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ADDENDUM: CONJUGATE EXPANSIONS FOR ULTRASPHERICAL FUNCTIONS

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I have recently become aware of Horváth [1], where, among other things, another approach to the conjugacy for ultraspherical function expansions is suggested and investigated. To be precise, a Hilbert transform that occurs there is given by the mapping

$$\varphi_n^\lambda(\theta) \mapsto \sqrt{\frac{n}{n+2\lambda}}\,\varphi_{n-1}^{\lambda+1}(\theta)\,. \tag{1}$$

Our Hilbert transform in [3, (1.3)] was given by

$$\varphi_n^\lambda(\theta) \mapsto \frac{\sqrt{n(n+2\lambda)}}{n+\lambda}\,\varphi_{n-1}^{\lambda+1}(\theta) \tag{2}$$

and the factors $\sqrt{n(n+2\lambda)/(n+\lambda)}$ were crucial to tie both, the Poisson and conjugate Poisson integrals, into a partial differential equation playing a role of a Cauchy-Riemann equation (cf. the remarks on p. 463 of [3]). Note also that the Hilbert transform appearing in Muckenhoupt and Stein [2], from which both [1] and [3] took their origins, was defined by

$$\tilde{P}_n^\lambda(\cos\theta) \mapsto \sqrt{\frac{n}{n+2\lambda}}\,\tilde{P}_{n-1}^{\lambda+1}(\cos\theta)\cdot\sin\theta \tag{3}$$

(here $\tilde{P}_n^\lambda$ denotes the n-th orthonormalized ultraspherical polynomial) so it is clear that (1) arises from (3) just by multiplying both sides of (3) by $(\sin\theta)^\lambda$, the square root of the weight function.

Let T denote the multiplier operator given by the "adjusting" sequence $m_n = (n+2\lambda)/(n+\lambda)$

$$T\left(\sum_{n=0}^{\infty} b_n\varphi_n^\lambda\right) = \sum_{n=0}^{\infty} m_n b_n \varphi_n^\lambda\,.$$

Consequently, the Hilbert transforms H_1 and H_2 defined by (1) and (2) are related by $H_2 = H_1 T$ and $H_1 = H_2 T^{-1}$. Since both sequences $\{m_n\}$ and $\{m_n^{-1}\}$ have 1 as the limit at infinity and satisfy $\sum_{k=0}^{\infty}(k+1)^N |\Delta^{N+1} a_k| \le C_N$ for every $N = 0, 1, \ldots$ it follows from Trebels [4, p. 21 and p. 88] that T and T^{-1} are bounded on $L^p((0,\pi), d\theta)$ for $1 \le p < \infty$.

Thus, for $1<p<\infty$ the equivalence of L^p-boundedness of H_1 and H_2 is clear. The L^1-boundedness of T and T^{-1} also easily implies the equivalence of weak type (1, 1) estimates for associated conjugate maximal operators ([1, Lemma 3] and [3, Theorem 3.2]).

It follows from the remarks on weighted inequalities in [2, §17] that [2, Theorem 4(a)] is extendible in the sense discussed there, cf. [2, (17.1)], for $1<p<\infty$ and $\alpha=2\lambda(1/2-1/p)$ which satisfies the (R_1) condition in [2, p. 89]. This simply means that Horváth's conjugacy result [1, Theorem 2(a)] is implied by the weighted conjugacy result of Muckenhoupt and Stein ([2, Theorem 4(a)] in the sense of (17.1)). The weak type (1, 1) estimates, [1, Lemma 3] and, equivalently, [3, Theorem 3.2], require, however, an independent argument.

I would also like to acknowledge that our results concerning Poisson integrals, [3, Theorem 2.2], are identical with those in [1, Lemma 1 and Theorem 1]. These, as is easily seen in [3, §2], are rather straightforward consequences of the fundamental estimates in [2]. Finally, I thank Walter Trebels to whom I owe a lot of insightful comments.

References

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