Tôhoku Math. J.
46 (1994), 341–355

INFINITESIMAL ISOMETRIES OF FRAME BUNDLES WITH A NATURAL RIEMANNIAN METRIC II

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(Received March 15, 1993, revised July 6, 1993)

Abstract. We consider the bundle of all oriented orthonormal frames over an orientable Riemannian manifold. This bundle has a natural Riemannian metric which is defined by the Riemannian connection of the base manifold. The purpose of the present paper is to clarify the structure of the Lie algebra of the group of all isometries of the bundle with the Riemannian metric.

1. Introduction. Let $(M, \langle\ ,\ \rangle)$ be a connected orientable Riemannian manifold of dimension $n\geqq 2$ and $SO(M)$ the bundle of all oriented orthonormal frames over M. $SO(M)$ has a Riemannian metric $\langle\ ,\ \rangle$ defined naturally as follows:

$$\begin{aligned}\langle X, Y\rangle &= \langle\theta(X), \theta(Y)\rangle + \langle\omega(X), \omega(Y)\rangle \\ &= {}^t(\theta(X))\theta(Y) + \mathrm{trace}({}^t(\omega(X))\omega(Y)),\end{aligned}$$

where ω and θ are the Riemannian connection form and the canonical form on $SO(M)$, respectively.

In [5], we gave a decomposition of a Killing vector field on $(SO(M), \langle\ ,\ \rangle)$ which is fiber preserving (see Proposition A of § 2) and we proved that M has constant curvature $1/2$, if $(SO(M), \langle\ ,\ \rangle)$ admits a horizontal Killing vector field which is not fiber preserving (see Proposition B of § 2). In the present paper, we give a decomposition of an arbitrary Killing vector field on $(SO(M), \langle\ ,\ \rangle)$ under the assumption that M is complete. The result is stated in the following theorem.

Let p be the projection $SO(M) \to M$. The canonical form θ is an $\boldsymbol{R}^n$-valued 1-form defined by $\theta_u(X) = u^{-1} \circ p(X)$, where u is regarded as a linear isometry of $(\boldsymbol{R}^n, \langle\ ,\ \rangle)$ onto the tangent space at $p(u)$. Let $\mathfrak{o}(n)$ be the Lie algebra of the special orthogonal group $SO(n)$. For each $A \in \mathfrak{o}(n)$, we define a vector field A^* on $SO(M)$ by $\omega(A^*) = A$ and $\theta(A^*) = 0$. A^* is called the fundamental vector field corresponding to A. For each $\xi \in \boldsymbol{R}^n$, we define a vector field $B(\xi)$ on $SO(M)$ by $\omega(B(\xi)) = 0$ and $\theta(B(\xi)) = \xi$. $B(\xi)$ is called the standard horizontal vector field corresponding to ξ. Let ϕ be a 2-form on M and F the tensor field of type (1, 1) on M defined by $\langle FY, Z\rangle = \phi(Y, Z)$. We define an $\mathfrak{o}(n)$-valued function $F^\sharp$ on $SO(M)$ and a vector field ϕ^L or F^L on $SO(M)$ by $F^\sharp(u) =$

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* Partly supported by the Grants-in-Aid for Scientific Research, The Ministry of Education, Science and Culture, Japan.

1991 *Mathematics Subject Classification*. Primary 53C20; Secondary 53B20, 53C07.

$u^{-1} \circ F_{p(u)} \circ u$ and $\omega(F^L) = F^\sharp$, $\theta(F^L) = 0$ (see [5]). ϕ^L and F^L are called the natural lift of ϕ and F, respectively. We note that ϕ^L or F^L is an infinitesimal gauge transformation of the bundle $SO(M)$. For a vector field Y on M, we define a vector field Y^H on $SO(M)$ by $\omega(Y^H) = 0$, $p(Y^H) = Y$. Y^H is called the horizontal lift of Y. Let Y be a Killing vector field on M and DY the covariant differential of Y. We denote the vector field $Y^H + (DY)^L$ by Y^L. Y^L is called the natural lift of Y. It is easy to see that the set of all parallel 2-forms on M is a Lie algebra, which is denoted by $(\bigwedge^2 M)_0$. It is a subalgebra of the algebra $\bigwedge^2 M$ of all 2-forms on M. We denote by $\mathfrak{i}(M)$ and $\mathfrak{i}(SO(M))$ the Lie algebras of all Killing vector fields on M and $SO(M)$, respectively.

THEOREM. (i) *For every* $Y \in \mathfrak{i}(M)$, $\phi \in (\bigwedge^2 M)_0$ *and* $A \in \mathfrak{o}(n)$, Y^L, ϕ^L *and* A^* *are all Killing vector fields.*

(ii) *If* $B(\xi)$ *is a Killing vector field for some non-zero* $\xi \in \boldsymbol{R}^n$, *then* M *has constant curvature* $1/2$. *Conversely, if* M *has constant curvature* $1/2$, *then* $B(\xi)$ *is a Killing vector field for any* $\xi \in \boldsymbol{R}^n$.

(iii) *Let* X *be an arbitrary Killing vector field on* $SO(M)$. *If* M *is complete, then there exist unique* $Y \in \mathfrak{i}(M)$, $\phi \in (\bigwedge^2 M)_0$, $A \in \mathfrak{o}(n)$ *and* $\xi \in \boldsymbol{R}^n$ *such that* $X = Y^L + \phi^L + A^* + B(\xi)$, *except when* $\dim M = 2, 3, 4$ *or* 8.

(iv) *For all* $Y, Z \in \mathfrak{i}(M)$, ϕ, $\psi \in (\bigwedge^2 M)_0$ *and* $A, C \in \mathfrak{o}(n)$, *we have* $[DY, \phi] \in (\bigwedge^2 M)_0$ *and*

$$[A^*, C^*] = [A, C]^*, \qquad [\phi^L, \psi^L] = -[\phi, \psi]^L, \qquad [Y^L, Z^L] = [Y, Z]^L,$$
$$[Y^L, \phi^L] = -[DY, \phi]^L, \qquad [Y^L, A^*] = 0, \qquad [\phi^L, A^*] = 0.$$

Especially if M *has constant curvature* $1/2$, *then* $(\bigwedge^2 M)_0 = \{0\}$ *and*

$$[B(\xi), B(\eta)] = -(1/2)(\xi \wedge \eta)^*,$$
$$[A^*, B(\xi)] = B(A\xi), \quad [Y^L, B(\xi)] = 0$$

for all $\xi, \eta \in \boldsymbol{R}^n$, *when* $\dim M > 2$.

COROLLARY. *If* M *is compact and does not have constant curvature* $1/2$, *then the Lie algebra* $\mathfrak{i}(SO(M))$ *is isomorphic to the direct sum*

$$\mathfrak{i}(M) + \mathfrak{o}(n) + (\textstyle\bigwedge^2 M)_0,$$

except when $n = \dim M = 2, 3, 4$ *or* 8.

REMARK. For our proof of (iii) of the above theorem, we need the following assumptions $(*)$ and $(**)$:

$(*)$ M is complete.

$(**)$ $n = \dim M \neq 2, 3, 4$ or 8.

The condition $(*)$ is necessary to prove the following Lemmas 7 and 11, while the condition $(**)$ is necessary to prove the following Proposition B (see [5]). More precisely, to prove Proposition B, we need the following conditions (a) and (b) on the Lie algebra $\mathfrak{o}(n)$:

(a) $\mathfrak{o}(n)$ is simple,

(b) the quotient group $\mathrm{Aut}(\mathfrak{o}(n))/\mathrm{Int}(\mathfrak{o}(n))$ is isomorphic to a cyclic group of order 1 or 2,

where $\mathrm{Aut}(\mathfrak{o}(n))$ and $\mathrm{Int}(\mathfrak{o}(n))$ are the group of all automorphisms and the group of all inner automorphisms of the Lie algebra $\mathfrak{o}(n)$, respectively. However, if $n=2$ or 4, then $\mathfrak{o}(n)$ is not simple and, if $n=8$, then $\mathrm{Aut}(\mathfrak{o}(n))/\mathrm{Int}(\mathfrak{o}(n))$ does not satisfy the condition (b) (see [5, p. 110]). When $n=3$, we cannot prove Lemma 11 of [5] which is necessary to prove Proposition B. We do not know whether this theorem is true or not when one of the conditions ($*$) and ($**$) is not satisfied.

2. Preliminaries. Let X be a vector field on $SO(M)$. X is said to be vertical (resp. horizontal) if $\theta(X)=0$ (resp. $\omega(X)=0$). X is decomposed uniquely as $X=X^H+X^V$, with X^H horizontal and X^V vertical. X is said to be fiber preserving if $[X, X']$ is vertical for any vertical vector field X'. X is determined by the functions $x(\xi)$ and $x(A)$ on $SO(M)$ defined by

$$x(\xi)=\langle X, B(\xi)\rangle=\langle\theta(X), \xi\rangle\,, \qquad \xi\in \boldsymbol{R}^n\,,$$

$$x(A)=\langle X, A^*\rangle=\langle\omega(X), A\rangle\,, \qquad A\in\mathfrak{o}(n)\,.$$

X is horizontal if and only if $x(A)=0$ for any $A\in\mathfrak{o}(n)$, while X is vertical if and only if $x(\xi)=0$ for any $\xi\in\boldsymbol{R}^n$. We denote by D the covariant differentiation with respect to the Riemannian connection of $SO(M)$ as well as M. We note that the right action of $SO(n)$ on $SO(M)$ is isometric, which is easily seen by the definition of $\langle\ ,\ \rangle$ on $SO(M)$ and by the fact that $R_a\omega=\mathrm{ad}(a^{-1})\omega$ and $R_a\theta=a^{-1}\theta$ for any $a\in SO(n)$. Let Ω be the curvature form of the Riemannian connection of M. It is well-known that $R_a\Omega=\mathrm{ad}(a^{-1})\Omega$ for any $a\in SO(n)$.

The following Lemmas 1, 2 and 3 are proved in [5].

LEMMA 1. *Let A, $C\in\mathfrak{o}(n)$ and $\xi, \eta, \zeta\in\boldsymbol{R}^n$. Then we have*

$$2D_{A^*}C^*=[A^*, C^*]=[A, C]^*\,, \qquad 2D_{B(\xi)}B(\eta)=[B(\xi), B(\eta)]\,,$$

$$\theta([B(\xi), B(\eta)])=0\,, \quad \omega([B(\xi), B(\eta)])=-2\Omega(B(\xi), B(\eta))\,,$$

$$[A^*, B(\xi)]=B(A\xi)\,, \qquad \omega(D_{B(\xi)}A^*)=\omega(D_{A^*}B(\xi))=0\,,$$

$$\langle D_{B(\xi)}A^*, B(\eta)\rangle=\langle D_{A^*}B(\xi), B(\eta)\rangle-\langle B(A\xi), B(\eta)\rangle=\langle\Omega(B(\xi), B(\eta)), A\rangle\,.$$

LEMMA 2. *Let X be an arbitrary vector field on $SO(M)$. Then we have*

$$\langle[A^*, X], B(\xi)\rangle=A^*(x(\xi))-x(A\xi)\,,$$

$$\langle[B(\xi), X], A^*\rangle=B(\xi)(x(A))-2\langle\Omega(B(\xi), X), A\rangle\,,$$

$$\langle[A^*, X], C^*\rangle=A^*(x(C))-x([A, C])\,,$$

$$\langle [B(\xi), X], B(\eta)\rangle = B(\xi)(x(\eta)) - \langle \omega(X)\xi, \eta\rangle$$

for all $\xi, \eta \in \boldsymbol{R}^n$ *and* $A, C \in \mathfrak{o}(n)$.

LEMMA 3. *Let* X *be a vector field on* $SO(M)$. *Then* X *is a Killing vector field if and only if*

$$A^*(x(C)) + C^*(x(A)) = 0\,, \qquad B(\xi)(x(\eta)) + B(\eta)(x(\xi)) = 0\,,$$
$$A^*(x(\xi)) - x(A\xi) + B(\xi)(x(A)) - 2\langle \Omega(B(\xi), X), A\rangle = 0$$

for all $\xi, \eta \in \boldsymbol{R}^n$ *and* $A, C \in \mathfrak{o}(n)$.

LEMMA 4. *Let* X *be a Killing vector field satisfying* $C^*(x(A)) = 0$ *for all* $A, C \in \mathfrak{o}(n)$. *Then we have*

$$\langle [C^*, [A^*, X]], B(\xi)\rangle = 2\langle \Omega(B(\xi), X), [C, A]\rangle + 2\langle \Omega(B(\xi), [C^*, X]), A\rangle\,.$$

PROOF. First we note that A^* is a Killing vector field for any $A \in \mathfrak{o}(n)$, since R_a is an isometry for any $a \in SO(n)$. We have $(L_{A^*}\Omega)(B(\xi), X) = -[A, \Omega(B(\xi), X)]$ for any $A \in \mathfrak{o}(n)$, since $R_a\Omega = \mathrm{ad}(a^{-1})\Omega$ for any $a \in SO(n)$. Then, by Lemmas 1, 2 and 3, we have

$$\begin{aligned}
\langle [C^*, [A^*, X]], B(\xi)\rangle &= C^*\langle [A^*, X], B(\xi)\rangle - \langle [A^*, X], [C^*, B(\xi)]\rangle \\
&= C^*(A^*(x(\xi)) - x(A\xi)) - (A^*(x(C\xi)) - x(AC\xi)) \\
&= C^*(-B(\xi)(x(A)) + 2\langle \Omega(B(\xi), X), A\rangle) + B(C\xi)(x(A)) \\
&\quad - 2\langle \Omega(B(C\xi), X), A\rangle \\
&= -B(\xi)C^*(x(A)) + 2C^*\langle \Omega(B(\xi), X), A\rangle - 2\langle \Omega(B(C\xi), X), A\rangle \\
&= -2\langle [C, \Omega(B(\xi), X)], A\rangle + 2\langle \Omega([C^*, B(\xi)], X), A\rangle \\
&\quad + 2\langle \Omega(B(\xi), [C^*, X]), A\rangle - 2\langle \Omega(B(C\xi), X), A\rangle \\
&= 2\langle \Omega(B(\xi), X), [C, A]\rangle + 2\langle \Omega(B(\xi), [C^*, X]), A\rangle\,,
\end{aligned}$$

where we used the fact $-\langle [C, F], A\rangle = \langle F, [C, A]\rangle$ for any $F \in \mathfrak{o}(n)$. ■

LEMMA 5. *Let* X *be an arbitrary Killing vector field on* $SO(M)$. *Then we have*

$$\begin{aligned}
2\langle [A^*, [C^*, X]], B(\xi)\rangle = \langle [[A, C]^*, X], B(\xi)\rangle &+ 2\langle \Omega(B(\xi), [A^*, X]), C\rangle \\
&+ 2\langle \Omega(B(\xi), [C^*, X]), A\rangle
\end{aligned}$$

for all $A, C \in \mathfrak{o}(n)$ *and* $\xi \in \boldsymbol{R}^n$.

PROOF. By Lemmas 2 and 3, we have

$$\begin{aligned}
&2\langle \Omega(B(\xi), [A^*, X]), C\rangle + 2\langle \Omega(B(\xi), [C^*, X]), A\rangle \\
&\quad = \langle [C^*, [A^*, X]], B(\xi)\rangle + B(\xi)(A^*(x(C)) - x([A, C])) + \langle [A^*, [C^*, X]], B(\xi)\rangle
\end{aligned}$$

$$+B(\xi)(C^*(x(A))-x([C, A]))$$
$$=-\langle[[A, C]^*, X], B(\xi)\rangle+2\langle[A^*, [C^*, X]], B(\xi)\rangle .$$

■

LEMMA 6. *Let X be a vector field on $SO(M)$. Then, X is parallel if and only if X is the horizontal lift Y^H of a parallel vector field Y on M, when $n\geqq 3$.*

PROOF. Assume $DX=0$ and $X=X^H+X^V$. Then, $D_{A^*}X^H+D_{A^*}X^V=0$ for any $A\in\mathfrak{o}(n)$. By Lemma 1, $D_{A^*}X^H$ is horizontal, while $D_{A^*}X^V$ is vertical. Hence, $D_{A^*}X^H=D_{A^*}X^V=0$. It follows that $X^V=0$, since each fiber of $SO(M)$ is totally geodesic and is isometric to the symmetric Riemannian manifold $(SO(n), \langle\ ,\ \rangle)$ of semi-simple type, if $n\geqq 3$. Since X is horizontal Killing vector field, we have

$$0=2\langle D_{A^*}X, B(\xi)\rangle=-2\langle D_{B(\xi)}X, A^*\rangle=2\langle X, D_{B(\xi)}A^*\rangle=2\langle\Omega(B(\xi), X), A\rangle$$
$$=A^*(x(\xi))-x(A\xi)=\langle[A^*, X], B(\xi)\rangle$$

for any $\xi\in\boldsymbol{R}^n$, by Lemmas 1, 2 and 3. It follows that $[A^*, X]=0$ for any $A\in\mathfrak{o}(n)$ and hence $R_aX=X$ for any $a\in SO(n)$, which shows that X is the horizontal lift Y^H of a certain vector field Y on M. Hence we have

$$B(\xi)(x(\eta))=B(\xi)(x(\eta))-\langle X, D_{B(\xi)}B(\eta)\rangle=\langle D_{B(\xi)}X, B(\eta)\rangle=0 .$$

Here, we note that $\langle D_{u(\xi)}Y, u(\eta)\rangle=B_u(\xi)(x(\eta))$ for all $\xi, \eta\in\boldsymbol{R}^n$ and $u\in SO(M)$ (see [5, p. 108]). Consequently, $DY=0$.

Assume $X=Y^H$ and $DY=0$. Then we have

$$x(A)=0 , \qquad [A^*, X]=0 , \qquad B(\xi)(x(\eta))=0 , \qquad \Omega(B(\xi), X)=0$$

for all $\xi, \eta\in\boldsymbol{R}^n$ and $A\in\mathfrak{o}(n)$. The equality $\Omega(B(\xi), X)=0$ follows from the fact that, for all $\xi\in\boldsymbol{R}^n$ and $u\in SO(M)$, $2\Omega_u(B(\xi), X)=u^{-1}\circ R(u(\xi), Y)\circ u$ and $R(Y, u(\xi))=0$, where R is the curvature transformation of M. Hence we have $[B(\xi), X]=0$ for any $\xi\in\boldsymbol{R}^n$, by Lemma 2. Therefore, by Lemmas 1 and 3,

$$\langle D_{B(\xi)}X, B(\eta)\rangle=\langle D_XB(\xi), B(\eta)\rangle=0 ,$$
$$\langle D_{B(\xi)}X, A^*\rangle=\langle D_XB(\xi), A^*\rangle=-\langle\Omega(X, B(\xi)), A\rangle=0 ,$$
$$\langle D_{A^*}X, B(\eta)\rangle=\langle D_XA^*, B(\eta)\rangle=\langle\Omega(X, B(\eta)), A\rangle=0 ,$$
$$\langle D_{A^*}X, C^*\rangle=\langle D_XA^*, C^*\rangle=0$$

for all $\xi, \eta\in\boldsymbol{R}^n$ and $A, C\in\mathfrak{o}(n)$, which means $DX=0$. ■

LEMMA 7. *Let Y be a Killing vector field on M satisfying $D^2Y=0$. Then $Y=0$ identically when M is complete and has the irreducible restricted homogeneous holonomy group.*

PROOF. Assume $Y\neq 0$. Then $DY\neq 0$, since M is irreducible. First, we show that there exists non-zero $a \in \boldsymbol{R}$ such that $F^2 = -a^2 I$ (I is the identity) at each point $m \in M$, where we put $F = DY$. Let $\mathfrak{h}(u)$ be the Lie algebra of the holonomy group at $u \in SO(M)$ with $p(u) = m$. Since $DF = 0$, it follows that $[h, F^{\#}(u)] = 0$ for every $h \in \mathfrak{h}(u)$. Making use of this fact, it is easy to see that $F^{\#}(u)$ has maximal rank n. Otherwise, for a suitable u, $F^{\#}(u)$ is written as

$$F^{\#}(u) = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix},$$

where $\det J \neq 0$. If $[h, F^{\#}(u)] = 0$ for such $F^{\#}(u)$ and

$$h = \begin{pmatrix} A & B \\ -{}^t B & C \end{pmatrix} \in \mathfrak{h}(u),$$

then $B = 0$, which shows that $\mathfrak{h}(u)$ is reducible. Hence, for a suitable u, $F^{\#}(u)$ is written as

$$F^{\#}(u) = \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_n \\ & & & -a_n & 0 \end{pmatrix}, \qquad a_1 a_2 \cdots a_n \neq 0,$$

where we note $a_1, a_2, \ldots, a_n$ are constant on M, since $DF = 0$. Then we have

$$(F^{\#}(u))^2 = -\begin{pmatrix} a_1^2 & & & & \\ & a_1^2 & & & \\ & & \ddots & & \\ & & & a_n^2 & \\ & & & & a_n^2 \end{pmatrix}.$$

Unless $a_1^2 = a_2^2 = \cdots = a_n^2 = a^2$, the distributions defined by the eigenvectors for eigenvalues $a_1^2, a_2^2, \ldots, a_n^2$ of F^2 are invariant by any parallel displacement, since $DF^2 = 0$, which contradicts the irreducibility of $\mathfrak{h}(u)$.

So, we put $f = (1/2a^2)\langle Y, Y\rangle$ and $Z = \operatorname{grad} f$. Since $(1/a^2)F^2 = -I$ and $D^2 Y = 0$, we get the equality

$$(D^2 f)(V, W) = \langle V, W\rangle \qquad \text{or} \qquad D_V Z = V,$$

for all vector fields V and W on M. In particular, we have $D_Z Z = Z$, which shows that each trajectory of Z is geodesic. Let $c: \boldsymbol{R} \to M$ be a geodesic paramerized by an arc length such that Z is tangent to c and its tangent vector $c'(s)$ satisfies the condition $\langle Z(c(s)), c'(s)\rangle > 0$ for a sufficiently large s. Then there exists $k \in \boldsymbol{R}$ such that $Z(c(s)) = (s-k)c'(s)$ which follows from the equality $Z\langle Z, Z\rangle = 2\langle Z, Z\rangle$. On the other hand, the distribution defined by the vectors orthogonal to Z is involutive. For, if V

and W are vector fields satisfying $\langle V, Z\rangle=\langle W, Z\rangle=0$, then

$$\begin{aligned}\langle Z, [V, W]\rangle &= \langle Z, D_V W - D_W V\rangle = -\langle D_V Z, W\rangle + \langle D_W Z, V\rangle \\ &= -\langle V, W\rangle + \langle W, V\rangle = 0 .\end{aligned}$$

Let $\{s, x^1, x^2, \dots, x^{n-1}\}$ be a local coordinate system in a neighborhood of $c(k)$ satisfying the following conditions:

(1) $c(k)$ has the coordinates $(k, 0, 0, \dots, 0)$,

(2) S is tangent to some trajectory of Z at each point,

(3) $\langle S, S\rangle=1$ and $\langle S, X_i\rangle=0$ $(1\leqq i\leqq n-1)$,

where we put $S=\partial/\partial s$ and $X_i=\partial/\partial x^i$ $(1\leqq i\leqq n-1)$. Here, we may assume $Z=(s-k)S$ by virtue of the argument as above and the fact $X_i\langle Z, Z\rangle=2\langle X_i, Z\rangle=0$. Consequently, if $s\neq k$, then we have

$$S\langle X_i, X_i\rangle=2\langle D_S X_i, X_i\rangle=2\langle D_{X_i} S, X_i\rangle=\{2/(s-k)\}\langle D_{X_i} Z, X_i\rangle=\{2/(s-k)\}\langle X_i, X_i\rangle ,$$

which shows that the function $S\langle X_i, X_i\rangle$ is not continuous at the points where $s=k$. This is a contradiction. ■

The following propositions are also proved in [5].

PROPOSITION A. *Let X be a fiber preserving Killing vector field on $SO(M)$. Then there exist unique $Y\in\mathfrak{i}(M)$, $\phi\in(\bigwedge^2 M)_0$ and $A\in\mathfrak{o}(n)$ such that $X=Y^L+\phi^L+A^*$, if $n\geqq 3$.*

REMARK. In Proposition A, the uniqueness is obvious by the following Lemma 8, though we did not state this fact in [5].

PROPOSITION B. *If $SO(M)$ has a horizontal Killing vector field which is not fiber preserving, then M has constant curvature $1/2$, except when $n=2, 3, 4$ or 8.*

3. Proof of the theorem.

LEMMA 8. *Let $X=X^H+X^V$ be a Killing vector field on $SO(M)$. If $n\geqq 3$, then X^V is decomposed uniquely as $X^V=X_1+X_2$, where $[A^*, X_1]=0$ and $A^*\langle X_2, C^*\rangle=0$ for all $A, C\in\mathfrak{o}(n)$.*

PROOF. This lemma follows from the fact that each fiber of $SO(M)$ is totally geodesic and is isometric to the symmetric Riemannian manifold $(SO(n), \langle\ ,\ \rangle)$, on which any Killing vector field is decomposed uniquely as the sum of a right invariant vector field and a left invariant vector field (see [5, p. 106]). ■

LEMMA 9. *Let X be a Killing vector field which is not fiber preserving. Then, there exists $A\in\mathfrak{o}(n)$ such that $[A^*, X]$ is not fiber preserving and $F^*\langle[A^*, X], C^*\rangle=0$ for all $C, F\in\mathfrak{o}(n)$, when $n\geqq 3$.*

PROOF. Assume that $[A^*, X]$ is fiber preserving for any $A\in\mathfrak{o}(n)$. By Lemma 8, X is decomposed as $X=X^H+X_1+X_2$ with $[A^*, X_1]=0$ and $A^*\langle X_2, C^*\rangle=0$ for all $A, C \in \mathfrak{o}(n)$. Then, by virtue of Proposition A, for each $A \in \mathfrak{o}(n)$, there exist $\phi_A \in (\bigwedge^2 M)_0$, $Y_A \in \mathfrak{i}(M)$ and $F_A \in \mathfrak{o}(n)$ such that

$$[A^*, X] = [A^*, X^H] + [A^*, X_2] = (Y_A)^H + (DY_A)^L + (\phi_A)^L + (F_A)^* ,$$

or equivalently,

$$[A^*, X^H] = (Y_A)^H , \qquad [A^*, X_2] = (F_A)^* + (DY_A + \phi_A)^L .$$

If we put $J_A = DY_A + \phi_A$, then we have

$$\begin{aligned}[C, F_A]^* &= [C^*, (F_A)^*] = [C^*, (F_A)^* + (J_A)^L] = [C^*, [A^*, X_2]] \\ &= [[C, A]^*, X_2] + [A^*, [C^*, X_2]] = (F_{[C,A]})^* + (J_{[C,A]})^L + [A, F_C]^* ,\end{aligned}$$

or equivalently,

$$(J_{[C,A]})^L = [C, F_A]^* - [A, F_C]^* - (F_{[C,A]})^* .$$

This equality is possible only when both sides are equal to zero, by Lemma 8. Since $\mathfrak{o}(n)$ is semi-simple, $J_A = 0$ for any $A \in \mathfrak{o}(n)$. It follows that $[A^*, X_2] = (F_A)^*$ is fundamental and hence $[A^*, X^H] = (Y_A)^H$ is a Killing vector field satisfying $[C^*, [A^*, X^H]] = 0$ for any $C \in \mathfrak{o}(n)$. This implies that $\langle \Omega(B(\xi), [A^*, X^H]), C \rangle = 0$ for any $\xi \in \boldsymbol{R}^n$, by Lemma 3. Then, by Lemma 5, $\langle [[A, C]^*, X^H], B(\xi) \rangle = 0$ for all $A, C \in \mathfrak{o}(n)$ and $\xi \in \boldsymbol{R}^n$. Since $\mathfrak{o}(n)$ is semi-simple, we have $[A^*, X^H] = 0$ for any $A \in \mathfrak{o}(n)$, which shows that X is fiber preserving. On the other hand, the equality $F^* \langle [A^*, X], C^* \rangle = 0$ holds good, since

$$\langle [A^*, X], C^* \rangle = \langle [A^*, X_2], C^* \rangle = A^* \langle X_2, C^* \rangle - \langle X_2, [A, C]^* \rangle = -\langle X_2, [A, C]^* \rangle$$

■

For $\xi, \eta \in \boldsymbol{R}^n$, we define $\xi \wedge \eta \in \mathfrak{o}(n)$ by

$$(\xi \wedge \eta)(\zeta) = \langle \eta, \zeta \rangle \xi - \langle \xi, \zeta \rangle \eta \quad (\zeta \in \boldsymbol{R}^n) .$$

It is easy to prove the following equalities:

$$\langle \xi \wedge \eta, A \rangle = -2 \langle A\xi, \eta \rangle ,$$

$$\mathrm{ad}(a)(\xi \wedge \eta) = a\xi \wedge a\eta ,$$

$$[A, \xi \wedge \eta] = A\xi \wedge \eta + \xi \wedge A\eta , \tag{☆}$$

where $A \in \mathfrak{o}(n)$ and $a \in SO(n)$.

LEMMA 10. *Suppose that there exists a Killing vector field X on $SO(M)$ satisfying the following conditions* (a) *and* (b):

(a) $A^*(x(C)) = 0$ *for all* $A, C \in \mathfrak{o}(n)$

(b) X^H *is not a Killing vector field.*

If $n \neq 4$, *then there exists a Killing vector field W on $SO(M)$ written as* $W = W^H + f(e_1 \wedge e_2)^*$ *in terms of a function f satisfying the following conditions* (a′) *and* (b′):

(a′) $A^*f=0$ *for any* $A\in\mathfrak{o}(n)$

(b′) $B(\xi)f\neq 0$ *for some* $\xi\in\boldsymbol{R}^n$,

where $\{e_1, e_2, \dots, e_n\}$ *is an orthonormal basis of* $\boldsymbol{R}^n$.

Proof. We shall construct W from X as follows. First, we give the proof under the assumption that $n\geqq 5$. X is written as

$$X=X^H+\sum_{r<s} x_{rs}(e_r\wedge e_s)^* ,$$

where $2x_{rs}=x(e_r\wedge e_s)=\langle X, (e_r\wedge e_s)^*\rangle$, since $\{e_r\wedge e_s;\ 1\leqq r<s\leqq n\}$ is an orthogonal basis of $\mathfrak{o}(n)$. By the condition (a), we have $A^*x_{rs}=0$ for all r, s and $A\in\mathfrak{o}(n)$, which is equivalent to the condition $R_a^*x_{rs}=x_{rs}$ for all r, s and $a\in SO(n)$. However, by (b), we have $B(\xi)x_{rs}\neq 0$ for some r, s and $\xi\in\boldsymbol{R}^n$, for otherwise X^V is a fundamental vector field and hence X^H is a Killing vector field. Here, we may assume $B(\xi)x_{13}\neq 0$ for some $\xi\in\boldsymbol{R}^n$. This follows from the following facts:

$$R_aX=R_aX^H+\sum_{r<s} x_{rs}(\mathrm{ad}(a^{-1})(e_r\wedge e_s))^*$$

is a Killing vector field for any $a\in SO(n)$, since R_a is isometry, $R_a^*x_{rs}=x_{rs}$ and

$$R_a(e_r\wedge e_s)^*=(\mathrm{ad}(a^{-1})(e_r\wedge e_s))^*=(a^{-1}e_r\wedge a^{-1}e_s)^* .$$

Moreover, for each pair (r, s) satisfying $1\leqq r<s\leqq n$, we can choose $a=a(r, s)\in SO(n)$ in such a way that $a^{-1}e_r\wedge a^{-1}e_s=e_1\wedge e_3$.

Now, let

$$W=[(e_2\wedge e_4)^*, [(e_1\wedge e_5)^*, [(e_1\wedge e_4)^*, [(e_3\wedge e_5)^*, X]]]] .$$

Then, we get $W=W^H-x_{13}(e_1\wedge e_2)^*$, which is the desired vector field. The last equality is given by the following facts (i) and (ii):

(i) $[e_2\wedge e_4, [e_1\wedge e_5, [e_1\wedge e_4, [e_3\wedge e_5, e_1\wedge e_3]]]]=-e_1\wedge e_2$,

(ii) $[e_2\wedge e_4, [e_1\wedge e_5, [e_1\wedge e_4, [e_3\wedge e_5, A]]]]=0$,

if $\langle e_1\wedge e_3, A\rangle=0$, which are checked easily by (☆). This proves the lemma for $n\geqq 5$. When $n=2$, the assertion is trivial. When $n=3$, the assertion follows from

$$[e_2\wedge e_3, [e_1\wedge e_2, [e_1\wedge e_2]]]=0 ,$$

$$[e_2\wedge e_3, [e_1\wedge e_2, [e_2\wedge e_3]]]=e_1\wedge e_2 ,$$

$$[e_2\wedge e_3, [e_1\wedge e_2, [e_1\wedge e_3]]]=0 .$$

∎

Lemma 11. *Let W be a Killing vector field on $SO(M)$ written as $W=W^H+f(e_1\wedge e_2)^*$ in terms of a function f satisfying $A^*f=0$ for any $A\in\mathfrak{o}(n)$. If M is complete and does not have constant curvature* $1/2$, *then f is constant on $SO(M)$, except when $n=2, 3, 4$ or* 8.

PROOF. The tangent bundle $T(SO(M))$ is decomposed as

$$T(SO(M))=T_0(SO(M))+T_1(SO(M))+\cdots+T_d(SO(M)),$$

which satisfies the following conditions:

(a) For each r $(0\leqq r\leqq d)$, $T_r(SO(M))$ is invariant under any parallel displacement.

(b) Any maximal integral manifold of $T_0(SO(M))$ is locally flat.

(c) For each r $(1\leqq r\leqq d)$, any maximal integral manifold of $T_r(SO(M))$ is irreducible.

Consequently, W is decomposed as

$$W=W_0+W_1+\cdots+W_d, \qquad W_r\in T_r(SO(M)),$$

where W_r is a Killing vector field for each r $(0\leqq r\leqq d)$. By Lemma 6, $T_0(SO(M))$ is contained in the horizontal subspace at each point of $SO(M)$. Hence W_0 is horizontal. Hence, $V=W-W_0=W_1+\cdots+W_d$ is a Killing vector field written also as $V=V^H+f(e_1\wedge e_2)^*$.

Now, for each $A\in\mathfrak{o}(n)$, we define a Killing vector field $X_A=(X_A)^H+fA^*$ as follows:

(i) For each $1\leqq r<s\leqq n$, choose an $a(r,s)\in SO(n)$ so that $\mathrm{ad}((a(r,s))^{-1})(e_1\wedge e_2)=e_r\wedge e_s$. Define $X_{e_r\wedge e_s}$ by

$$X_{e_r\wedge e_s}=R_{a(r,s)}V=R_{a(r,s)}V^H+f(e_r\wedge e_s)^*.$$

(ii) For each $A=\sum_{r<s}a_{rs}e_r\wedge e_s\in\mathfrak{o}(n)$, define X_A by

$$X_A=\sum_{r<s}a_{rs}X_{e_r\wedge e_s}=(X_A)^H+fA^*.$$

Then we have

(1) $X_{A+C}=X_A+X_C$, $X_{kA}=kX_A$,

(2) $[C^*,X_A]=X_{[C,A]}$,

for all $A, C\in\mathfrak{o}(n)$ and $k\in\boldsymbol{R}$. (1) is trivial by definition. The proof of (2) is as follows. $[C^*,X_A]-X_{[C,A]}$ is a horizontal Killing vector field, which is fiber preserving, by virtue of Proposition B, since we assumed that M does not have constant curvature $1/2$. It follows that $[C^*,X_A]-X_{[C,A]}$ is equal to the horizontal lift Y^H of a certain Killing vector field Y on M, which satisfies the condition $D^2Y=0$, by Proposition A. Here, the tangent bundle $T(M)$ is decomposed as

$$T(M)=T_0(M)+T_1(M)+\cdots+T_e(M),$$

which satisfies the conditions similar to the above (a), (b) and (c). Consequently, Y is decomposed as

$$Y=Y_0+Y_1+\cdots+Y_e, \qquad Y_r\in T_r(M),$$

and hence

$$0=D^2Y=D^2Y_0+D^2Y_1+\cdots+D^2Y_e,$$

which implies $D^2Y_r=0$ for $0\leqq r\leqq e$. Then we have $Y_r=0$ for $1\leqq r\leqq e$, by Lemma 7. By Lemma 6, the horizontal lift of $T_0(M)$ is contained in $T_0(SO(M))$ and hence $Y^H=(Y_0)^H\in T_0(SO(M))$. On the other hand, we have

$$[C^*, X_A]-X_{[C,A]}=Y^H\in T_1(SO(M))+\cdots+T_d(SO(M)),$$

since $V\in T_1(SO(M))+\cdots+T_d(SO(M))$ and this subbundle is invariant under R_a for every $a\in SO(n)$. Consequently, $Y^H=0$.

Therefore, by Lemma 3, we have

$$(*)\qquad \langle X_{[C,A]}, B(\xi)\rangle+\langle C, A\rangle B(\xi)f-2\langle\Omega(B(\xi), X_A), C\rangle=0,$$

for all $\xi\in\boldsymbol{R}^n$ and $A, C\in\mathfrak{o}(n)$. In particular,

$$(**)\qquad \langle A, A\rangle B(\xi)f-2\langle\Omega(B(\xi), X_A), A\rangle=0,$$

which implies that $\langle A, A\rangle(X_A)^Hf=2\langle\Omega(X_A, X_A), A\rangle=0$ or $(X_A)^Hf=0$ for any $A\in\mathfrak{o}(n)$.

Finally, let Z be a vector field orthogonal to $(X_A)^H$ for any $A\in\mathfrak{o}(n)$. We shall show $Zf=0$, which completes the proof of the lemma. If $\langle C, A\rangle=0$, then $\langle\Omega(Z, X_A), C\rangle=0$, by $(*)$. Hence, for each $A\in\mathfrak{o}(n)$, there exists a 1-form ψ_A satisfying $\Omega(Z, X_A)=\psi_A(Z)A$. However, by $(**)$, we have

$$\langle A, A\rangle Zf=2\langle\Omega(Z, X_A), A\rangle=2\psi_A(Z)\langle A, A\rangle,$$

which shows $\Omega(Z, X_A)=(1/2)(Zf)A$. Here, if there exists Z such that $Zf\neq 0$, then

$$\dim\{\Omega_u(Z, X_A)\in\mathfrak{o}(n)\,;\ A\in\mathfrak{o}(n)\}=n(n-1)/2,$$

$$\dim\{((X_A)^H)_u\in Q_u\,;\ A\in\mathfrak{o}(n)\}\leqq n,$$

for some $u\in SO(M)$, which is a contradiction. Consequently, $Zf=0$, if $\langle Z, (X_A)^H\rangle=0$ for any $A\in\mathfrak{o}(n)$. ■

It is easy to prove the following lemma.

LEMMA 12. *Let $q\colon(N, \langle\ ,\ \rangle)\to(M, \langle\ ,\ \rangle)$ be a Riemannian covering. Then, the induced bundle homomorphism $q_*\colon SO(N)\to SO(M)$ has the following properties* (a)～(e):

(a) *$q_*\colon(SO(N), \langle\ ,\ \rangle)\to(SO(M), \langle\ ,\ \rangle)$ is a Riemannian covering.*

(b) *For any Killing vector field X on $SO(M)$, there exists a unique Killing vector field on $SO(N)$ which is q_*-related to X.*

(c) *For any Killing vector field Y on M, there exists a unique Killing vector field Z on N such that Z is q-related Y and that Z^L is q_*-related to Y^L.*

(d) *Let Z be a Killing vector field on N. If Z^L is q_*-related to some Killing vector field on $SO(M)$, then Z is q-related to some Killing vector field on M.*

(e) *For any $\xi\in\boldsymbol{R}^n$ and $A\in\mathfrak{o}(n)$, the vector fields A^* and $B(\xi)$ on $SO(N)$ are q_*-related* *to A^* and $B(\xi)$ on $SO(M)$, respectively.*

LEMMA 13. *If $B(\xi)$ is a Killing vector field for some non-zero $\xi \in \boldsymbol{R}^n$, then M has constant curvature $1/2$. Conversely, if M has constant curvature $1/2$, then $B(\xi)$ is a Killing vector field for any $\xi \in \boldsymbol{R}^n$.*

PROOF. If $B(\xi)$ is a Killing vector field for $\xi \neq 0$, then $R_a B(\xi) = B(a^{-1}\xi)$ is a Killing vector field for every $a \in SO(n)$, since R_a is an isometry. This shows that $B(\xi)$ is a Killing vector field for every $\xi \in \boldsymbol{R}^n$. Hence, by Lemma 3, we have

$$-\langle A\xi, \eta\rangle - 2\langle \Omega(B(\eta), B(\xi)), A\rangle = 0\,,$$

or

$$\langle 2\Omega(B(\eta), B(\xi)) - (1/2)\eta \wedge \xi, A\rangle = 0\,,$$

for all $\xi, \eta \in \boldsymbol{R}^n$ and $A \in \mathfrak{o}(n)$. It follows that

$$2\Omega(B(\eta), B(\xi)) = (1/2)\eta \wedge \xi\,,$$

which means that M has constant curvature $1/2$.

Conversely, suppose that M has constant curvature $1/2$, that is, $2\Omega(B(\eta), X) = (1/2)\eta \wedge \theta(X)$ for all vector field X and $\eta \in \boldsymbol{R}^n$. Then, by Lemma 3, $B(\xi)$ is a Killing vector field for any $\xi \in \boldsymbol{R}^n$. ■

LEMMA 14. *Let M be a sphere of curvature $1/2$. Then, for any Killing vector field X on $SO(M)$, there exist unique $Y \in \mathfrak{i}(M)$, $A \in \mathfrak{o}(n)$ and $\xi \in \boldsymbol{R}^n$ such that $X = Y^L + A^* + B(\xi)$ when $n \geqq 3$.*

PROOF. In this case, $SO(M)$ is the Lie group $SO(n+1)$ and the metric $\langle\ ,\ \rangle$ of $SO(M)$ is bi-invariant, which follows from

$$[A^*, B(\xi)] = B(A\xi)\,, \qquad [A^*, C^*] = [A, C]^*\,,$$
$$[B(\xi), B(\eta)] = -(1/2)(\xi \wedge \eta)^*\,.$$

Thus, we get the assertion by the theory of symmetric Riemannian manifolds (see the proof of Lemma 8). ■

LEMMA 15. (i) *If $Y \in \mathfrak{i}(M)$ and $\phi \in (\bigwedge^2 M)_0$, then $[DY, \phi] \in (\bigwedge^2 M)_0$.* (ii) *If M has constant curvature $c \neq 0$, then $(\bigwedge^2 M)_0 = \{0\}$, when $= \dim M > 2$.*

PROOF. (i) Since $Y \in \mathfrak{i}(M)$, for any vector field W on M,

$$D_W([DY, \phi]) = [D_W DY, \phi] = -[R(Y, W), \phi] = -D_Y D_W \phi + D_W D_Y \phi + D_{[Y,W]}\phi = 0\,,$$

where $R(Y, W)$ is the curvature transformation.

(ii) If $D\phi = 0$, then $[R(W, Z), \phi] = 0$, where $R(W, Z) = cW \wedge Z$ for all vector fields W and Z on M. Hence we have $[W \wedge Z, \phi] = 0$, which implies that $\phi = 0$, since $\mathfrak{o}(n)$ is semi-simple for $n > 2$. ■

LEMMA 16. *Let $Y, Z \in \mathfrak{i}(M)$, $\phi, \psi \in (\bigwedge^2 M)_0$, $A \in \mathfrak{o}(M)$ and $\xi \in \boldsymbol{R}^n$. Then we have*
(i) $[Y^L, A^*]=0$, (ii) $[Y^L, B(\xi)]=0$, (iii) $[\phi^L, A^*]=0$, (iv) $[\phi^L, \psi^L]=-[\phi, \psi]^L$, (v) $[Y^L, Z^L]=[Y, Z]^L$, (vi) $[Y^L, \phi^L]=-[DY, \phi]^L$.

PROOF. For the proof of this lemma, we use the following equalities (a)~(i), whose proofs can be found in [2] and [5].

Let Y and Z be arbitrary vector fields on M. Then, for all $A \in \mathfrak{o}(M)$, $\xi \in \boldsymbol{R}^n$, $\phi, \psi \in \bigwedge^2 M$ and $u \in SO(M)$, we have

(a) $A^*(\omega(\phi^L))=-[A, \omega(\phi^L)]$,
(b) $B_u(\xi)(\omega(\phi^L))=u^{-1}\circ(D_{u(\xi)}\phi)\circ u$,
(c) $A^*(\theta(Y^H))=-A(\theta(Y^H))$,
(d) $B_u(\xi)(\theta(Y^H))=u^{-1}(D_{u(\xi)}Y)$,
(e) $Y^H_u(\theta(Z^H))=u^{-1}(D_Y Z)_{p(u)}$,
(f) $Y^H_u(\omega(\phi^L))=u^{-1}\circ(D_Y\phi)_{p(u)}\circ u$,
(g) $[\omega(\phi^L), \omega(\psi^L)]=\omega([\phi, \psi]^L)$.

Especially if $Y \in \mathfrak{i}(M)$, then we have

(h) $(DY)^L_u(\theta(Z^H))=-u^{-1}(D_Z Y)_{p(u)}$,
(i) $(DY)^L_u(\omega(\phi^L))=-u^{-1}\circ[DY, \phi]_{p(u)}\circ u$.

Now, we give a proof of (v). It is similar to our proofs of the other five, so we omit them. We assume that (i) and (ii) are true. Since $Y^L, Z^L \in \mathfrak{i}(SO(M))$, for each $A \in \mathfrak{o}(M)$, $\xi \in \boldsymbol{R}^n$ and $u \in SO(M)$, we have

$$\begin{aligned}
\langle [Y^L, Z^L], B(\xi)\rangle(u) &= Y^L_u\langle Z^L, B(\xi)\rangle - \langle Z^L, [Y^L, B(\xi)]\rangle(u) = Y^L_u\langle \theta(Z^L), \xi\rangle \\
&= \langle Y^H_u(\theta(Z^H)), \xi\rangle + \langle (DY)^L_u(\theta(Z^H)), \xi\rangle \\
&= \langle u^{-1}(D_Y Z)_{p(u)}, \xi\rangle - \langle u^{-1}(D_Z Y)_{p(u)}, \xi\rangle \\
&= \langle u^{-1}(D_Y Z - D_Z Y)_{p(u)}, \xi\rangle = \langle [Y, Z]_{p(u)}, u(\xi)\rangle \\
&= \langle [Y, Z]^H, B(\xi)\rangle(u)\,,
\end{aligned}$$

$$\begin{aligned}
\langle [Y^L, Z^L], A^*\rangle(u) &= Y^L_u\langle Z^L, A^*\rangle - \langle Z^L, [Y^L, A^*]\rangle(u) = Y^L_u\langle \omega(Z^L), A\rangle \\
&= \langle Y^H_u(\omega((DZ)^L)), A\rangle + \langle (DY)^L_u(\omega((DZ)^L)), A\rangle \\
&= \langle u^{-1}\circ(D_Y DZ)_{p(u)}\circ u, A\rangle - \langle u^{-1}\circ[DY, DZ]_{p(u)}\circ u, A\rangle \\
&= \langle u^{-1}\circ(D_Y DZ - [DY, DZ])_{p(u)}\circ u, A\rangle \\
&= \langle (D_Y DZ - [DY, DZ])^{\#}(u), A\rangle \\
&= \langle \omega_u((D_Y DZ - [DY, DZ])^L), \omega_u(A^*)\rangle \\
&= \langle (-R(Z, Y) - [DY, DZ])^L, A^*\rangle(u)\,.
\end{aligned}$$

These equalities show

$$(*) \qquad [Y^L, Z^L]=[Y, Z]^H-(R(Z, Y)+[DY, DZ])^L\,.$$

On the other hand, for any vector field W on M, we have

$$
\begin{aligned}
D_W(D_Y Z - D_Z Y) &= D_W((DZ)(Y) - (DY)(Z)) \\
&= (D_W DZ)(Y) + (DZ)(D_W Y) - (D_W DY)(Z) - (DY)(D_W Z) \\
&= -R(Z, W)Y + R(Y, W)Z + (DZ)((DY)(W)) - (DY)((DZ)(W)) \\
&= R(Y, Z)W - [DY, DZ]W,
\end{aligned}
$$

by the first Bianchi identity. This shows

$$(\star) \qquad D([Y, Z]) = R(Y, Z) - [DY, DZ].$$

From $(*)$ and $(\star)$, we get

$$[Y^L, Z^L] = [Y, Z]^H + (D([Y, Z]))^L = [Y, Z]^L.$$

■

PROOF OF THEOREM. For the proof of (i), see [5]. We proved (ii) in Lemma 13. (iii) Lemmas 12 and 14 prove the theorem, under the assumption that M has constant curvature $1/2$. So we assume that M does not have constant curvature $1/2$. It is sufficient to prove that any Killing vector field X is fiber preserving. Suppose X is not fiber preserving. By Lemma 9, we may assume $A^*(x(C)) = 0$ for all $A, C \in \mathfrak{o}(n)$. If X^H is not a Killing vector field, then there exists a Killing vector field W as in Lemma 10, which contradicts Lemma 11. If X^H is a Killing vector field, then it is not fiber preserving, since X is not, which contradicts Proposition B. The uniqueness is obvious by what we have seen so far. (iv) follows from Lemmas 1, 14, 15 and 16. ■

PROOF OF COROLLARY. If M is compact, then, by the theorem of B. Kostant, DY is contained in the holonomy algebra at each point of M for every $Y \in \mathfrak{i}(M)$ (see [2, vol. 1, p. 247, Theorem 4.5]). Then, by Lemma 16, for any $Y \in \mathfrak{i}(M)$ and $\phi \in (\bigwedge^2 M)_0$,

$$[Y^L, \phi^L] = -[DY, \phi]^L = 0,$$

which proves the corollary by virtue of the theorem. ■

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