for any $x \in \Gamma$ (cf. [11]).

(2) $a(x) = a(y)$ and $x \leq_L y$ (resp. $x \leq_R y) \Rightarrow x \sim_L y$ (resp. $x \sim_R y$) (cf. [12]).

(3) Let $\delta(z) = \deg P_{e,z}$ for $z \in W$, where e is the identity of the group W. Then the inequality

$$\ell(z) - 2\delta(z) - a(z) \geq 0 \tag{1.6.2}$$

holds for any $z \in W$. The set

$$\mathscr{D} = \{ w \in W \mid \ell(w) - 2\delta(w) - a(w) = 0 \} \tag{1.6.3}$$

is a finite set of involutions. Each left (resp. right) cell of W contains a unique element of $\mathscr{D}$ (cf. [12]).

1.7. Let W be an irreducible affine Weyl group of type $\tilde{X}$. Let G be the connected reductive algebraic group over $\boldsymbol{C}$ of type $X^\vee$, where $X^\vee$ is the dual of X. Then the following result is due to Lusztig (cf. [14]).

THEOREM. *There exists a bijection $\boldsymbol{u} \mapsto c(\boldsymbol{u})$ from the set $\mathfrak{U}(G)$ of unipotent conjugacy* *classes in G to the set* $\mathrm{Cell}(W)$ *of two-sided cells in W satisfying*

(1) $a(\boldsymbol{c}(\boldsymbol{u}))=\dim \mathfrak{B}_u$;

(2) $\boldsymbol{c}(\boldsymbol{u})$ *is finite if and only if $Z_G^\circ(u)$ is unipotent*;

(3) *For any $\boldsymbol{c}\in \mathrm{Cell}(W)$, there exists some $I\subsetneqq S$ with $\boldsymbol{c}\cap W_I\neq\emptyset$,*

where u is any element in $\boldsymbol{u}$, $\dim \mathfrak{B}_u$ *is the dimension of the variety of Borel subgroups of G containing u, $Z_G^\circ(u)$ is the identity component of the centralizer of u in G, and W_I is the subgroup of W generated by I.*

2. A property of left cells of W. It is known that the elements in the same left cell of W have the same generalized τ-invariant (see [23] and also 4.2 for the definition). But the generalized τ-invariant does not determine a left cell uniquely in general. In this section, we shall show a property of a left cell of W which conjecturally characterizes a left cell of W. In particular, this property enables us to design an algorithm to find a representative set of left cells in a given two-sided cell of W and, furthermore, in the whole group W.

To each element $x\in W$, we associate a set $\Sigma(x)$ of all the left cells Γ of W satisfying the condition that there exists an element $y\in\Gamma$ with y—x, $\mathscr{R}(y)\not\subseteq\mathscr{R}(x)$ and $a(y)=a(x)$. It is obvious that any $\Gamma\in\Sigma(x)$ is in the two-sided cell of W containing x. Thus for any $x, y\in W$, we have $\Sigma(x)\cap\Sigma(y)\neq\emptyset$ only if $x\sim_{LR} y$.

The following result is crucial in this paper.

THEOREM 2.1. *If $x\sim_L y$ in W, then $\mathscr{R}(x)=\mathscr{R}(y)$ and $\Sigma(x)=\Sigma(y)$.*

The assertion $\mathscr{R}(x)=\mathscr{R}(y)$ in the theorem is known already (see [6]). So we need only to show $\Sigma(x)=\Sigma(y)$. To do so, we need the following:

LEMMA 2.2. *If two elements $x, y\in W$ satisfy the conditions x—y, $\mathscr{L}(x)\not\subseteq\mathscr{L}(y)$ and $a(x)=a(y)$, then $\Sigma(x)\subseteq\Sigma(y)$.*

PROOF. Take any $\Gamma\in\Sigma(x)$. We must show $\Gamma\in\Sigma(y)$. Now there exists an element $w\in\Gamma$ with w—x, $\mathscr{R}(w)\not\subseteq\mathscr{R}(x)$ and $a(w)=a(x)$. Choose any $t\in\mathscr{R}(w)\setminus\mathscr{R}(x)$ and any $s\in\mathscr{L}(x)\setminus\mathscr{L}(y)$. (Such elements t, s do exist by our assumption.) Then we see from (1.3.1) and (1.3.2) that

$$h_{x,t,w}\neq 0\neq h_{s,y,x}\,. \tag{2.2.1}$$

By the associativity of the algebra $\mathscr{H}$, we have an expression

$$C_sC_yC_t=\sum_{v,z} h_{s,y,v}h_{v,t,z}C_z=\sum_{v',z} h_{y,t,v'}h_{s,v',z}C_z\,. \tag{2.2.2}$$

Thus the $\mathscr{A}$-coefficient of C_w in (2.2.2) is

$$\sum_{v} h_{s,y,v}h_{v,t,w}=\sum_{v'} h_{y,t,v'}h_{s,v',w}\,. \tag{2.2.3}$$

The left hand side of (2.2.3) can be rewritten as

$$h_{s,y,x}h_{x,t,w}+\sum_{\substack{v\\ v\neq x}} h_{s,y,v}h_{v,t,w}\,, \tag{2.2.4}$$

which is non-zero by (2.2.1) and by the positivity of the $\boldsymbol{Z}$-coefficients of the $h_{\alpha,\beta,\gamma}$'s in u. This implies that on the right hand side of (2.2.3), there must exist some $z'\in W$ with

$$h_{y,t,z'}\neq 0\neq h_{s,z',w}\,. \tag{2.2.5}$$

By 1.6, (1), we have inequalities

$$a(y)\leq a(z')\leq a(w)=a(x)=a(y)\,, \tag{2.2.6}$$

and hence $a(z')=a(w)$. This implies $z'\sim_L w$ by 1.6, (2) and by the fact $w\leq_L z'$, that is, $z'\in\Gamma$. Note that z'—y and $\mathscr{R}(z')\nsubseteq\mathscr{R}(y)$ (since $t\in\mathscr{R}(z')\setminus\mathscr{R}(y)$). This implies $\Gamma\in\Sigma(y)$. ■

THE PROOF OF THEOREM 2.1. Since $\sim_L$ is an equivalence relation on W, it is enough to show $\Sigma(x)\subseteq\Sigma(y)$. By our hypothesis, there exists a sequence of elements $x_0=x, x_1,\ldots,x_r=y$ in W with $r\geq 0$ such that for every i, $1\leq i\leq r$, all the conditions x_{i-1}—x_i, $\mathscr{L}(x_{i-1})\nsubseteq\mathscr{L}(x_i)$ and $a(x_{i-1})=a(x_i)$ are satisfied. Thus by Lemma 2.2, we have $\Sigma(x_{i-1})\subseteq\Sigma(x_i)$ for $1\leq i\leq r$. But this implies $\Sigma(x)\subseteq\Sigma(y)$ immediately. ■

By Theorem 2.1, we can use the notation $\Sigma(\Gamma)$ for any left cell Γ of W, which is by definition $\Sigma(x)$ for any $x\in\Gamma$.

We have the following conjecture which asserts that the converse of Theorem 2.1 should also be true.

CONJECTURE 2.3. *For $x, y\in W$, we have an equivalence*

$$x\sim_L y \Leftrightarrow \mathscr{R}(x)=\mathscr{R}(y) \text{ and } \Sigma(x)=\Sigma(y)\,.$$

REMARK 2.4. (1) In the above conjecture, the condition $\mathscr{R}(x)=\mathscr{R}(y)$ on the right hand side is necessary. For example, let (W, S) be the affine Weyl group of type $\tilde{B}_2$ with $S=\{s_0, s_1, s_2\}$ such that the order $o(s_0s_2)$ of the product s_0s_2 is 2. Then $s_0\nsim_L s_2$ but $\Sigma(s_0)=\Sigma(s_2)=\{\Gamma_{s_1}\}$. Also let $x=s_0s_2s_1s_2$ and $y=s_0s_2s_1s_0$. We have $x\nsim_L y$ but $\Sigma(x)=\Sigma(y)=\{\Gamma_{s_0s_2s_1}, \Gamma_{s_0s_2}\}$. Note that the notation Γ_w ($w\in W$) stands for the left cell of W containing w.

(2) The above conjecture has been verified in the following cases:

(a) W is any Weyl group;

(b) W is any irreducible affine Weyl group of type $\neq\tilde{F}_4$;

(c) W is an affine Weyl group of type $\tilde{F}_4$, and the element x satisfies either

$$\mathscr{R}(x)\notin\{\{s_0, s_1, s_2\}, \{s_3, s_4\}\}\,,$$

or $\mathscr{R}(x)=\{s_0, s_1, s_2\}$ with $a(x)\notin\{7, 9, 10, 13, 16\}$,

or $\mathscr{R}(x)=\{s_3, s_4\}$ with $a(x)\notin\{6, 7, 9, 10, 13, 16\}$,

where $o(s_0s_1)=o(s_1s_2)=o(s_3s_4)=3$ and $o(s_2s_3)=4$ (see [22]).

(3) If for any $x\in W$, we define a set $\Sigma'(x)$ of left cells in the same way as that for the set $\Sigma(x)$ but by changing the condition $\mathscr{R}(y)\nsubseteq\mathscr{R}(x)$ to $\mathscr{R}(y)\nsupseteq\mathscr{R}(x)$, then we have a result similar to Theorem 2.1: If $x\sim_L y$ in W, then $\mathscr{R}(x)=\mathscr{R}(y)$ and $\Sigma'(x)=\Sigma'(y)$. But this result is less important than Theorem 2.1, because for a given element x, it is easier to find all the elements y satisfying y—x, $\mathscr{R}(y)\nsubseteq\mathscr{R}(x)$ and $a(y)=a(x)$ than to find y satisfying the same conditions but with $\mathscr{R}(y)\nsupseteq\mathscr{R}(x)$ instead of $\mathscr{R}(y)\nsubseteq\mathscr{R}(x)$.

3. Algorithm for finding a representative set of left cells. A subset $K\subset W$ is called a representative set of left cells of W (resp. of W in a two-sided cell Ω), if $|K\cap\Gamma|=1$ for any left cell Γ of W (resp. of W in Ω), where the notation $|X|$ stands for the cardinality of a set X.

It is known that the set $\mathscr{D}$ (see 1.6, (3)) is a representative set of left cells of W. But it is not easy to find the whole set $\mathscr{D}$ in general. In this section, we shall apply Theorem 2.1 to design an algorithm for finding a representative set of left cells of W in a given two-sided cell Ω. When W is an affine Weyl group, the number of two-sided cells of W and the a-values of these cells are known, and our algorithm could actually be used to find a representative set of left cells of the whole group W.

The algorithm will be based on the following:

THEOREM 3.1. *Let Ω be a two-sided cell of W. Then a non-empty subset $M\subset\Omega$ is a representative set of left cells of W in Ω, if M satisfies the following conditions*:

(1) *$x\nsim_L y$ for any $x\neq y$ in M*;

(2) *Let $y\in W$. Suppose that there exists an element $x\in M$ such that y—x, $\mathscr{R}(y)\nsubseteq\mathscr{R}(x)$ and $a(y)=a(x)$. Then there exists an element $z\in M$ with $y\sim_L z$.*

PROOF. By condition (1), it suffices to show that for any given left cell Γ of W in Ω, the intersection $\Gamma\cap M$ is non-empty. It is known that in Ω, the intersection of a left cell with a right cell is non-empty. Thus for any element $w\in M$, there exists a sequence of elements $w_0=w', w_1, \ldots, w_r=w$ in Ω with $r\geq 0$ such that $w'\in\Gamma$ and that for every i, $1\leq i\leq r$, we have w_{i-1}—w_i and $\mathscr{R}(w_{i-1})\nsubseteq\mathscr{R}(w_i)$. We choose $w\in M$ and a sequence $w_0=w', w_1, \ldots, w_r=w$ as above with r as small as possible. We must show $r=0$. Suppose $r>0$. Then by the condition (2), there exists some $z\in M$ with $w_{r-1}\sim_L z$. Then by repeated application of Theorem 2.1, there exists a sequence of elements $z_{r-1}=z$, $z_{r-2}, \ldots, z_1, z_0$ in W such that for every i, j with $0\leq i\leq r-1$ and $1\leq j\leq r-1$, we have z_j—z_{j-1} and $z_i\sim_L w_i$. Now the sequence $z_0, z_1, \ldots, z_{r-1}=z$ satisfies all the above conditions but with shorter length, which contradicts the minimality of the integer r. ∎

3.2. In principle, Theorem 3.1, (2) provides us with a method of finding a representative set of left cells of W in any given two-sided cell Ω from a non-empty subset of Ω.

A subset $P \subset W$ is said to be distinguished if $P \neq \emptyset$ and $x \not\sim_L y$ for any $x \neq y$ in P.

Now assume that P is a subset of Ω. We introduce the following two processes.

(A) For each $x \in P$, find elements $y \in W$ such that there exists a sequence of elements $x_0 = x, x_1, \ldots, x_r = y$ in W with $r > 0$, where for every i, $1 \leq i \leq r$, the conditions $x_i^{-1} x_{i-1} \in S$ and $\mathscr{R}(x_{i-1}) \underset{\not\subseteq}{\not\supseteq} \mathscr{R}(x_i)$ are satisfied, add these elements y on the set P to form a set P' and then take a largest possible subset Q from P' with Q distinguished.

(B) For each $x \in P$, find elements $y \in W$ such that y—x, $\mathscr{R}(y) \supsetneq \mathscr{R}(x)$ and $a(y) = a(x)$, add these elements y on the set P to form a set P' and then take a largest possible subset Q from P' with Q distinguished.

Note that the resulting sets in both Processes (A) and (B) are automatically in the two-sided cell Ω.

A subset P of Ω is said to be A-saturated (resp. B-saturated) if Process (A) (resp. Process (B)) cannot produce an element z satisfying $z \not\sim_L x$ for all $x \in P$.

Thus a representative set of left cells of W in a two-sided cell Ω is exactly a distinguished subset of Ω which is both A- and B-saturated. So to get such a set, we may use the following:

ALGORITHM 3.3. (1) Find a non-empty subset P of Ω (usually we take P to be distinguished for avoiding unnecessary complication if possible);

(2) Perform Processes (A) and (B) alternately on P until the resulting distinguished set cannot be further enlarged by both processes.

REMARK 3.4. The above algorithm has been applied by several person to classify the left cells of affine Weyl groups W in the following cases.

(1) For W of type $\tilde{D}_4$, by myself [20] (I understand that Chen Chengdong [3] also dealt with this case but his method is different from mine).

(2) For W of type $\tilde{C}_4$, by myself [21].

(3) For W of type $\tilde{B}_4$, by Zhang [24].

(4) For all the left cells with their a-values equal to 3 in any irreducible affine Weyl group W, by Rui [15].

(5) For all the left cells with their a-values equal to 4 and 5 in W of type $\tilde{F}_4$, by myself.

We shall deal with the case (5) and one of the cases (4) in §5 to illustrate our method.

3.5. We need some techniques in applying our algorithm. The following known results may be useful in this respect.

(1) Let I be a subset of S such that the subgroup W_I of W generated by I is finite, and let w_I be the longest element in W_I. Then $a(w_I) = \ell(w_I)$.

(2) Suppose that a two-sided cell Ω of W contains an element of the form w_I for some $I\subset S$. Then the set $\{w\in\Omega \mid \mathscr{R}(w)=I\}$ forms a single left cell of W.

(3) Assume that $x=yz$ with $\ell(x)=\ell(y)+\ell(z)$ for $x, y, z\in W$. Then we have $x\leq_L z$, $x\leq_R y$ and hence $a(x)\geq a(y), a(z)$. In particular, if $I=\mathscr{R}(x)$ (resp. $I=\mathscr{L}(x)$), then $a(x)\geq \ell(w_I)$.

(4) If $x, y\in W$ satisfy x—y and $\mathscr{R}(x)\underset{\not\subseteq}{\overset{\not\supseteq}{}}\mathscr{R}(y)$, then $x^{-1}y\in S$. More precisely, we have $x^{-1}y\in\mathscr{R}(x)\vee\mathscr{R}(y)$, where the notation $X\vee Y$ stands for the symmetric difference of two sets X and Y.

(5) If $x, y\in W$ satisfy y—x, $\mathscr{R}(y)\not\subseteq\mathscr{R}(x)$ and $a(x)=a(y)$, then we have either $y^{-1}x\in S$ or $y<x$ with $\ell(x)-\ell(y)$ odd, and we also have $\mathscr{L}(y)=\mathscr{L}(x)$.

Result (5) tells us the extent of elements y with y—x, $\mathscr{R}(y)\not\subseteq\mathscr{R}(x)$ and $a(y)=a(x)$ for a fixed element x. This result is of particular importance in performing Process (B). Correspondently, the result (4) is special for performing Process (A). Besides, the result (1) is good for choosing a starting distinguished subset in Algorithm 3.3; the results (2) and (3) are often used in checking whether a given set is distinguished or not.

4. Graphs, generalized τ-invariants and strings. In order to perform Algorithm 3.3 effectively, we shall develop more concepts as well as some related results in this section.

4.1. Given an element $x\in W$, we consider the set $M(x)$ of all elements y such that there exists a sequence of elements $x_0=x, x_1, \ldots, x_r=y$ in W with some $r\geq 0$, where for every i, $1\leq i\leq r$, the conditions $x_{i-1}^{-1}x_i\in S$ and $\mathscr{R}(x_{i-1})\underset{\not\subseteq}{\overset{\not\supseteq}{}}\mathscr{R}(x_i)$ are satisfied. The set $M(x)$ could be either finite or infinite. Clearly, we always have $x\in M(x)$. For $x, x'\in W$, we have either $M(x)=M(x')$ or $M(x)\cap M(x')=\varnothing$. The following well-known result will be useful in § 5.

PROPOSITION. *$x, x'\in W$ satisfy $x\sim_R x'$ if there exist $y, z\in M(x)$ and $y', z'\in M(x')$ such that y—y', z—z', $\mathscr{R}(y)\not\subseteq\mathscr{R}(y')$ and $\mathscr{R}(z')\not\subseteq\mathscr{R}(z)$. In particular, we have $a(x)=a(x')$.*

Now we define a graph $\mathfrak{M}(x)$ associated to x as follows. Its vertex set is $M(x)$. Its edge set consists of all two-element subsets $\{y, z\}\subset M(x)$ with $y^{-1}z\in S$ and $\mathscr{R}(y)\underset{\not\subseteq}{\overset{\not\supseteq}{}}\mathscr{R}(z)$. To each vertex $y\in M(x)$, we are given a subset $\mathscr{R}(y)$ of S. To each edge $\{y, z\}$ of $\mathfrak{M}(x)$, we are given an element $s\in S$ with $s=y^{-1}z$.

Two graphs $\mathfrak{M}(x)$ and $\mathfrak{M}(x')$ are said to be quasi-isomorphic if there exists a bijection ϕ from the set $M(x)$ to the set $M(x')$ satisfying the following conditions.

(1) $\mathscr{R}(w)=\mathscr{R}(\phi(w))$ for all $w\in M(x)$.

(2) For $y, z\in M(x)$, $\{y, z\}$ is an edge of $\mathfrak{M}(x)$ if and only if $\{\phi(y), \phi(z)\}$ is an edge of $\mathfrak{M}(x')$.

Note that in the above definition we make no requirement on the labelings of the corresponding edges. This is why we put the prefix "quasi".

4.2. By a path in the graph $\mathfrak{M}(x)$, we mean a sequence of vertices $z_0, z_1, \ldots, z_t$

in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathfrak{M}(x)$ for any i, $1 \le i \le t$. Two elements $x, x' \in W$ are said to have the same generalized τ-invariant if for any path $z_0 = x, z_1, \ldots, z_t$ in the graph $\mathfrak{M}(x)$, there exists a path $z'_0 = x', z'_1, \ldots, z'_t$ in $\mathfrak{M}(x')$ with $\mathscr{R}(z'_i) = \mathscr{R}(z_i)$ for every i, $0 \le i \le t$, and if the same condition holds when the roles of x and x' are interchanged.

Note that our definition of the generalized τ-invariant is slightly different from the one given by Vogan [23]. The following result could be shown simply by Theorem 2.1.

PROPOSITION. *All the elements in the same left cell of W have the same generalized τ-invariant.*

The converse of the above proposition is not true in general. It may happen that two elements in different left cells of W have the same generalized τ-invariant.

4.3. A set Σ of left cells of W is said to be represented by a set M of elements of W if Σ is the set of all the left cells Γ of W with $\Gamma \cap M \neq \varnothing$.

Given a non-empty subset X of a two-sided cell Ω of W, we want to find from X an A-saturated and distinguished subset $\bar{X}$ of W in Ω by performing Process (A), where the set of left cells represented by $\bar{X}$ contains the one represented by X. This can be done in virtue of the graphs $\mathfrak{M}(x)$, $x \in X$, by picking out a largest possible distinguished vertex set from those graphs $\mathfrak{M}(x)$.

4.4. A sequence of elements in W of the form

$$\underbrace{ys,\ yst,\ ysts, \ldots}_{m-1 \text{ terms}} \tag{4.4.1}$$

is called an $\{s, t\}$-string (or just a string) if $s, t \in S$ and $y \in W$ satisfy the conditions that the order $o(st)$ of the product st is m and $\mathscr{R}(y) \cap \{s, t\} = \varnothing$. The number $m-1$ is called the length of this string. Clearly, when (4.4.1) is an $\{s, t\}$-string, the sequence

$$\underbrace{yt,\ yts,\ ytst, \ldots}_{m-1 \text{ terms}} \tag{4.4.2}$$

is also an $\{s, t\}$-string.

Suppose that we are given two $\{s, t\}$-strings $x_1, x_2, \ldots, x_{m-1}$ and $y_1, y_2, \ldots, y_{m-1}$ with $o(st) = m$. We denote the integers $\mu(x_i, y_j)$ by a_{ij} for $1 \le i, j \le m-1$. Then the following is known:

PROPOSITION 4.5 (cf. [11]). *Let the situation be as above.*

(1) *When $m=3$, we have $a_{12} = a_{21}$, $a_{11} = a_{22}$;*

(2) *When $m=4$, we have $a_{12} = a_{21} = a_{23} = a_{32}$, $a_{11} = a_{33}$, $a_{13} = a_{31}$ and $a_{22} = a_{11} + a_{13}$.*

We have the following result corresponding to this.

PROPOSITION 4.6. *Let the situation be as above.*

(1) *If* $m=3$, *then*

$$x_1 \sim_L y_1 \Leftrightarrow x_2 \sim_L y_2\,;$$
$$x_1 \sim_L y_2 \Leftrightarrow x_2 \sim_L y_1\,.$$

(2) *If* $m=4$, *then*

(a) $x_1 \sim_L y_2 \Leftrightarrow x_2 \sim_L y_1 \Leftrightarrow x_2 \sim_L y_3 \Leftrightarrow x_3 \sim_L y_2$;

(b) $x_1 \sim_L y_1 \Leftrightarrow x_3 \sim_L y_3$;

(c) $x_1 \sim_L y_3 \Leftrightarrow x_3 \sim_L y_1$;

(d) $x_2 \sim_L y_2 \Leftrightarrow$ *either* $x_1 \sim_L y_1$ *or* $x_1 \sim_L y_3 \Leftrightarrow$ *either* $x_1 \sim_L y_1$ *or* $x_3 \sim_L y_1$
$\Leftrightarrow$ *either* $x_3 \sim_L y_1$ *or* $x_3 \sim_L y_3 \Leftrightarrow$ *either* $x_1 \sim_L y_3$ *or* $x_3 \sim_L y_3$.

PROOF. Assertion (1) was shown by Kazhdan and Lusztig [6]. Now we shall show (2). We may assume $t \notin \mathscr{R}(y_2)$ without loss of generality. For $f \in \mathscr{H}$ and $w \in W$, we write $f \succ C_w$ if the term C_w occurs in the expression

$$f = \sum_z a_z C_z\,, \qquad a_z \in \mathscr{A}\,,$$

with $a_w \neq 0$.

Let us first show the implication $x_1 \sim_L y_2 \Rightarrow x_2 \sim_L y_1$.

We have $y_2 \leq_L x_1$ and hence there exists some $z \in W$ with $C_z C_{x_1} \succ C_{y_2}$. By the assumption $t \notin \mathscr{R}(y_2)$, we have $C_{y_2} C_t \succ C_{y_1}$ and then $C_z C_{x_1} C_t \succ C_{y_1}$ by the positivity of the $\boldsymbol{Z}$-coefficients of $h_{x,y,w}$'s in u^i ($i \in \boldsymbol{Z}$). Thus by the associativity of the Hecke algebra $\mathscr{H}$, there must exist some $x \in W$ with $C_{x_1} C_t \succ C_x$ and $C_z C_x \succ C_{y_1}$. So $y_1 \leq_L x$ and $a(y_1) \geq a(x) \geq a(x_1) = a(y_2) = a(y_1)$. This implies $y_1 \sim_L x$. Now by Theorem 2.1, we have $\mathscr{R}(x) = \mathscr{R}(y_1)$ and $\mathscr{R}(x_1) = \mathscr{R}(y_2)$. Since x—x_1 and since y_1, y_2 are neighboring terms in an $\{s, t\}$-string, this implies from 3.5, (4) that x, x_1 also must be neighboring terms in an $\{s, t\}$-string and hence $x = x_2$. So $x_2 \sim_L y_1$.

The remaining part of the assertion (2), (a) as well as (2), (d) can be proved similarly.

Next we show the implication $x_1 \sim_L y_1 \Rightarrow x_3 \sim_L y_3$.

By (2), (d), we have $x_2 \sim_L y_2$ and hence either $x_3 \sim_L y_3$ or $x_3 \sim_L y_1$ holds. If $x_3 \sim_L y_3$, then we are done. Now assume $x_3 \sim_L y_1$. Then $x_1 \sim_L x_3$. Again by (2), (d), we have either $y_3 \sim_L x_3$ or $y_3 \sim_L x_1$. In either case, we get $y_3 \sim_L x_3$.

The remaining part of (2), (b) and part (2), (c) can be shown similarly. ∎

REMARK 4.7. Results analogous to 4.5 and 4.6 hold for arbitrary m.

5. Applications to W of type $\tilde{F}_4$. In this section, we assume (W, S) to be the affine Weyl group of type $\tilde{F}_4$ with $S = \{s_0, s_1, s_2, s_3, s_4\}$ the Coxeter generator set, where $o(s_0 s_1) = o(s_1 s_2) = o(s_3 s_4) = 3$ and $o(s_2 s_3) = 4$. We shall apply the algorithm provided in §3 to find a representative set of left cells in certain two-sided cells of W.

5.1. Let $W_{(i)} = \{w \in W \mid a(w) = i\}$ for any non-negative integer i. Then from the knowledge of unipotent classes of the complex connected reductive algebraic group of type F_4 and from Theorem 1.7, we see that for each $i=4, 5$, the set $W_{(i)}$ forms a single two-sided cell of W, and that the set $W_{(3)}$ is a union of two two-sided cells of W. We want to find representative sets of left cells of all the two-sided cells Ω of W with $a(\Omega)=3, 4, 5$.

5.2. We first consider the two-sided cell $W_{(4)}$. Let

$$X=\{w_I \mid I\in\{\{2,3\},\{0,3,4\},\{1,3,4\},\{0,1,3\},\{0,1,4\},\{1,2,4\}\}\},$$

where we denote s_i by i for brevity. Then X is a distinguished subset of $W_{(4)}$ (see 3.2). We perform Process (A) on X. The graphs $\mathfrak{M}(w_I)$ (see 4.1) with

$$I\in\{\{0,3,4\},\{1,3,4\},\{0,1,3\},\{0,1,4\},\{1,2,4\}\}$$

are all finite. By Proposition 4.2, we see that the set of vertices in each of these graphs are distinguished. Any two of these vertex sets represent either the same set of left cells or disjoint sets of left cells. The graph $\mathfrak{M}(w_{\{2,3\}})$ is infinite. By 3.5, (2) and Proposition 4.6, we may pick out a finite subgraph $\mathfrak{M}'(w_{\{2,3\}})$ from it such that the vertex set of this subgraph forms a maximal distinguished subset in the vertex set $M(w_{\{2,3\}})$. Thus we get A-saturated and distinguished sets $\bar{X}$, which are the set of all vertices of the graphs in Figures 1, 2 and 3, where the vertices x are represented by boxes, inside which we

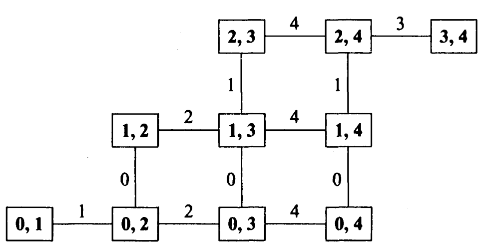

FIGURE 1. $\mathfrak{M}'(w_{\{2,3\}})$

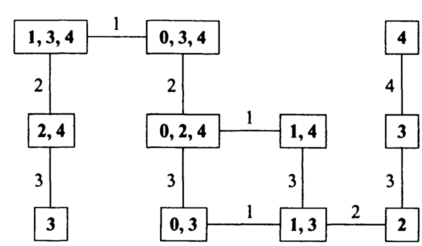

FIGURE 2. $\mathfrak{M}(w_{\{0,3,4\}})$

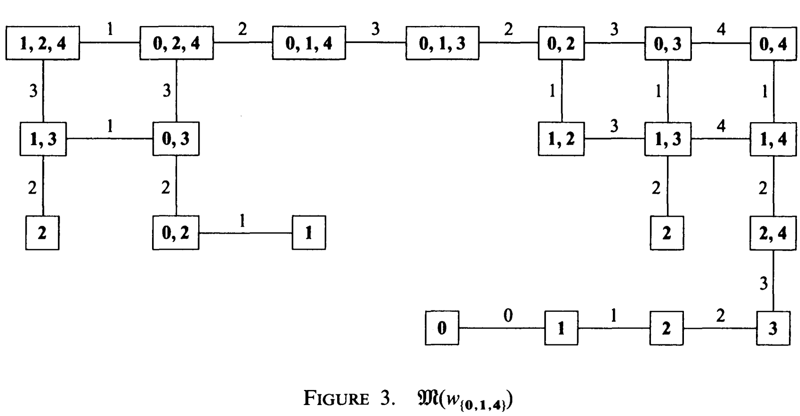

FIGURE 3. $\mathfrak{M}(w_{\{0,1,4\}})$

describe the corresponding subset $\mathscr{R}(x)$ of S. The vertices x with $\mathscr{R}(x)=I\in\{\{\mathbf{2},\mathbf{3}\},\{\mathbf{0},\mathbf{3},\mathbf{4}\},\{\mathbf{0},\mathbf{1},\mathbf{4}\}\}$ are the elements w_I. The labeling i of an edge indicates that its two terminals x, y have the relation $x=ys_i$. We can check that the set $\bar{X}$ is also B-saturated. Hence by Theorem 3.1, we assert that $\bar{X}$ forms a representative set of left cells of W in $W_{(4)}$.

REMARK 5.3. As a starting distinguished set in the algorithm, X is usually chosen with larger cardinality and shorter elements if possible. This may make our process easier and faster. Thus the elements of the form w_I, $I\subset S$, are ideal candidates to be selected into the set X. This is because of their shorter lengths, computable a-values and being distinguished (see 3.5, (1), (2)). But we should be cautious when a given two-sided cell Ω of W is not of the form $W_{(i)}$. One should make sure that the set X is indeed wholly in a two-sided cell. The next example will tell us something about this.

5.4. Now let us consider the two-sided cells of W with a-values equal to 3. As mentioned in 5.1, $W_{(3)}$ is a union of two two-sided cells of W. $W_{(3)}$ contains four elements of the form w_I, where $I\in\{\{\mathbf{0},\mathbf{1}\},\{\mathbf{1},\mathbf{2}\},\{\mathbf{3},\mathbf{4}\},\{\mathbf{0},\mathbf{2},\mathbf{4}\}\}$. At this stage, we do not know how to distribute these four elements into two two-sided cells. Let $X=\{w_{\{\mathbf{0},\mathbf{1}\}}=\mathbf{010}\}$ and let Ω_1 be the two-sided cell of W containing $\mathbf{010}$. First we want to find a representative set of left cells of W in Ω_1 by performing Processes (A) and (B) on X. The graph $\mathfrak{M}(\mathbf{010})$ is displayed in Figure 4, where the vertex x with $\mathscr{R}(x)=\{\mathbf{0},\mathbf{1}\}$ is the element $\mathbf{010}$. Since $\mathfrak{M}(\mathbf{010})$ contains the vertex $y=\mathbf{01021}$ with $\mathscr{R}(y)=\{\mathbf{1},\mathbf{2}\}$, this implies $w_{\{\mathbf{1},\mathbf{2}\}}=\mathbf{121}\in\Omega_1$ by 3.5, (2). By Proposition 4.2, we see that the vertex set $M(\mathbf{010})$ of $\mathfrak{M}(\mathbf{010})$ is distinguished. $M(\mathbf{010})$ is A-saturated but not B-saturated. Indeed, let $z=\mathbf{010232}\in M(\mathbf{010})$ and $z'=z\mathbf{4}$. Then z'—z and $\mathscr{R}(z')=\{\mathbf{0},\mathbf{2},\mathbf{4}\}\supsetneqq\{\mathbf{0},\mathbf{2}\}=\mathscr{R}(z)$. Observe the graph in Figure 5, where the fourth and the fifth vertices from the left are the elements z, z', respectively. By

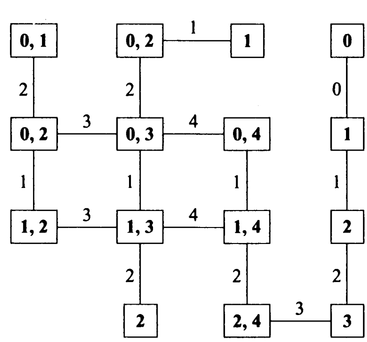

FIGURE 4. $\mathfrak{M}(\mathbf{010})$

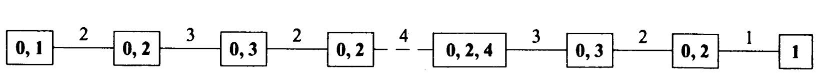

FIGURE 5.

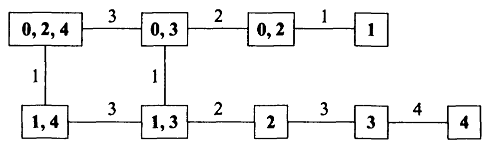

FIGURE 6. $\mathfrak{M}(\mathbf{024})$

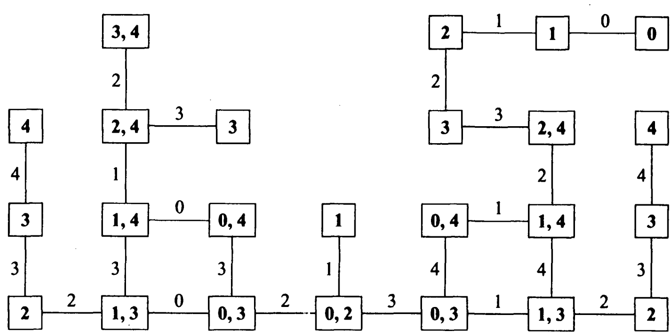

FIGURE 7. $\mathfrak{M}(\mathbf{343})$

Propositions 4.1 and 4.5, we see that $z \sim_R z'$ and hence $z' \in \Omega_1$. This implies $w_{\{0,2,4\}} = \mathbf{024} \in \Omega_1$ by 3.5, (2). The graph $\mathfrak{M}(\mathbf{024})$ is as in Figure 6, where the vertex x with $\mathscr{R}(x) = \{\mathbf{0}, \mathbf{2}, \mathbf{4}\}$ is the element $\mathbf{024}$. Let $M(\mathbf{024})$ be the vertex set of $\mathfrak{M}(\mathbf{024})$. Then by Proposition 4.2, we see that the union $M(\mathbf{010}) \cup M(\mathbf{024})$ is distinguished. By a case-by-case checking, we also see that $M(\mathbf{010}) \cup M(\mathbf{024})$ is both A- and B-saturated and hence forms a representative set of left cells of W in Ω_1 by Theorem 3.1.

Since the union $M(\mathbf{010}) \cup M(\mathbf{024})$ contains no element x with $\mathscr{R}(x) = \{\mathbf{3}, \mathbf{4}\}$, this implies that $w_{\{3,4\}} = \mathbf{343} \notin \Omega_1$. So the element $\mathbf{343}$ is contained in another two-sided cell Ω_2 of W in $W_{(3)}$. Now we want to find a representative set of left cells of W in Ω_2 by performing Processes (A) and (B) on the set $X = \{\mathbf{343}\}$. The graph $\mathfrak{M}(\mathbf{343})$ is as in Figure 7, where the vertex x with $\mathscr{R}(x) = \{\mathbf{3}, \mathbf{4}\}$ is the element $\mathbf{343}$. By Proposition 4.2, the vertex set $M(\mathbf{343})$ of the graph $\mathfrak{M}(\mathbf{343})$ is distinguished. We can check that the set $M(\mathbf{343})$ is both A- and B-saturated. So $M(\mathbf{343})$ forms a representative set of left cells of W in Ω_2 by Theorem 3.1.

5.5. Now we consider the two-sided cell $W_{(5)}$ of W. There is only one element, i.e. $\mathbf{02323}$, in $W_{(5)}$ which has the form w_I. The graph $\mathfrak{M} = \mathfrak{M}(\mathbf{02323})$ is as in Figure 8, where there are two vertices x with $\mathscr{R}(x) = \{\mathbf{0}, \mathbf{2}, \mathbf{3}\}$ in the graph, the one on the left hand side is the element $\mathbf{02323}$. By 3.5, (2), the vertices $\mathbf{023234323}$ and $\mathbf{02323}$ are in the same left cell of W. This implies that the vertex set $M(\mathbf{02323})$ of the graph $\mathfrak{M}$ is not distinguished. Let $\mathfrak{M}'$ be the subgraph of $\mathfrak{M}$ consisting of the part of $\mathfrak{M}$ located on the left side of the dotted line. Then by Propositions 4.2 and 4.6, the vertex set $M'(\mathbf{02323})$

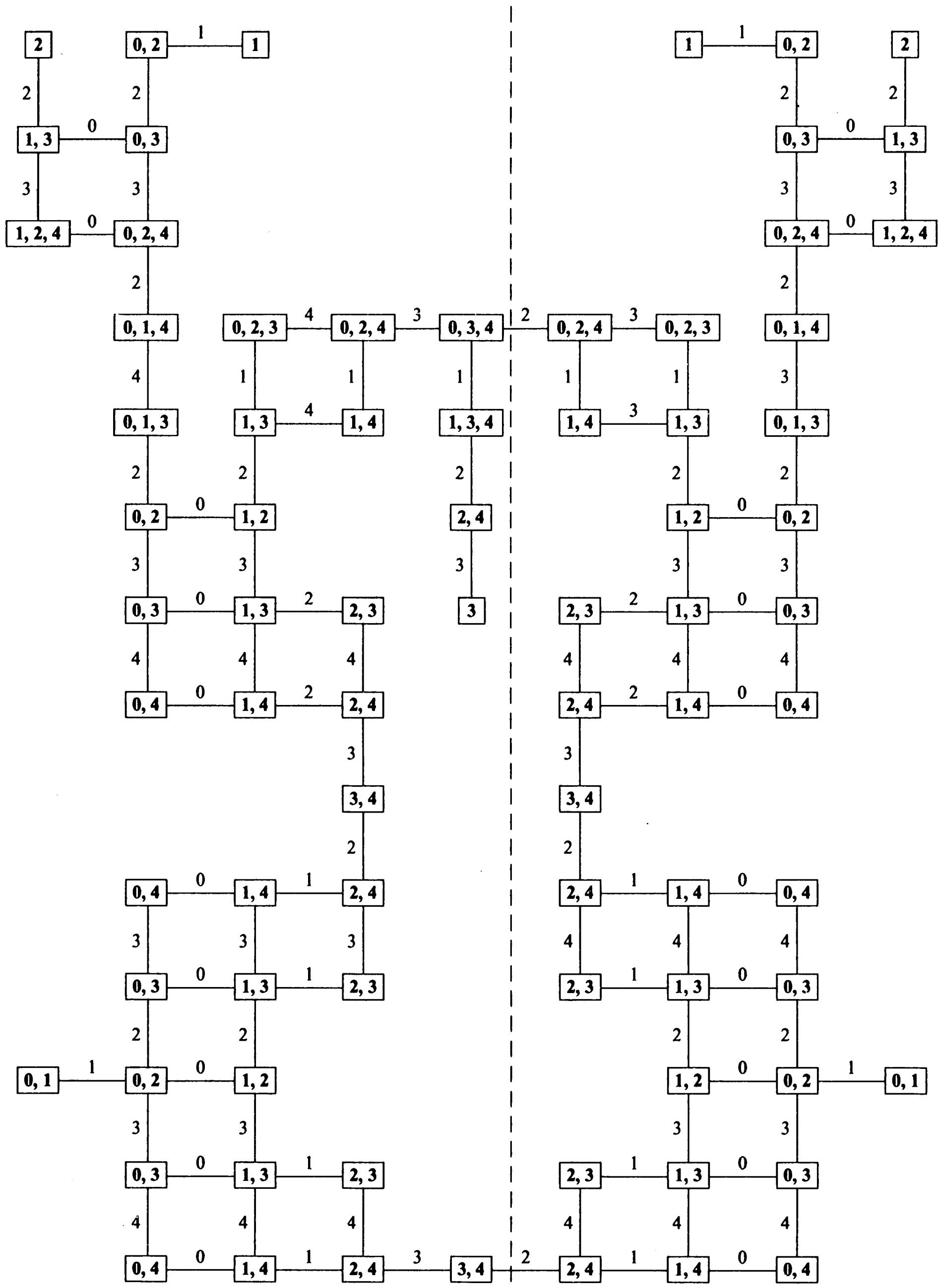

FIGURE 8. $\mathfrak{M}(02323)$

of $\mathfrak{M}'$ is a distinguished subset of $M(\mathbf{02323})$ with the maximal cardinality.

5.6. The set $M'(\mathbf{02323})$ is A-saturated but not B-saturated. Indeed, to the vertex $y=\mathbf{023231234}$, we have an element $y'=y\mathbf{3}$ which satisfies y'—y and $\mathscr{R}(y')=\{\mathbf{1},\mathbf{4},\mathbf{3}\}\supsetneqq\{\mathbf{1},\mathbf{4}\}=\mathscr{R}(y)$; to the vertex $z=\mathbf{0232343123}$, we have an element $z'=z\mathbf{2}$ which satisfies z'—z and $\mathscr{R}(z')=\{\mathbf{2},\mathbf{3}\}\supsetneqq\{\mathbf{3}\}=\mathscr{R}(z)$. We also have $a(y')=a(z')=5$ by applying Propositions 4.1 and 4.5 on the graphs $\mathfrak{M}(\mathbf{02323})$, $\mathfrak{M}(y')$ and $\mathfrak{M}(z')$. The set $M'(\mathbf{02323})\cup\{y',z'\}$ is distinguished. The graphs $\mathfrak{M}(y')$ and $\mathfrak{M}(z')$ are displayed in Figures 9 and 10, respectively, where the x with $\mathscr{R}(x)=\{\mathbf{1},\mathbf{3},\mathbf{4}\}$ in $\mathfrak{M}(y')$ is the element y', the vertex x' with $\mathscr{R}(x')=\{\mathbf{2},\mathbf{3}\}$ in $\mathfrak{M}(z')$ is the element z'. Note that $\mathfrak{M}(y')$ is quasi-isomorphic to $\mathfrak{M}(w_{\{\mathbf{0},\mathbf{3},\mathbf{4}\}})$ (see Figure 2). By Proposition 4.2, we see that the vertex set $M(y')$ of $\mathfrak{M}(y')$ is distinguished and that the sets of left cells represented by the vertex sets $M'(\mathbf{02323})$, $M(y')$, $M(z')$ of $\mathfrak{M}'$, $\mathfrak{M}(y')$, $\mathfrak{M}(z')$ are pairwise disjoint. But it is not clear whether the vertex set $M(z')$ is distinguished or not.

5.7. Let $\alpha=z'\mathbf{42}$ and $\alpha'=z'\mathbf{43}$, $\beta=z'\mathbf{12}$, $\beta'=z'\mathbf{13}$, $\gamma=z'\mathbf{120}$, $\gamma'=z'\mathbf{130}$, $\delta=z'\mathbf{1201}$ and $\delta'=z'\mathbf{1301}$. They are vertices in $\mathfrak{M}(z')$. We shall check whether x, x' are in the same left cell of W or not for $x\in\{\alpha,\beta,\gamma,\delta\}$.

PROPOSITION. *In the above setup, we have*

$$(1)\ \alpha\not\sim_L\alpha';\quad (2)\ \beta\sim_L\beta';\quad (3)\ \gamma\sim_L\gamma';\quad (4)\ \delta\sim_L\delta'.$$

By Proposition 4.6, (1), we have the equivalence

$$(2)\Leftrightarrow(3)\Leftrightarrow(4). \tag{5.7.1}$$

So it suffices to show (1) and (2). Now we show (2) and postpone the proof of (1) to later stage.

5.8. The proof of (2). We have $\ell(\beta)=13$ and $\ell(\beta')=11$. Consider the element $\beta''=\mathbf{40232341232312}$. We have $\beta''=\mathbf{4}\beta=\mathbf{234}\beta'$, $\mathscr{R}(\beta'')=\{\mathbf{1},\mathbf{2}\}$ and $\ell(\beta'')=14$. Thus by 3.5,

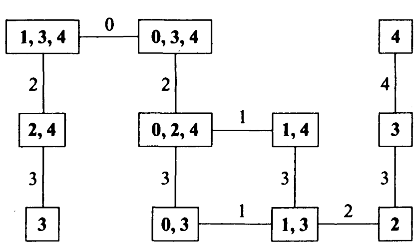

FIGURE 9. $\mathfrak{M}(y')$

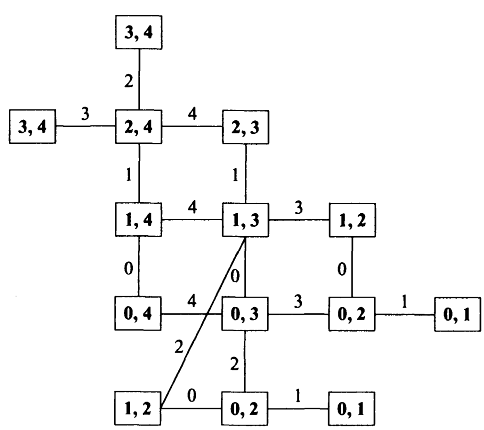

FIGURE 10. $\mathfrak{M}(z')$

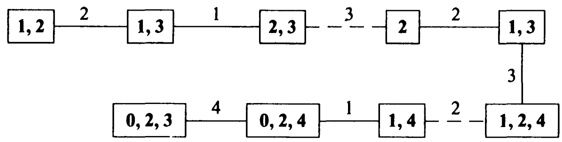

FIGURE 11.

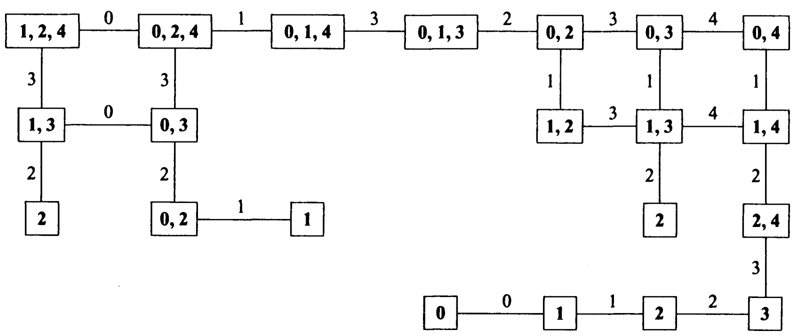

FIGURE 12. $\mathfrak{M}(w')$

(3), we have

$$\beta'' \leq_L \beta, \beta' . \tag{5.8.1}$$

Observe the graph in Figure 11, where the leftmost vertex is the element β''. The lengths of the elements x in this graph (which are represented by vertices with labeling $\mathscr{R}(x)$) monotone decrease along the path when getting farther from β''. The other terminal vertex is the element $v=\mathbf{402323}$. It is easily seen that $v\sim_L \mathbf{02323}=w_{\{\mathbf{0},\mathbf{2},\mathbf{3}\}}$ and hence $a(v)=5$. By Propositions 4.1 and 4.5, we see $\beta''\sim_R v$ and so $a(\beta'')=5$. Thus by 1.6, (2) and the fact $a(\beta)=a(\beta')=5$, this implies from (5.8.1) that

$$\beta \sim_L \beta'' \sim_L \beta' . \tag{5.8.2}$$

■

REMARK 5.9. The element β'' plays a key role in the above proof. It seems quite accidental in finding such an element β''. But if we express elements of W in their alcove forms (see [17] for the definition), then we see that the element β'' could be obtained from β and β' in a quite natural way. Actually, all the results of the present section are worked out by using the alcove forms of elements.

5.10. The set $M'(\mathbf{02323})\cup M(y')\cup M(z')$ is A-saturated but not B-saturated. For the element $w=y'\mathbf{2}\in M(y')$, we have $w'=w\mathbf{1}$ satisfying w'—w, $\mathscr{R}(w')=\{\mathbf{1},\mathbf{2},\mathbf{4}\}\supsetneqq\{\mathbf{2},\mathbf{4}\}=\mathscr{R}(w)$ and $a(w')=5$. The last condition on w' is obtained by applying Propositions 4.1 and 4.5 on $\mathfrak{M}(y')$ and $\mathfrak{M}(w')$. The graph $\mathfrak{M}(w')$ is displayed in Figure 12, where the

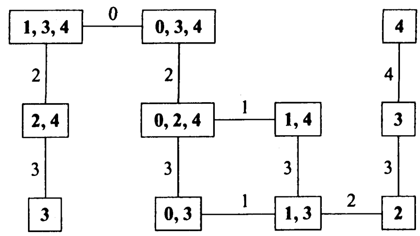

FIGURE 13. $\mathfrak{M}(\eta)$

vertex x with $\mathscr{R}(x)=\{\mathbf{1},\mathbf{2},\mathbf{4}\}$ is the element w'. Note that $\mathfrak{M}(w')$ is quasi-isomorphic to $\mathfrak{M}(w_{\{\mathbf{0},\mathbf{1},\mathbf{4}\}})$. The vertex set $M(w')$ of $\mathfrak{M}(w')$ is distinguished and the set of left cells represented by $M(w')$ is disjoint from those by $M'(\mathbf{02323})\cup M(y')\cup M(z')$ by Proposition 4.2.

5.11. On the other hand, for the element $\alpha'=z'\mathbf{43}\in M(z')$, we have $\eta=\alpha'\mathbf{1}$ satisfying η—α', $\mathscr{R}(\eta)=\{\mathbf{1},\mathbf{3},\mathbf{4}\}\supsetneqq\{\mathbf{3},\mathbf{4}\}=\mathscr{R}(\alpha')$ and $a(\eta)=5$. The last condition on η is obtained by applying Propositions 4.1 and 4.5 on $\mathfrak{M}(z')$ and $\mathfrak{M}(\eta)$. $\mathfrak{M}(\eta)$ is displayed in Figure 13, where the vertex x with $\mathscr{R}(x)=\{\mathbf{1},\mathbf{3},\mathbf{4}\}$ is the element η. We see that $\mathfrak{M}(\eta)$ is quasi-isomorphic to $\mathfrak{M}(y')$. One may ask whether the set of left cells represented by the vertex set $M(\eta)$ of $\mathfrak{M}(\eta)$ is the same as the one represented by the set $M(y')$. The answer is as follows.

PROPOSITION. *The sets $M(\eta)$ and $M(y')$ represent different sets of left cells of W which are disjoint.*

PROOF. Take $b=y'\mathbf{23}\in M(y')$ and $c=\eta\mathbf{23}\in M(\eta)$ which are in the corresponding positions of two graphs with $\mathscr{R}(b)=\mathscr{R}(c)=\{\mathbf{3}\}$. Thus to show our result, we need only to show $b\not\sim_L c$. By Theorem 2.1, it suffices to show $\Sigma(b)\neq\Sigma(c)$. Consider all the left cells Γ with $\mathscr{R}(\Gamma)=\{\mathbf{2},\mathbf{3}\}$ in the sets $\Sigma(b)$ and $\Sigma(c)$. We see that the element $c'=c\mathbf{2}$ satisfies c'—c, $\mathscr{R}(c')=\{\mathbf{2},\mathbf{3}\}\supsetneqq\{\mathbf{3}\}=\mathscr{R}(c)$ and $a(c')=5$. More precisely, we have $c'\in M(\mathbf{02323})$ and

$$c'\sim_L\mathbf{023231234234232312 31}=\mathbf{0232312343231 23}=d\in M'(\mathbf{02323}),$$

i.e. $\Gamma_d\in\Sigma(c)$ (see 2.4, (1)). Now we need only to show $\Gamma_d\notin\Sigma(b)$. Consider all the elements b' satisfying b'—b, $a(b')=5$ and $\mathscr{R}(b')=\{\mathbf{2},\mathbf{3}\}$. They are $b'_1=\mathbf{023231232}$, $b'_2=\mathbf{0232312343232}$ and z', where $b'_1, b'_2\in M'(\mathbf{02323})$ with $d\neq b'_1, b'_2$. So $z', b'_i\notin\Gamma_d$ for $i=1,2$, i.e. $\Gamma_d\notin\Sigma(b)$. ■

5.12. Now we are ready to show the assertion (1) of Proposition 5.7.

The proof of Proposition 5.7, (1). Recall that for $\eta=\alpha'\mathbf{1}$, we have $\Gamma_\eta\in\Sigma(\alpha')$ with $\mathscr{R}(\Gamma_\eta)=\{\mathbf{1},\mathbf{3},\mathbf{4}\}$. By Propositions 5.11 and 4.2, we see that the set of left cells represented by the set $M(\eta)$ is disjoint from those represented by the union $M'(\mathbf{02323})\cup M(y')\cup M(z')\cup M(w')$.

By Theorem 2.1, we need only to show that none of the left cells Γ with $\mathscr{R}(\Gamma)=\{1,3,4\}$ in $\Sigma(\alpha)$ is equal to Γ_η. It is equivalent to showing that none of the elements b with b—α, $\mathscr{R}(b)=\{1,3,4\}$ and $a(b)=5$ is in Γ_η. By direct computation, we see that such an element b must be one of the following elements: $b_1=\alpha 1$, $b_2=02323431$, $b_3=0232341234$. It is easily seen that $b_2\in M'(02323)$ and $b_3=y'$. Now our proof is completed by the following:

LEMMA 5.13. $b_1\sim_L y'$.

PROOF. We have $b_1=023234312341$ and $y'=0232341234$ with $\mathscr{L}(b_1)=\mathscr{L}(y')=\{0,2,3\}$. Let $b'=4b_1$. Then $\mathscr{L}(b')=\{0,2,4\}$ and so $b'\sim_L b_1$. On the other hand, we have $b'=4023234312341=234y'$ with $\ell(b')=\ell(y')+3$, and so $b'\leq_L y'$. This implies

$$b_1\leq_L y' . \tag{5.13.1}$$

Now consider the graph in Figure 14, where the vertices x, y with $\mathscr{R}(x)=\{3,4\}$ and $\mathscr{R}(y)=\{1,3,4\}$ are the elements α, b_1, respectively. By Propositions 4.1 and 4.5, we see that $\alpha\sim_R b_1$ and hence $a(b_1)=5=a(y')$. So by 1.6 (2), we get $b_1\sim_L y'$ from (5.13.1). ∎

5.14. Let $\mathfrak{M}'(z')$ be the subgraph of $\mathfrak{M}(z')$ as in Figure 15 and let $M'(z')$ be its vertex set. Then by the discussion in 5.5–5.13, we know that the union

$$M'(02323)\cup M(y')\cup M'(z')\cup M(w')\cup M(\eta)$$

is both distinguished and A-saturated. We can check that it is also B-saturated and hence forms a representative set of left cells of W in $W_{(5)}$ by Theorem 3.1.

REMARK 5.15. In the above examples, we always take a starting distinguished set

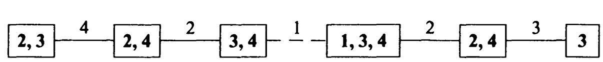

FIGURE 14.

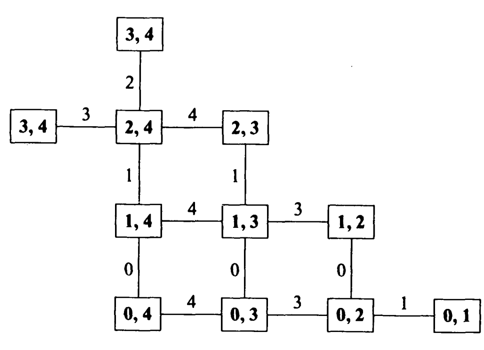

FIGURE 15. $\mathfrak{M}'(z')$

X in the algorithm to be a set of elements of the form w_I, $I \subset S$. This is because all the two-sided cells which we have dealt with contain such elements. But in general, it could happen that there is no element of such form in a given two-sided cell Ω, e.g. when Ω is the two-sided cell with $a(\Omega)=7$ in W of type $\tilde{D}_4$ or with $a(\Omega)=13$ in W of type $\tilde{F}_4$. When this happens, we may choose the elements of the set X in a standard parabolic subgroup of W by Theorem 1.7, (3), which is easier to find.

REFERENCES

[1] R. BÉDARD, Cells for two Coxeter groups, Comm. Algebra 14 (1986), 1253–1286.

[2] CHEN CHENGDONG, Two-sided cells in affine Weyl groups, Northeastern Math. J. 6 (1990), 425–441.

[3] CHEN CHENGDONG, Cells of the affine Weyl group of type $\tilde{D}_4$, to appear in J. Algebra.

[4] DU JIE, The decomposition into cells of the affine Weyl group of type $\tilde{B}_3$, Comm. Algebra 16 (1988), 1383–1409.

[5] DU JIE, Cells in the affine Weyl group of type $\tilde{D}_4$, J. Algebra 128 (1990), 384–404.

[6] D. KAZHDAN AND G. LUSZTIG, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), 165–184.

[7] D. KAZHDAN AND G. LUSZTIG, Schubert varieties and Poincaré duality, Proc. Sympos. Pure Math. 36, Amer. Math. Soc., (1980), 185–203.

[8] G. M. LAWTON, On cells in affine Weyl groups, Ph. D. Thesis, MIT, 1986.

[9] G. LUSZTIG, Some examples in square integrable representations of semisimple p-adic groups, Trans. Amer. Math. Soc. 277 (1983), 623–653.

[10] G. LUSZTIG, The two-sided cells of the affine Weyl group of type $\tilde{A}_n$, in "Infinite Dimensional Groups with Applications", (V. Kac, ed.), MSRI. Publications 4, Springer-Verlag, 1985, pp. 275–283.

[11] G. LUSZTIG, Cells in affine Weyl groups, in "Algebraic Groups and Related Topics", (R. Hotta, ed.) Advanced Studies in Pure Math., Kinokuniya and North Holland, 1985, pp. 255–287.

[12] G. LUSZTIG, Cells in affine Weyl groups, II, J. Algebra 109 (1987), 536–548.

[13] G. LUSZTIG, Cells in affine Weyl groups, III, J. Fac. Sci. Univ. Tokyo Sect. IA. Math. 34 (1987), 223–243.

[14] G. LUSZTIG, Cells in affine Weyl groups, IV, J. Fac. Sci. Univ. Tokyo Sect. IA. Math. 36 (1989), 297–328.

[15] RUI HEBING, Some cells in affine Weyl groups of types other than $\tilde{A}_n$ and $\tilde{G}_2$, preprint.

[16] JIAN-YI SHI, "The Kazhdan-Lusztig cells in certain affine Weyl groups," Lecture Notes in Math. 1179, Springer-Verlag, New York, Heidelberg and Berlin, 1986.

[17] JIAN-YI SHI, Alcoves corresponding to an affine Weyl group, J. London Math. Soc. 35 (1987), 42–55.

[18] JIAN-YI SHI, A two-sided cell in an affine Weyl group, J. London Math. Soc. 36 (1987), 407–420.

[19] JIAN-YI SHI, A two-sided cell in an affine Weyl group, II, J. London Math. Soc. 37 (1988), 253–264.

[20] JIAN-YI SHI, Left cells in the affine Weyl group $W_a(\tilde{D}_4)$, to appear in Osaka J. Math.

[21] JIAN-YI SHI, The classification of left cells of the affine Weyl group of type $\tilde{C}_4$, in preparation.

[22] JIAN-YI SHI, The verification of a conjecture on left cells of certain Coxeter groups, to appear in Hiroshima J. Math.

[23] D. VOGAN, A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra, Math. Ann. 242 (1979), 209–224.

[24] ZHANG XIN-FA, Cells decomposition in the affine Weyl group $W_a(\tilde{B}_4)$, to appear in Comm. Algebra.

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