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FLAT TOTALLY REAL SUBMANIFOLDS OF CP^n AND THE SYMMETRIC GENERALIZED WAVE EQUATION

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Abstract. In this paper we show that the projection of the Hopf fibration establishes a one-to-one correspondence between the set of symmetric flat submanifolds in Euclidean sphere and the set of totally real flat submanifolds in complex projective space with the same codimension. We also show that any complete totally real submanifold in complex projective space with mean curvature of constant length and equal dimension and codimension is a flat torus.

The *Generalized Wave equation* (GWE) has been extensively studied by Bianchi ([Bi_1] to [Bi_4], see also [BT]) as the system of partial differential equations associated to an *orthogonal system of Guichard-Darboux*. This is a coordinate system $(u_1, \dots, u_{n+1})$ with flat orthogonal metric $ds^2 = \sum_{j=1}^{n+1} v_j^2 du_j^2$ satisfying the quadratic condition $\sum v_j^2 = 1$. A solution of the GWE (see system (I)) is a pair (v, h), where $v = (v_1, \dots, v_{n+1})$ and $h = (h_{ij})$, $1 \le i \ne j \le n+1$, such that $\partial v_i / \partial u_j = h_{ji} v_j$. Bianchi showed that the GWE, which reduces to the standard wave equation for $n=1$, is a completely integrable system of Bourlet's type, so any real analytic solution is completely determined by $n(n+1)$ arbitrary real analytic functions in one variable.

Extrinsically, any orthogonal system of Guichard-Darboux on a simply connected flat manifold M_0^{n+1} can be made into a principal coordinate system of an isometric immersion $F: M_0^{n+1} \to S^{2n+1}(1)$ into the round Euclidean unit sphere, hence setting up a one-to-one correspondence between the set of solutions of the GWE and the set of isometric immersions of M_0^{n+1} into $S^{2n+1}(1)$. See [BT] and [DT_2] for details.

Bianchi also considered (see §25 of [Bi_4]) the *Symmetric Generalized Wave equation* (SGWE) where it is assumed further that $h_{ij} = h_{ji}$ for all $1 \le i \ne j \le n+1$. In this case, real analytic solutions are determined by $(1/2)n(n+1)$ functions. We will say that an isometric immersion $F: M_0^{n+1} \to S^{2n+1}(1)$ is *symmetric* when the corresponding principal coordinates generate a solution of the SGWE.

Given an isometric immersion $f: M^n \to \boldsymbol{CP}^n$, the fibre product

$$M^{n+1} = \{\{x\} \times \pi^{-1}(f(x)) \in M^n \times S^{2n+1}(1)\},$$

where $\pi: S^{2n+1} \to \boldsymbol{CP}^n$ is the Hopf projection, is an immersed submanifold of $M^n \times S^{2n+1}(1)$ of dimension $n+1$. The projection $F: M^{n+1} \to S^{2n+1}(1)$ is an immersion

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which is called the *lifting of* f by π. Our main result is the following.

THEOREM 1. *An isometric immersion of a flat manifold* $F: M_0^{n+1} \to S^{2n+1}(1)$ *is symmetric if and only if it is the lifting by the Hopf projection* $\pi: S^{2n+1} \to \boldsymbol{CP}^n$ *of a flat totally real isometric immersion* $f: M_0^n \to \boldsymbol{CP}^n$.

From classical differential geometry, flat surfaces in S^3 are in one-to-one correspondence with solutions $\Phi(u_1, u_2) = \Phi_1(u_1 + u_2) + \Phi_2(u_1 - u_2)$ of the wave equation

$$\frac{\partial^2 \Phi}{\partial u_1^2} - \frac{\partial^2 \Phi}{\partial u_2^2} = 0 ,$$

where Φ_1, Φ_2 are arbitrary smooth functions in one variable. When one of the functions vanishes identically, we get those flat surfaces which are also obtained as the lifts of any smooth curve in S^2 by the projection of the Hopf fibration $S^3 \to \boldsymbol{CP}^1 \cong S^2$.

The simplest examples of flat totally real submanifolds in $\boldsymbol{CP}^n$ are obtained as the image by the projection π of any flat torus $T^{n+1} = S^1(r_1) \times \cdots \times S^1(r_{n+1})$, $\sum_j r_j^2 = 1$, on $S^{2n+1}(1) \subset \boldsymbol{R}^{2n+2} \cong \boldsymbol{C}^{n+1}$, where the circles are contained in complex planes. Then $\pi(T^{n+1})$ is a flat torus T^n. Notice that an immersed T^{n+1} corresponds to a trivial solution of the SGWE.

It follows from the results in [Oh] and [Ur] (see also [Ke]) that any compact isometrically immersed $f: M_0^n \to \boldsymbol{CP}^n$ with parallel mean curvature vector is a flat torus T^n. We prove the following.

THEOREM 2. *Any complete flat totally real isometric immersion* $f: M_0^n \to \boldsymbol{CP}^n$ *with mean curvature vector of constant length is a flat torus* T^n.

In fact, for parallel mean curvature vector, the corresponding local result is also true but this seems to be already known.

1. The SGWE. Given an isometric immersion $F: M_0^{n+1} \to S^{2n+1}(1)$ of a flat simply connected Riemannian manifold into the unit Euclidean sphere, it is well known (see [Mo], [Te] or [DT$_2$]) that there exists a (unique up to sign) principal coordinate system $(u_1, \ldots, u_{n+1})$ where the metric verifies

$$ds^2 = \sum_j v_j^2 du_j^2 , \qquad \sum_j v_j^2 = 1 , \tag{1}$$

and the second fundamental form of the composite $G = i \circ F$, of F with the umbilical inclusion i of $S^{2n+1}(1)$ into $\boldsymbol{R}^{2n+2}$, is given by

$$\alpha_G\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = v_j \delta_{ij} \xi_j , \tag{2}$$

$\{\xi_1, \ldots, \xi_{n+1}\}$ being an orthonormal normal frame.

THEOREM 3 (cf. [DT$_2$], [Te]). *The pair* (v, h) *with* $v=(v_1, \dots, v_{n+1})$ *and* $h=(h_{ij})$, $1\le i\ne j\le n+1$, *satisfies the* GWE

$$(\mathrm{I})=\begin{cases} \text{(i)}\ \dfrac{\partial u_i}{\partial u_k}=h_{ki}v_k\,, & \text{(ii)}\ \dfrac{\partial h_{ij}}{\partial u_i}+\dfrac{\partial h_{ji}}{\partial u_j}+\sum_k h_{ki}h_{kj}=0\,, \\ \text{(iii)}\ \dfrac{\partial h_{ij}}{\partial u_k}=h_{ik}h_{kj}\,, & \text{(iv)}\ \dfrac{\partial h_{ij}}{\partial u_j}+\dfrac{\partial h_{ji}}{\partial u_i}+\sum_k h_{ik}h_{jk}=0\,, \end{cases}$$

where always $i\ne j\ne k$. *Conversely, let* (v, h) *be a pair verifying the above system on a simply connected open subset* $U\subset \boldsymbol{R}^{n+1}$. *Then there exists an immersion* $F\colon U\to S^{2n+1}(1)$ *with flat induced metric* $ds^2=\sum v_i^2du_i^2$ *whose second fundamental form* (*in* $\boldsymbol{R}^{2n+2}$) *is given by* (2).

Equation (i) is merely the definition of the function h_{ij}. The curvature condition is given by equations (ii) and (iii). Equation (iv) is a consequence of the others together with (1) and has been added in order to have a completely integrable system. For $n=1$, when setting $(v_1, v_2)=(\sin\Phi, \cos\Phi)$, system (I) reduces to the wave equation for Φ.

If the matrix (h) is symmetric, equations (ii) and (iv) are the same. In this case we call system (I) the *Symmetric Generalized Wave equation* (SGWE).

2. The proofs. Let $\pi\colon S^{2n+1}(1)\to \boldsymbol{CP}^n$ be the standard Riemannian submersion onto complex projective space of constant holomorphic curvature. The image $\tilde\eta=\tilde J N$ of the outward unit vector field N, normal to $S^{2n+1}(1)\subset \boldsymbol{R}^{2n+2}\cong \boldsymbol{C}^{n+1}$, by the complex structure in $\boldsymbol{C}^{n+1}$, defines a smooth unit vector field tangent to $S^{2n+1}(1)$ whose integral curves are the fibers $S^1(1)$ of π.

Given an isometric immersion $f\colon M^n\to \boldsymbol{CP}^n$, the lift

$$M^{n+1}=\{\{x\}\times\pi^{-1}(f(x))\in M^n\times S^{2n+1}(1)\}$$

is an immersed $(n+1)$-dimensional submanifold of $M^n\times S^{2n+1}(1)$ whose projection $F\colon M^{n+1}\to S^{2n+1}(1)$ is an immersion. Moreover, the projection $\pi\colon M^{n+1}\to M^n$ is a Riemannian submersion with fibers $S^1(1)$ with respect to the metric induced by F. We denote by η the smooth unit vector field in TM^{n+1} defined by $F_*\eta=\tilde\eta$.

LEMMA 4. *An isometric immersion* $f\colon M^n\to \boldsymbol{CP}^n$ *is totally real if and only if the vector field* $\eta\in TM^{n+1}$ *is parallel.*

PROOF. Recall that the complex structure J in $\boldsymbol{CP}^n$ is defined by

$$J\circ\pi_*=\pi_*\circ\Phi \tag{3}$$

where $\Phi\colon TS^{2n+1}\to TS^{2n+1}$ is given for all $Z\in TS^{2n+1}$ by

$$\Phi(Z)=\tilde J Z+\langle Z, \tilde\eta\rangle N\,.$$

Recall also that f being *totally real* means that $Jf_*TM^n\perp f_*TM^n$, which is equivalent by (3) to the condition

$$\Phi(F_* TM^{n+1}) \perp F_* TM^{n+1} .$$

Let ∇' and $\tilde{\nabla}$ be the correspondent Riemannian connections in $S^{2n+1}(1)$ and $\boldsymbol{R}^{2n+2}$. We have

$$\tilde{J}Z = \tilde{J}\tilde{\nabla}_Z N = \tilde{\nabla}_Z \tilde{J} N = \tilde{\nabla}_Z \tilde{\eta} .$$

Therefore,

$$\Phi(Z) = \tilde{\nabla}_Z \tilde{\eta} + \langle Z, \tilde{\eta} \rangle N = \nabla'_Z \tilde{\eta} .$$

Let $\alpha_F : TM \times TM \to T_F M^{\perp}$ denote the second fundamental form of $F : M^{n+1} \to S^{2n+1}(1)$ with values in the normal bundle. The Gauss formula for $X \in TM$, $\xi \in T_F M^{\perp}$ yields

$$\nabla'_X F_* \eta = F_* \nabla_X \eta + \alpha_F(X, \eta) ,$$

where ∇ is the Riemannian connection in M^{n+1}. Since

$$\nabla'_X F_* \eta = \nabla'_X \tilde{\eta} = \Phi(F_* X) ,$$

the proof follows.

Proof of Theorem 1. Suppose first that $F : M_0^{n+1} \to S^{2n+1}(1)$ is a symmetric isometric immersion and let $U \subset M_0^{n+1}$ be an open subset parametrized by our principal coordinates. We claim that the unitary vector field $\eta = \sum \partial / \partial u_i$ verifies: (i) (v, h) is constant along η, (ii) η is parallel. From $\sum v_k^2 = 1$, we get

$$0 = \sum_{k \neq i} v_k \frac{\partial v_k}{\partial u_i} + v_i \frac{\partial v_i}{\partial u_i} = \sum_{k \neq i} v_k h_{ik} v_i + v_i \frac{\partial v_i}{\partial u_i} = v_i \left(\sum_{k \neq i} h_{ik} v_k + \frac{\partial v_i}{\partial u_i} \right) .$$

Hence,

$$\frac{\partial v_i}{\partial u_i} = - \sum_{k \neq i} h_{ik} v_k . \tag{4}$$

We obtain from (4) that

$$\eta(v_i) = \sum_k \frac{\partial v_i}{\partial u_k} = \sum_{k \neq i} \frac{\partial v_i}{\partial u_k} + \frac{\partial v_i}{\partial u_i} = \sum_{k \neq i} (h_{ki} - h_{ik}) v_k = 0 .$$

Also

$$\eta(h_{ij}) = \sum_{k \neq i,j} \frac{\partial h_{ij}}{\partial u_k} + \frac{\partial h_{ij}}{\partial u_i} + \frac{\partial h_{ij}}{\partial u_j} = \sum_{k \neq i,j} h_{ik} h_{kj} + \frac{\partial h_{ji}}{\partial u_i} + \frac{\partial h_{ij}}{\partial u_j} = 0 ,$$

which proves the first part of the claim.

We define an orthonormal tangent frame by $\partial / \partial u_i = v_i X_i$, $1 \leq i \leq n+1$. A straightforward computation yields

$$\nabla_{\partial/\partial u_i} X_j = h_{ji} X_i\,, \qquad 1 \le i \ne j \le n+1\,. \tag{5}$$

Making use of (i) and (5), we have

$$\begin{aligned}\nabla_\eta \partial/\partial u_i &= \nabla_\eta v_i X_i = v_i \nabla_\eta X_i = v_i \sum_k \nabla_{\partial/\partial u_k} X_i \\ &= v_i\Big(\sum_{k\ne i} h_{ik} X_k - \sum_{k \ne i} \langle \nabla_{\partial/\partial u_i} X_k, X_i\rangle X_k\Big) \\ &= v_i \sum_{k\ne i} (h_{ik} - h_{ki}) X_k = 0\,,\end{aligned}$$

and the proof of the claim follows from $[\eta, \partial/\partial u_i] = 0$.

Now set $D_t = \{x \in U : x + t\eta \in U\}$. Because the v_i's are constant along the parallel unit vector field η, the translations $r_t\colon D_t \to U$ given by $r_t(x) = x + t\eta$, form a one-parameter group of isometries. By the fundamental theorem of submanifolds, since all data related to the isometric immersion are constant along η, there exist isometries R_t of $S^{2n+1}(1)$ such that

$$R_t \circ F = F\big|_{D_t} \circ r_t\,. \tag{6}$$

Since F cannot reduce codimension even locally, the isometries R_t form a one-parameter subgroup by (6). Thus $R_t = \exp tA$, where A is skew-symmetric. Moreover, η being unitary, the curve $\gamma_t(x) = \exp tA(F(x))$ is parametrized by arclength for any $x \in U$. Hence, we have $1 = \|\gamma_t'(x)\| = \|A \exp tA(F(x))\|$. Taking $t = 0$, we get that A is orthogonal also. Therefore, $A = P^t \circ D \circ P$, where P is orthogonal and

$$D = \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\end{pmatrix}.$$

We conclude that $R_t = \cos tI + \sin tD$ up to conjugation.

Set $\bar U = \{x + t\eta : x \in U, t \in \boldsymbol{R}\}$, and let $\bar F\colon \bar U \to S^{2n+1}(1)$ be the extension of F given by (6). Clearly, since η is constant, $\bar U = V \times \boldsymbol{R}$ isometrically, for an open subset $V \subset \boldsymbol{R}^n$. F being equivariant with respect to the actions of the one-parameter groups r_t on $\bar U$ and R_t on $S^{2n+1}(1)$, we conclude that $\bar F$ induces an immersion f between the quotients $\bar U/\{r_t\} \cong V$ and $S^{2n+1}/\{R_t\} \cong \boldsymbol{CP}^n$. Set $M_0^n = V$ with the metric induced by f. It follows from Lemma 4 that f is totally real since η is parallel along $\bar U$.

Conversely, assume that $F\colon M^{n+1} \to S^{2n+1}(1)$ is the lift by the Hopf fibration of a totally real isometric immersion $f\colon M_0^n \to \boldsymbol{CP}^n$. Since the vector field $\eta \in TM^{n+1}$ tangent to the vertical fibers $S^1(1)$ of the fibration is asymptotic, we conclude by a straightforward computation that it has the form $\eta=\sum_k \varepsilon_k \partial/\partial u_k$ for some $\varepsilon_k=\pm 1$. Moreover, by changing the signs of the coordinates, if necessary, we may assume all $\varepsilon_k=1$. f being totally real, we get that η is parallel with respect to the induced metric. Flatness of $M^{n+1}=M_0^{n+1}$ follows from flatness of M_0^n and parallelity of η. Finally, from the proof of (ii), we have that $h_{ij}=h_{ji}$, $1\le i\ne j\le n+1$. This concludes the proof.

REMARKS 5. (1) For the SGWE, we have that $\partial v_i^2/\partial u_j=\partial v_j^2/\partial u_i$, $1\le i,j\le n+1$, so there exists a function $\theta=\theta(u_1,\dots,u_{n+1})$ such that $ds^2=\theta_{u_1}du_1^2+\cdots+\theta_{u_n}du_n^2$. Clearly, $\eta=\nabla\theta$. Thus, the fact that η is parallel is equivalent to Hess $\theta=0$, i.e. the function θ is linear.

(2) An intrinsic version of Theorem 1 has been given in §25 of [Bi_4]. It was shown that solutions of the SGWE correspond to orthogonal systems of Guichard-Darboux invariant by a one-parameter family of translations.

PROOF OF THEOREM 2. Consider the lift F: $M_0^{n+1}\to S^{2n+1}(1)$ of f. Using the fact that the projection of a Riemannian submersion is an isometry horizontally, and that the vertical fibers $S^1(1)$ are asymptotic, we easily conclude that the norm of the mean curvature vector of F is the same constant. Moreover, M_0^{n+1} is complete. We obtain from Corollary 3 of [DT_1] that $F(M_0^{n+1})$ is a torus T^{n+1} and this concludes the proof.

REMARKS 6. (1) In the local case, if we assume that f has parallel mean curvature vector, we easily see by similar arguments as above that F must also have parallel mean curvature vector. Then the image of F is contained in a torus T^{n+1} as follows from Theorem 1 of [Er] or Theorem 1 of [DT_1].

(2) It was shown in [Ej] that an n-dimensional totally real minimal submanifold of constant sectional curvature in any n-dimensional complex space form must be totally geodesic or flat.

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