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LIAPUNOV FUNCTIONALS AND PERIODICITY IN INTEGRAL EQUATIONS

THEODORE A. BURTON

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Abstract. Liapunov's direct method has been used very effectively for a hundred years on various types of differential equations. It has not, however, been used with much success on non-differentiated equations. In this paper we construct a Liapunov function for a nonlinear integral equation with an infinite delay which is nonconvolution type. From that Liapunov function we deduce conditions for boundedness, stability, and the existence of periodic solutions. The kernel of the integral equation is a perturbation of a positive kernel and there are estimates showing how large the perturbation can be. The advantage of the Liapunov approach over classical methods for integral equations is the simplicity of analysis, once a Liapunov function is constructed.

1. Introduction. Liapunov functions and functionals have been used very effectively on ordinary, functional, and partial differential equations, but have had little application to nondifferentiated equations (cf. Miller [12; p. 337] and Gripenberg et al. [5; p. 426]). The reason for this is simple. Given

$$x' = f(t, x), \quad ' = d/dt,$$

and any differentiable scalar function

$$V(t, x),$$

if $x(t)$ is a solution, then $V(t, x(t))$ is a scalar function of t and we can compute

$$dV(t, x(t))/dt = \operatorname{grad} V \cdot f + \partial V/\partial t.$$

That is, we can find the derivative of V along the solution directly from the differential equation. If it turns out, for example, that $dV/dt \leq 0$, then this may yield much information about the behavior of the unknown solution.

By contrast, if

$$x(t) = a(t) + \int_{-\infty}^{t} g(t, s, x(s))ds,$$

it seems unclear how to relate the derivative of a scalar function $V(t, x)$ to the unknown solution. Indeed, Miller [12; p. 337] proceeds only under the assumption that the inte-

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gral equation can be differentiated. Gripenberg et al. [5; p. 426] dismiss Liapunov's direct method out of hand saying that the analogues "for integral and functional equations are of little practical interest."

Our thesis here is that the direct method of Liapunov is of great interest in functional equations and we present examples to support that view.

It this paper we construct Liapunov functionals for equations of the form

$$x(t)=a(t)-\int_{-\infty}^{t} D(t,s)g(s,x(s))ds\,, \tag{1}$$

which are closely releated to a combination of Liapunov functionals constructed by Levin ([7], [8]) for variants of

$$x'(t)=-\int_{0}^{t} D(t,s)g(x(s))ds$$

and by the author [1] for

$$x'(t)=Ax+\int_{0}^{t} B(t,s)x(s)ds\,.$$

These Liapunov functionals have properties in sharp contrast to those for differential equations.

In the classical theory of Liapunov's direct method for a functional differential equation of the form

$$x'=F(t,x_t)$$

(see Lakshmikantham and Leela [6] or Yoshizawa [14] for standard theory and explanation of notation), one seeks a functional $V(t,\phi)$ with at least the property that

$$W(|\phi(0)|)\le V(t,\phi)\,,\quad V(t,0)=0\,,$$

where W is a strictly increasing function. Thus, if $V'(t,x_t)\le 0$, then the zero solution is stable. Such functions W are prominently missing for integral equations and one is forced to other methods. The interesting part is that one can frequently derive the required W along solutions; and that is all that is needed to prove the classical relations.

Equations of this sort are often written as

$$x(t)=A(t)-\int_{0}^{t} D(t,s)g(s,x(s))ds\,, \tag{F}$$

where $A(t)$ now contains both $a(t)$ and $\int_{-\infty}^{0} D(t,s)g(s,\phi(s))ds$, and where ϕ is a given initial function. Conditions commonly required on D will ensure that $A(t)\in L^1[0,\infty)$ for a bounded initial function ϕ.

Much has been written about this equation when D is of convolution type, often using the theory of positive kernels. A selection may be found in Levin [9], London [10], MacCamy [11], and Staffans [13]. Physical problems described by such equations are found in Gripenberg et al. [5; pp. 4–13], MacCamy [11; pp. 570–574], and Miller [12; pp. 62–73], for example.

2. A scalar integral equation. Let $D: R\times R\to R$ with both D and $\int_{-\infty}^{t}|D(t,s)|ds$ being continuous, let $a: R\to R$ be continuous, and let $g: R\times R\to R$ and $g_i: R\to R$ all be continuous with $xg(t,x)>0$ if $x\neq 0$, $|g_1(x)|\le|g(t,x)|\le|g_2(x)|$, $xg_1(x)>0$ if $x\neq 0$. Consider the equation

$$x(t)=a(t)-\int_{-\infty}^{t} D(t,s)g(s,x(s))ds\,. \tag{1}$$

If ϕ: $(-\infty, t_0]\to R$ is a given bounded and continuous initial function, then there is a continuous solution $x(t, t_0, \phi)$ defined on an interval $[t_0, \alpha)$ and satisfying (1) on that interval, while agreeing with ϕ on $(-\infty, t_0]$, provided that φ is chosen so that (1) is an identity at $t=t_0$ (see, [12] and [5; p. 538]). If the solution remains bounded then it can be continued for all future time. It is always assumed that φ is chosen so that the solution is continuous.

We suppose that there are continuous functions B, Q: $R\times R\to R$ with

$$B(t,s)=D(t,s)+Q(t,s)\,, \tag{2}$$

$$B_s(t,s)\ge 0\,,\quad B_{st}(t,s)\le 0\,, \tag{3}$$

$$\int_{-\infty}^{t}[|B(t,s)|+B_s(t,s)(t-s)^2+|B_{st}(t,s)|+|Q(t,s)|]ds \quad \text{continuous}\,, \tag{4}$$

$$\lim_{s\to-\infty}(t-s)B(t,s)=0 \quad \text{for fixed } t\,, \tag{5}$$

$$\int_{0}^{\infty}|Q(u+t,t)|du+\int_{-\infty}^{t}\int_{t-s}^{\infty}|Q(u+s,s)|duds \quad \text{exists for } t\ge 0\,. \tag{6}$$

Much can be deduced from the following result. We shall give a few possibilities.

THEOREM 1. *If $x(t)$ is a solution of (1) on $[t_0, \alpha)$, then the functional*

$$V(t,x(\cdot))=\int_{-\infty}^{t} B_s(t,s)\left(\int_{s}^{t} g(v,x(v))dv\right)^2 ds+k\int_{-\infty}^{t}\int_{t-s}^{\infty}|Q(u+s,s)|du\,g^2(s,x(s))ds \tag{7}$$

satisfies

$$\left[a(t)-x(t)+\int_{-\infty}^{t} Q(t,s)g(s,x(s))ds\right]^2\le V(t,x(\cdot))B(t,t) \tag{8}$$

and

$$V'_{(1)}(t, x(\cdot)) \le 2g(t, x(t))[a(t)-x(t)] - (k-1)\int_{-\infty}^{t} |Q(t, s)|g^2(s, x(s))ds \tag{9}$$
$$+\left[\int_{-\infty}^{t} |Q(t, s)|ds + k\int_{0}^{\infty} |Q(u+t, t)|du\right]g^2(t, x).$$

Proof. We apply Schwarz's inequality to (7) and have

$$V(t, x(\cdot)) \ge \left(\int_{-\infty}^{t} B_s(t, s)\int_{s}^{t} g(v, x(v))dvds\right)^2 \Big/ B(t, t).$$

Integrate by parts and use (5), together with the fact that there is a bounded initial function to obtain

$$V(t, x(\cdot))B(t, t) \ge \left[B(t, s)\int_{s}^{t} g(v, x(v))dv\,\Big|_{s=-\infty}^{s=t} + \int_{-\infty}^{t} B(t,s)g(v,x(v))dv\right]^2.$$

The first term on the right is zero. When B is separated as in (2) and (1) is used, we have

$$V(t, x(\cdot))B(t, t) \ge \left[a(t)-x(t)+\int_{-\infty}^{t} Q(t, s)g(s, x(s))ds\right]^2$$

so that (8) holds.

Denote the last term in V by $Z(t)$ and compute

$$V'(t, x(\cdot)) = \int_{-\infty}^{t} B_{st}(t, s)\left(\int_{s}^{t} g(v, x(v))dv\right)^2 ds$$
$$+\int_{-\infty}^{t} B_s(t, s)\,2\int_{s}^{t} g(v, x(v))dv\,ds\,g(t, x(t)) + Z'(t)$$
$$\le 2g(t, x(t))\left[B(t, s)\int_{s}^{t} g(v, x(v))dv\,\Big|_{s=-\infty}^{s=t} + \int_{-\infty}^{t} B(t, s)g(s, x(s))ds\right] + Z'(t)$$
$$= 2g(t, x(t))\left[\int_{-\infty}^{t} D(t, s)g(s, x(s))ds + \int_{-\infty}^{t} Q(t, s)g(s, x(s))ds\right] + Z'(t)$$
$$\le 2g(t, x(t))[a(t)-x(t)] + g^2(t, x(t))\int_{-\infty}^{t} |Q(t, s)|ds$$
$$+\int_{-\infty}^{t} |Q(t, s)|g^2(s, x(s))ds + k\int_{0}^{\infty} |Q(u+t, t)|du\,g^2(t, x(t))$$
$$-k\int_{-\infty}^{t} |Q(t, s)|g^2(s, x(s))ds$$

$$=2g(t, x(t))[a(t)-x(t)]+g^2(t, x(t))\left[\int_{-\infty}^{t} |Q(t, s)|ds+k\int_{0}^{\infty} |Q(u+t, t)|du\right]$$

$$-(k-1)\int_{-\infty}^{t} |Q(t, s)|g^2(s, x(s))ds\,,$$

as required.

Many interesting consequences can be derived from (8) and (9). We begin with two extreme cases.

COROLLARY 1. *If $a(t)=Q(t, s)=0$ and if $\int_{-\infty}^{t} D_s(t, s)ds \le 1/M$ for some M, then along any solution $x(t)$ we have*

$$Mx^2(t) \le V(t, x(\cdot))$$

and

$$V'(t, x(\cdot)) \le -2g(t, x(t))x(t) \le -2g_1(x(t))x(t)\,.$$

Thus, $x(t)$ is bounded, $x=0$ is stable, and

$$\int^{\infty} g_1(x(t))x(t)dt < \infty\,.$$

We later give three kinds of conditions to ensure that $x(t)\to 0$ as $t\to\infty$ when $\alpha=0$.

REMARK. Notice that Cor. 1 has no growth condition on g, but there will be in Cor. 2 and 3 when $Q(t, s)\neq 0$. In effect, Q is a "perturbation term" and the bounds on Q offer a measure of how far D can deviate from the conditions on B. In addition, Condition (10) will itself be a growth condition on g when $Q\neq 0$. To start the completion of Cor. 1, examine Cor. 4 and note that when $Q=0$, then no growth condition on g is required to conclude that $x(t)\to 0$. Cor. 5 asks that g satisfy a local Lipschitz condition in order to conclude that $x(t)\to 0$. Compare this with the discussion of MacCamy [11; pp. 556–7] who surveys convolution counterparts of our problems. In those result, $Q(t, s)=0$ and growth conditions on g are required of the form $|g(u)| \le M(1+\int_0^u g(\xi)d\xi)$ and, sometimes, $\limsup_{u\to 0} g(u)/u < \infty$.

COROLLARY 2. *If $k=1$, $B(t, s)=a(t)=0$ (so that $Q(t, s)=-D(t, s)$), and if there is a $\beta<2$ with*

$$\beta x g(t, x) \ge \left[\int_{-\infty}^{t} |D(t, s)|ds+\int_{0}^{\infty} |D(u+t, t)|du\right]g^2(t, x)\,,$$

then $\int^{\infty} x(t)g(t, x(t))dt < \infty$ holds for any solution $x(t)$ on $[t_0, \infty)$.

There are many possible variants of the next lemma. It is the natural extension of the statement that the convolution of an L^1-function with a function tending to zero, itself tends to zero.

LEMMA 1. *Let* $h:[0, \infty)\to[0, \infty)$ *with* $\int_0^\infty h(s)ds<\infty$ *and let* $C: R\times R\to R$ *be continuous with* $|C(t, s)|\le K$ *if* $0\le s\le t$ *for some* $K>0$. *Suppose also that for each* $P>0$ *we have* $\limsup_{t\to\infty}\,_{0\le s\le P}|C(t, s)|=0$. *Then* $\int_0^t C(t, s)h(s)ds\to 0$ *as* $t\to\infty$.

PROOF. Let $\varepsilon>0$ be given and choose $P>0$ so that $\int_P^\infty Kh(t)dt<\varepsilon/2$. Then

$$\int_0^t |C(t, s)|h(s)ds\le\int_0^P |C(t, s)|h(s)ds+K\int_P^\infty h(s)ds$$

$$\le \sup_{0\le s\le P}|C(t, s)|\int_0^P h(s)ds+\varepsilon/2.$$

The next lemma will be used repeatedly.

LEMMA 2. *Let* $xg(t, x)>0$ *if* $x\neq 0$, g *be continuous and bounded for* x *bounded,* $|a(t)|\le A/2$ *for some* $A>0$ *and all* t, *and let* $c_1>0$. *Then there is an* $M>0$ *with*

$$-2c_1xg(t, x)+2|g(t, x)||a(t)|\le -c_1xg(t, x)+M|a(t)|.$$

PROOF. We have

$$K(t, x):=-2c_1xg(t, x)+2|g(t, x)a(t)|$$

$$\le -c_1xg(t, x)+|g(t, x)|[-c_1|x|+2|a(t)|].$$

If $|x|\ge 2A/c_1$, then $-c_1|x|+2|a(t)|\le -2A+A<0$. If $|x|\le 2A/c_1$, then $2|g(t, x)|\le M$ for some $M>0$, and the proof is complete.

COROLLARY 3. *Suppose there is a* $k\ge 1$ *and a* $\beta<2$ *such that*

$$\beta xg(t, x)\ge\left[\int_{-\infty}^t |Q(t, s)|ds+k\int_0^\infty |Q(u+t, t)|du\right]g^2(t, x) \tag{10}$$

and that $a(t)$ *is both bounded and* L^1 $[0, \infty)$. *Then for any solution* $x(t)$ *of* (1) *on* $[t_0, \infty)$ *we have* $\int_{t_0}^\infty x(t)g(t, x(t))dt<\infty$. *If, in addition,* $\int_{-\infty}^t D_s(t, s)ds$ *is bounded,* $|g(t,x)|\le J|x|$ *for some* $J>0$, *if* $\int_{-\infty}^{t_0}|D(t, s)|ds\to 0$ *as* $t\to\infty$, *if* $\int_{t_0}^t |D(t, s)|ds$ *is bounded, and if for each* $P>0$ *we have* $\limsup_{t\to\infty}\,_{0\le s\le P}|D(t, s)|=0$, *then* $x(t)\to a(t)$ *as* $t\to\infty$.

PROOF. By (9), (10), and Lemma 2 we have $V'(t, x(\cdot))\le -c_1x(t)g(t, x(t))+M|a(t)|$. Since $V\ge 0$, the first conclusion holds. Next, if ϕ is the bounded initial function on $(-\infty, t_0]$ with $g^*\ge|g(t, \phi(t))|$ on $(-\infty, t_0]$, then

$$(*)\qquad \left|\int_{-\infty}^{t} D(t,s)g(s,x(s))ds\right| \le \left[\int_{t_0}^{t} |D(t,s)|ds \int_{t_0}^{t} |D(t,s)|g^2(s,x(s))ds\right]^{1/2} + \int_{-\infty}^{t_0} |D(t,s)|g^* ds\,.$$

Since $|g(t,x)|\le J|x|$, it follows that $\int_{t_0}^{\infty} g^2(s,x(s))ds<\infty$ because $\int_{t_0}^{\infty} x(s)g(s,x(s))ds<\infty$. Thus, by Lemma 1, $(*)$ tends to zero as $t\to\infty$ and the conclusion follows from (1).

A classical results for a finite delay equation $x'=F(t,x_t)$ states (see [14; p. 191]) that if there is a $V(t,\phi)$ and increasing function W_i with

(i) $W_1(|\phi(0)|)\le V(t,\phi)\le W_2(\|\phi\|)$,

(ii) $V'(t,x_t)\le -W_3(|x(t)|)$, and

(iii) $|F(t,\phi)|$ is bounded for ϕ bounded,

then $x=0$ is uniformly asymptotically stable. Condition (iii) assures us that a bounded solution is Lipschitz; hence, $\int^{\infty} W_3(|x(t)|)dt<\infty$ implies that $x(t)$ must tend to zero. The following result is a counterpart for integral equations and it leads us to a priori bounds for periodic solutions.

COROLLARY 4. *Let $a(t)\in L^1[0,\infty)$, $a(t)\to 0$ as $t\to\infty$, and either $Q(t,s)=0$ or $|g(t,x)|\le J|x|$ for some $J>0$. Also, for each $t_0\in R$ and each $P>0$ let both*

$$\int_{-\infty}^{t_0} |Q(t,s)|ds\to 0 \qquad \text{as}\quad t\to\infty \quad \text{and} \quad \limsup_{t\to\infty}\ \sup_{0\le s\le P} Q(t,s)=0\,.$$

Finally, suppose there are $k\ge 1$ and $\beta<2$ such that (10) *holds and an $M>0$ such that $|a(t_1)-a(t_2)|\le M|t_1-t_2|$, $\int_{-\infty}^{t}|B_s(t,s)|ds\le M$, $\int_{t_0}^{t}|Q(t,s)|ds\le M$, and*

$$(11)\qquad \int_{-\infty}^{t_1} |D(t_1,s)-D(t_2,s)|ds\le M|t_1-t_2| \quad \text{for}\quad 0\le t_1\le t_2<\infty$$

and $|t_1-t_2|$ small. Then every solution $x(t)$ is defined on $[t_0,\infty)$ and $x(t)\to 0$ as $t\to\infty$.

PROOF. By the proof of Cor. 3 we have V bounded and $\int_{t_0}^{\infty} x(t)g(t,x(t))dt<\infty$. By assumption $|g_1(x)|\le|g(t,x)|\le|g_2(x)|$ where $xg_1(x)>0$ if $x\ne 0$. If $x(t)\not\to 0$, then there is an $\varepsilon>0$ and a sequence $\{t_n\}\uparrow\infty$ with $|x(t_n)|\ge\varepsilon$. Since V is bounded, if $Q=0$, then from (8) and $a(t)\to 0$, we have $x(t)$ bounded. If $Q\ne 0$, then $|g(t,x)|\le J|x|$ so $\int^{\infty} x(t)g(t,x(t))dt<\infty$ yields $\int^{\infty} g^2(t,x(t))dt<\infty$; and this implies that

$$\left|\int_{-\infty}^{t} Q(t,s)g(s,x(s))ds\right| \le \left|\int_{-\infty}^{t_0} Q(t,s)g(s,x(s))ds\right| + \left[\int_{t_0}^{t} |Q(t,s)|ds\int_{t_0}^{t} |Q(t,s)|g^2(s,x(s))ds\right]^{1/2}$$

and this tends to zero. By (8), again, $x(t)$ is bounded and so, in any case, $|g(t, x(t))| \leq L$ for some $L > 0$.

From the boundedness of $x(t)$ and $\int^{\infty} x g_1(x) dt < \infty$, we can suppose that $|x(t_n)| = \varepsilon$ and choose another sequence $\{s_n\} \uparrow \infty$ with $|x(s_n)| = \varepsilon/2$ and $\varepsilon \geq |x(t)| \geq \varepsilon/2$ for $t_n \leq t \leq s_n$. Thus,

$$\begin{aligned}
\varepsilon/2 \leq |x(t_n) - x(s_n)| &\leq |a(t_n) - a(s_n)| \\
&\quad + \left| \int_{-\infty}^{t_n} D(t_n, s) g(s, x(s)) ds - \int_{-\infty}^{t_n} D(s_n, s) g(s, x(s)) ds \right| \\
&\quad + \left| \int_{-\infty}^{t_n} D(s_n, s) g(s, x(s)) ds - \int_{-\infty}^{s_n} D(s_n, s) g(s, x(s)) ds \right| \\
&\leq M |t_n - s_n| + L \int_{-\infty}^{t_n} |D(t_n, s) - D(s_n, s)| ds \\
&\quad + \left| \int_{s_n}^{t_n} D(s_n, s) g(s, x(s)) ds \right| \leq (M + LM + LB) |t_n - s_n| ,
\end{aligned}$$

where $\int_{t_n}^{s_n} |D(s_n, s)| ds \leq B$. This yields $|t_n - s_n| \geq \delta$ for some $\delta > 0$, contradicting $\int^{\infty} x(t) g_1(x(t)) dt < \infty$ while $|x(t)| \geq \varepsilon/2$ on $[t_n, s_n]$. This completes the proof.

There is a third way to drive $x(t)$ to zero.

COROLLARY 5. *Let $Q(t, s) = 0$, $a(t) \to 0$ as $t \to \infty$, $a(t) \in L^1 [0, \infty)$, and $\int_{-\infty}^{t} D_s(t, s) ds$ be bounded. Suppose also that for each $P > 0$ we have $\int_{-\infty}^{P} D_s(t, s)(t-s)^2 ds \to 0$ as $t \to \infty$ and that there is an M independent of P with $\int_{P}^{t} D_s(t, s)(t-s) ds \leq M$. If, in addition, for each $K > 0$ there is a $J > 0$ such that $|x| \leq K$ implies $|g(t, x)| \leq J|x|$, then $x(t) \to 0$ as $t \to \infty$.*

PROOF. From (8), if $V(t, x(\cdot)) \to 0$, so does $x(t)$. Since $x(t)$ is bounded by (8) and the fact that V is bounded, we have $\int^{\infty} g^2(t, x(t)) dt < \infty$. By Schwarz's inequality we obtain

$$\begin{aligned}
V(t, x(\cdot)) &\leq \int_{-\infty}^{t} D_s(t, s)(t-s) \int_{s}^{t} g^2(v, x(v)) dv ds \\
&\leq \int_{-\infty}^{P} D_s(t, s)(t-s) \int_{s}^{t} g^2(v, x(v)) dv ds + \int_{P}^{t} D_s(t, s)(t-s) \int_{P}^{\infty} g^2(v, x(v)) dv ds \\
&\leq J^2 \int_{-\infty}^{P} D_s(t, s)(t-s)^2 ds + \left(\int_{P}^{\infty} g^2(v, x(v)) dv \right) \int_{P}^{t} D_s(t, s)(t-s) ds .
\end{aligned}$$

The last integral is bounded by M, while its coefficient tends to zero as $P \to \infty$. This completes the proof.

REMARK. In the next result, notice that the a priori bound does not require V to be positive, as in [3]. The a priori bound comes from V' alone.

COROLLARY 6. *Let* (10) *hold and suppose there is a* $T>0$ *with* $a(t+T)=a(t)$, $g(t+T,x)=g(t,x)$, $D(t+T,s+T)=D(t,s)$, *and* $B_s(t+T,s+T)=B_s(t,s)$. *Suppose, in addition, that* $xg_1(x)\to\infty$ *as* $|x|\to\infty$, *and that there is an* $M>0$ *with* $|g(t,x)|\le M|x|$, $\sup_{0\le t\le T}\int_{-\infty}^{t}|D(t,s)|ds\le M$, $|a(t_1)-a(t_2)|\le M|t_1-t_2|$, $|g(t,x_1)-g(t,x_2)|\le M|x_1-x_2|$, *and that* $\int_{-\infty}^{t_1}|D(t_1,s)-D(t_2,s)|ds\le M|t_1-t_2|$ *for* $0\le t_1\le t_2\le T$. *Then there is a* $K>0$ *such that if* $x(t)$ *in any* T-*periodic solution of* (1), *then* $\sup_{0\le t\le T}|x(t)|=:\|x\|\le K$ *and there is a* T-*periodic solution.*

PROOF. Let $0\le\lambda\le 1$ and write (1) as

$$x(t)=\lambda a(t)-\int_{-\infty}^{t} D(t,s)\lambda g(s,x(s))ds \tag{1_λ}$$

so that if

$$V(t,x(\cdot))=\int_{-\infty}^{t} B_s(t,s)\left(\int_s^t \lambda g(v,x(v))dv\right)^2 ds\,,$$

then we obtain

$$V'(t,x(\cdot))\le -c_1x(t)\lambda g(t,x(t))+M\lambda|a(t)|\,.$$

We now show that there is an a priori bound on any T-periodic solution $x(t)$ of (1_λ). If $\lambda=0$, then $\|x\|=0$. If $\lambda>0$, since V is also T-periodic, $0=V(T,x(\cdot))-V(0,x(\cdot))\le -c_1\int_0^T x(s)\lambda g(s,x(s))ds+G\lambda$ where $G=MT\|a\|$; thus, λ divides out and we have $\int_0^T x(s)g(s,x(s))ds\le G/c_1$. Next, let $0\le t_1\le t_2\le T$, $|x(t_1)|=\|x\|$, and consider (recalling that $|g(t,x)|\le M|x|$)

$$\begin{aligned}|x(t_1)-x(t_2)|&\le\lambda|a(t_1)-a(t_2)|+\lambda\left|\int_{-\infty}^{t_1}[D(t_1,s)-D(t_2,s)]g(s,x(s))ds\right|\\&\quad+\lambda\left|\int_{-\infty}^{t_1}D(t_2,s)g(s,x(s))ds-\int_{-\infty}^{t_2}D(t_2,s)g(s,x(s))ds\right|\\&\le M|t_1-t_2|+M^2|t_1-t_2|\,\|x\|+M\|x\|\int_{t_1}^{t_2}|D(t_2,s)|ds\\&\le(M+M^2\|x\|+BM\|x\|)|t_1-t_2|\le J(1+\|x\|)|t_1-t_2|\end{aligned}$$

for $|D(t_2,s)|\le B$ if $0\le s\le T$ and some $J>0$. If $J|t_1-t_2|\le 1/2$, then $|x(t_1)|-|x(t_2)|\le|x(t_1)-x(t_2)|\le(1+\|x\|)/2$ and $|x(t_1)|=\|x\|$ so $\|x\|/2\le 1/2+|x(t_2)|$. If $\|x\|\le 2$, then this is an a priori bound. If $\|x\|\ge 2$, then $\|x\|\le 1+2|x(t_2)|\le\|x\|/2+2|x(t_2)|$ or $\|x\|/2\le 2|x(t_2)|$ so that $|x(t_2)|\ge\|x\|/4$ if $|t_1-t_2|\le 1/2J$. But

$$\int_0^T x(t)g_1(x(t))dt\le\int_0^T x(t)g(t,x(t))dt\le G/c_1$$

and $xg_1(x)\to\infty$ as $|x|\to\infty$, while $|x(t_2)|\geq\|x\|/4$ if $|t_1-t_2|\leq 1/2J$. Thus, the required bound on $\|x\|$ exists for $0\leq\lambda\leq 1$.

Next, let $(P, \|\cdot\|)$ be the Banach space of continuous T-periodic functions with the supremum norm. For $0\leq\lambda\leq 1$ we define a mapping $H_\lambda\colon P\to P$ by $\phi\in P$ implies that

$$H_\lambda(\phi)(t)=\lambda\left[a(t)-\int_{-\infty}^{t} D(t, s)g(s, \phi(s))ds\right]. \tag{12}$$

The degree-theoretic work of Granas [4], as discussed in [2] and [3], will show that (1) has a T-periodic solution provided we can show that:

(a) $H_\lambda\colon P\to P$;

(b) For fixed λ, H_λ maps bounded subsets of P into compact subsets of P;

(c) H_λ is jointly continuous in (λ, ϕ); and

(d) There is a number B such that any T-periodic solution x of (1_λ) satisfies $\|x\|\leq B$.

We have already shown (d). To show (a) we compute

$$\begin{aligned}
H_\lambda(\phi)(t+T)&=\lambda\left[a(t+T)-\int_{-\infty}^{t+T} D(t+T, s)g(s, \phi(s))ds\right]\\
&=\lambda\left[a(t)-\int_{-\infty}^{t} D(t+T, u+T)g(u+T, \phi(u+T))du\right]\\
&=\lambda\left[a(t)-\int_{-\infty}^{t} D(t, u)g(u, \phi(u))du\right]=H_\lambda(\phi)(t),
\end{aligned}$$

whenever $\phi\in P$. To show that $H_\lambda(\phi)$ is continuous in t and lies in a compact set we let $\phi\in P$ with $\|\phi\|\leq K$, where K is an arbitrary positive number. Then

$$\begin{aligned}
|H_\lambda(\phi)(t_1)-H_\lambda(\phi)(t_2)|&\leq\lambda\left[|a(t_1)-a(t_2)|+\left|\int_{-\infty}^{t}[D(t_1, s)-D(t_2, s)]g(s, \phi(s))ds\right|\right.\\
&\qquad\left.+\left|\int_{t_1}^{t_2} D(t_2, s)g(s, \phi(s))ds\right|\right]\\
&\leq\lambda[M|t_1-t_2|+M^2\|\phi\|\,|t_1-t_2|+D^*M\|\phi\|\,|t_2-t_2|],
\end{aligned}$$

where $D^*=\sup_{0\leq s\leq T\,0\leq t_2\leq T}|D(t_2, s)|$. Hence, $H_\lambda(\phi)$ is equicontinuous and bounded by a function of K. This establishes both (a) and (b). To show that $H(\lambda, \phi)$ is jointly continuous in λ and ϕ, for fixed t and for $\phi_i\in P$ we have

$$\begin{aligned}
|H_\lambda(\phi_1)(t)-H_\lambda(\phi_2)(t)|&=\lambda\left|\int_{-\infty}^{t} D(t, s)[g(s, \phi_1(s))-g(s, \phi_2(s))]ds\right|\\
&\leq\lambda\int_{-\infty}^{t} |D(t, s)|M|\phi_1(s)-\phi_2(s)|ds
\end{aligned}$$

$$\leq \lambda M \|\phi_1 - \phi_2\| \int_{-\infty}^{t} |D(t, s)| ds$$

$$\leq U\|\phi_1 - \phi_2\| \qquad \text{for some} \quad U > 0 .$$

Hence, H is continuous in ϕ for fixed λ, uniformly continuous in λ for fixed ϕ, and so is jointly continuous in (λ, ϕ). This completes the proof.

REMARK. When $B(t, s) = 0$, a more flexible Liapunov functional is

$$H(t, x(\cdot)) = k \int_{-\infty}^{t} \int_{t-s}^{\infty} |D(u+s, s)| du |g(s, x(s))| ds$$

with

$$H'(t, x(\cdot)) \leq -\delta \left[|g(t, x)| + \int_{-\infty}^{t} |C(t, s) g(s, x(s))| ds \right] + |a(t)|$$

with $\delta > 0$. From this we conclude that if $a \in L^1$, then $|g(t, x)|$ and $|x|$ are L^1. In the previous corollaries we did not yet use the term $-(k-1)\int_{-\infty}^{t} |C(t, s) g(s, x(s))| ds$ in the derivative of V. But here it can be used effectively and we see that under suitable assumptions relating D to one of its integrals we can obtain

$$H'(t, x(\cdot)) \leq -\gamma H(t, x(\cdot)) + |a(t)| , \qquad \gamma > 0$$

and

$$\mu[|x| - |a(t)|] \leq H(t, x(\cdot)) , \qquad \mu > 0 .$$

Other uses of the $(k-1)$-term are illustrated in Burton [1].

3. A linear vector equation. Let D be a continuous $n \times n$ matrix with $\int_{-\infty}^{t} |D(t, s)| ds$ continuous, $a\colon R \to R^n$ be continuous, and consider the equation

$$x(t) = a(t) - \int_{-\infty}^{t} D(t, s) x(s) ds . \tag{13}$$

It turns out that all of the work in Section 2 can be done for (13) except that we have been unable to obtain a counterpart of (8). Thus, we readily prove that solutions are L^2, that they converge to $a(t)$, and that there are periodic solutions. But we must rely on techniques independent of (8) to show boundedness. Formal counterparts of (2)–(6) are needed. The symbol $|\cdot|$ will denote absolute value as well as compatible vector and matrix norms.

Suppose there are continuous matrix functions B and Q with

$$B(t, s) = D(t, s) + Q(t, s) , \quad B^T(t, s) = B(t, s) , \tag{14}$$

$$x^T B_s(t,s)x \geq 0\,, \quad x^T B_{st}(t,s)x \leq 0\,, \tag{15}$$

$$\int_{-\infty}^{t} [|B(t,s)|+|B_s(t,s)|(t-s)^2+|B_{st}(t,s)|+|Q(t,s)|]ds \quad \text{continuous}\,, \tag{16}$$

$$\lim_{s\to-\infty} |t-s||B(t,s)|=0 \quad \text{for fixed } t\,, \tag{17}$$

$$\int_0^\infty |Q(u+t,t)|du+\int_{-\infty}^{t}\int_{t-s}^{\infty}|Q(u+s,s)|duds \quad \text{exists for } t\geq 0\,. \tag{18}$$

THEOREM 2. *If $x(t)$ is a solution of (13) on $[t_0, \infty)$, then the functional*

$$\begin{aligned} V(t,x(\cdot)) = {} & \int_{-\infty}^{t}\left\{\left[\int_s^t x^T(q)dq\right]B_s(t,s)\int_s^t x(q)dq\right\}ds \\ & +k\int_{-\infty}^{t}\int_{t-s}^{\infty}|Q(u+s,s)|\,du\,|x(s)|^2 ds \end{aligned} \tag{19}$$

satisfies

$$\begin{aligned} V'(t,x(\cdot)) \leq {} & 2|a(t)||x(t)|-\left[2-\int_{-\infty}^{t}|Q(t,s)|ds-k\int_0^\infty|Q(u+t,t)|du\right]|x(t)|^2 \\ & -(k-1)\int_{-\infty}^{t}|Q(t,s)||x(s)|^2ds\,. \end{aligned} \tag{20}$$

PROOF. We have

$$\begin{aligned} V'(t,x(\cdot)) \leq {} & \int_{-\infty}^{t} 2x^T(t)B_s(t,s)\int_s^t x(q)dqds \\ & +k\int_0^\infty|Q(u+t,t)|du|x(t)|^2-k\int_{-\infty}^{t}|Q(t,s)||x(s)|^2ds \\ = {} & 2x^T(t)\left[B(t,s)\int_s^t x(q)dq\,\Big|_{s=-\infty}^{s=t}+\int_{-\infty}^{t}B(t,s)x(s)ds\right] \\ & +k\int_0^\infty|Q(u+t,t)|du|x(t)|^2-k\int_{-\infty}^{t}|Q(t,s)||x(s)|^2ds \\ = {} & 2x^T(t)\left[a(t)-x(t)+\int_{-\infty}^{t}Q(t,s)x(s)ds\right] \\ & +k\int_0^\infty|Q(u+t,t)|du|x(t)|^2-k\int_{-\infty}^{t}|Q(t,s)||x(s)|^2ds \end{aligned}$$

$$\begin{aligned}
&\le 2|a(t)||x(t)| - 2|x(t)|^2 + |x(t)|^2 \int_{-\infty}^{t} |Q(t,s)|\,ds \\
&\quad + \int_{-\infty}^{t} |Q(t,s)||x(s)|^2 ds + k\int_0^{\infty} |Q(u+t,t)|\,du\,|x(t)|^2 \\
&\quad - k\int_{-\infty}^{t} |Q(t,s)||x(s)|^2 ds \\
&\le 2|a(t)||x(t)| - \left[2 - \int_{-\infty}^{t} |Q(t,s)|\,ds - k\int_0^{\infty} |Q(u+t,t)|\,du\right]|x(t)|^2 \\
&\quad - (k-1)\int_{-\infty}^{t} |Q(t,s)||x(s)|^2 ds,
\end{aligned}$$

as required.

At this point we do not have a lower bound parallel to (8); but for linear systems this is not so crucial since solutions can always be defined for all future time. We can prove results for the system parallel to the ones for (1) as follows. In Cor. 1 and 2 we conclude only that $x \in L^2[0, \infty)$. Cor. 3 and 5 say little about the system. Cor. 4 and Cor. 6 hold exactly as they did for (1).

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DEPARTMENT OF MATHEMATICS
SOUTHERN ILLINOIS UNIVERSITY
CARBONDALE, ILLINOIS
U.S.A.