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FUNCTIONS WHICH OPERATE ON ALGEBRAS OF FOURIER MULTIPLIERS

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Abstract. We study functions which operate on a Banach space of bounded functions defined on a discrete space. As a consequence we characterize functions which operate on the algebra of the translation invariant operators from $L^p(G)$ to $L^2(G)$ for $1<p<2$ and for a compact abelian group G.

1. Introduction. Let G be a compact abelian group and $\hat{G}$ the dual group of G. Let $1\leq p, q\leq\infty$. A bounded operator T from $L^p(G)$ to $L^q(G)$ is called a (L^p,L^q)-multiplier if $TT_\gamma=T_\gamma T$ for every $\gamma\in G$, where $T_\gamma f(x)=f(x-\gamma)$. The set of all (L^p, L^q)-multipliers is denoted by $M(p, q)$. Since G is a compact abelian group the Fourier transform $\hat{T}$ for $T\in M(p, q)$ is a complex-valued bounded function defined on the discrete group $\hat{G}$. We denote $M(p, q)\hat{}=\{\hat{T}\colon T\in M(p, q)\}$. If $p\leq q$, then $M(p, q)$ is a Banach algebra and $M(p, q)\hat{}$ is a Banach algebra of bounded functions on $\hat{G}$. Let E be a space of complex-valued functions defined on a set X. We say a complex-valued function φ defined on a subset S of $\boldsymbol{C}$ operates on E if $\varphi\circ f\in E$ for every $f\in E$ such that $f(X)\subset S$.

The algebra $M(1, 1)$ is isometric and isomorphic to the algebra $M(G)$ of all the bounded regular Borel measures on G and the operating functions on $M(G)\hat{}$ is characterized by Kahane and Rudin [10]. The result is extended to the case of $p=q\neq 2$ by Igari [8]. Igari and Sato [9] consider the case of $1\leq p<q\leq\infty$. They prove, for example, that if $1\leq p<q\leq 2$ or $2\leq p<q\leq\infty$, n_0 is the smallest n integer such that $n\geq\beta_0=(1/q-1/2)/(1/p-1/q)$ or $n\geq\beta_0=(1/2-1/p)/(1/p-1/q)$ respectively, and φ_0 is a bounded function on $[-1, 1]$, then for any constants $\alpha_1, \alpha_2, \dots, \alpha_{n_0}$ the function

$$\varphi(t)=\alpha_1 t+\alpha_2 t^2+\cdots+\alpha_{n_0}t^{n_0}+|t|^{\beta_0+1}\varphi_0(t)$$

defined on $[-1, 1]$ operates on $M(p, q)\hat{}$. They also prove that if $1\leq p<2\leq q\leq\infty$, $\beta_1=\min\{(1/2-1/q)/(1/p-1/2), (1/p-1/2)/(1/2-1/q)\}$ and φ_0 is a bounded function on $[-1, 1]$, then for any constant α the function

$$\varphi(t)=\alpha t+|t|^{\beta_1+1}\varphi_0(t)$$

operates on $M(p, q)\hat{}$. The converse of the last result when G is the circle group is also proven by Igari and Sato [9]. One of the essential arguments they use in their proof

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is to estimate the norm of certain trigonometric polynomials when G is the circle group. By their results the operating functions on $M(p, 2)^{\wedge}$ in the case of the circle group are characterized: a complex-valued function φ defined on $[-1, 1]$ operates on $M(p, 2)^{\wedge}$ if and only if $\varphi(0)=0$, φ is bounded on $[-1, 1]$ and $\limsup_{w\to 0}|\varphi(w)/w|<\infty$.

In this paper we study operating functions in a different approach, that is, we first consider the operating functions on a Banach space of bounded functions defined on a discrete space, which is modeled after the space $M(p, 2)^{\wedge}$. Then by using the results for the abstract function spaces we characterize the operating functions on $M(p, 2)^{\wedge}$ for an arbitrary compact abelian group G in a fashion similar to the case of the circle group G by Igari and Sato [9]. Suppose that $1<p<2$ and S is a subset of the complex plane such that $0\in S$ and 0 is an accumulation point of S. Then a function φ defined on S operates on $M(p, 2)^{\wedge}$ if and only if $\varphi(0)=0$, φ is bounded on any bounded subset of S and $\limsup_{w\to 0}|\varphi(w)/w|<\infty$.

2. Operating functions on a space of bounded functions. Let $\boldsymbol{Z}$ be a discrete space. We denote by $C_0(\boldsymbol{Z})$ the space of all complex-valued functions on $\boldsymbol{Z}$ which vanish at infinity. The space of all complex-valued bounded functions on $\boldsymbol{Z}$ is denoted by $C^b(\boldsymbol{Z})$. For a subset X of $\boldsymbol{Z}$ $\|\cdot\|_{\infty(x)}$ is the supremum norm on X. The complex plane is denoted by $\boldsymbol{C}$.

We study the feature at the origin of the operating functions on a Banach space A of bounded functions on $\boldsymbol{Z}$ which satisfies the following four conditions: (1) $1\notin A$; (2) $A\cap C_0(\boldsymbol{Z})\neq C_0(\boldsymbol{Z})$; (3) $\chi_{\{z\}}\in A$ for every $z\in\boldsymbol{Z}$ and $\sup\{\|\chi_{\{z\}}\|_A : z\in\boldsymbol{Z}\}<\infty$; (4) $f\chi_X\in A$ for every $f\in A$ and for every countable subset X of $\boldsymbol{Z}$, where we donte the characteristic function of a subset Y of $\boldsymbol{Z}$ by χ_Y. The condition (3) plays an essential role in this and the following sections. If $1<p<2$, then the algebra $M(p, 2)^{\wedge}$ is contained in $C^b(\boldsymbol{Z})$ and satisfies the above four properties.

LEMMA 1. *Let $\boldsymbol{Z}$ be a discrete space and A a complex Banach space continuously embedded in $C_0(\boldsymbol{Z})$ such that $\chi_{\{z\}}\in A$ for every $z\in\boldsymbol{Z}$. Then A coincides with $C_0(\boldsymbol{Z})$ if and only if there is a positive number M such that the inequality*

$$\inf\{\|f\|_A : f\in A,\ f|X=1,\ f|Y=0\}<M$$

holds for every pair X and Y of disjoint compact subsets of $\boldsymbol{Z}$.

PROOF. We may suppose without loss of generality that $\|\cdot\|_{\infty(\boldsymbol{Z})}\leq\|\cdot\|_A$. The necessity is trivial by the open mapping theorem. So we show the sufficiency. Let f be a function in $C_0(\boldsymbol{Z})$ such that $\|f\|_{\infty(\boldsymbol{Z})}=1$. Put $X_1=\{z\in\boldsymbol{Z} : \operatorname{Re}f(z)\geq 1/2\}$, $X_2=\{z\in\boldsymbol{Z} : \operatorname{Re}f(z)\leq -1/2\}$, $X_3=\{z\in\boldsymbol{Z}\setminus(X_1\cup X_2) : \operatorname{Im}f(z)\geq 1/2\}$, $X_4=\{z\in\boldsymbol{Z}\setminus(X_1\cup X_2) : \operatorname{Im}f(z)\leq -1/2\}$. Then each X_i is a compact subset of $\boldsymbol{Z}$. By the condition there exist functions f_1, f_2, f_3 and f_4 in A such that $f_i=1$ on X_i, $f_i=0$ on $(\bigcup_{k=1}^4 X_k)\setminus X_i$ and $\|f_i\|_A<M$ for every $1\leq i\leq 4$. Put $\delta=1/20M$ and $h=\delta(f_1+if_3-f_2-if_4)$. Then $h\in A$ and $\|h\|_A\leq 1/5$. We also see by simple calculation that $\|f-h\|_{\infty(\boldsymbol{Z})}\leq\sqrt{1-\delta+\delta^2}$ and

$\sqrt{1-\delta+\delta^2}<1$. It follows that for every $f \in C_0(\boldsymbol{Z})$ there exists a function $h \in A$ such that $\|h\|_A \leq \|f\|_{\infty(\boldsymbol{Z})}/5$ and $\|f-h\|_{\infty(\boldsymbol{Z})} \leq \sqrt{1-\delta+\delta^2}\,\|f\|_{\infty(\boldsymbol{Z})}$. Thus by a standard argument on Banach spaces we see that $A = C_0(\boldsymbol{Z})$. □

PROPOSITION 2. *Let $\boldsymbol{Z}$ be a discrete space and A a complex Banach space which is continuously embedded in $C^b(\boldsymbol{Z})$. Suppose that A satisfies the following four conditions: (1) $1 \notin A$; (2) $A \cap C_0(\boldsymbol{Z}) \neq C_0(\boldsymbol{Z})$; (3) A contains the characteristic function $\chi_{\{z\}}$ for every $z \in \boldsymbol{Z}$ and $\sup\{\|\chi_{\{z\}}\|_A : z \in \boldsymbol{Z}\} = L < \infty$; (4) A contains the function $f\chi_X$ for every $f \in A$ and for every countable subset X of $\boldsymbol{Z}$. Let S be a subset of $\boldsymbol{C}$ such that $0 \in S$ and 0 is an accumulation point of S. Suppose that a complex-valued function φ defined on S operates on A. Then we have $\varphi(0)=0$ and*

$$\limsup_{w\to 0}\left|\frac{\varphi(w)}{w}\right| < \infty\,.$$

The condition $\sup\{\|\chi_{\{z\}}\|_A : z \in \boldsymbol{Z}\} < \infty$ is essential, that is, the conclusions of Proposition 2 and Theorem 5 are false unless $\sup\{\|\chi_{\{z\}}\|_A : z \in \boldsymbol{Z}\} < \infty$ is assumed.

EXAMPLE (cf. [5, Proposition 24], [6, Example]). Let $\boldsymbol{N}$ be the discrete space of all the positive integers and $E = \{f \in C_0(\boldsymbol{N}) : \sum_{n=1}^{\infty} |f(n)| C_n < \infty\}$, where $C_n = 2^{n^2}$. Then E is a Banach algebra with respect to the norm $\|f\| = \sum_{n=1}^{\infty} |f(n)| C_n$ for $f \in E$ such that the conditions (1), (2) and (4) in Proposition 2 hold and $\chi_{\{z\}} \in E$ for every $z \in \boldsymbol{N}$. Let $d_n = 1/(2C_{n+1}) + 1/(2C_{n+1}-2)$, $r_n = -1/(2C_{n+1}) + 1/(2C_{n+1}-2)$ and $h_n = 2^{-n^2-n}$. Let φ be a complex-valued continuous function on $\{w \in \boldsymbol{C} : |w| < 1\}$ such that

$$\varphi(w) = \begin{cases} 0\,, & |w-d_n| > r_n \quad \text{for all} \quad n \in \boldsymbol{N} \\ (r_n - |w-d_n|)h_n/r_n\,, & |w-d_n| \leq r_n \quad \text{for some} \quad n \in \boldsymbol{N}\,. \end{cases}$$

Then we see that $\limsup_{w\to 0} |\varphi(w)/w| = \infty$. We also see that φ operates on E.

Inspired by this example we may consider the problem involving the function φ such that $\varphi(0)=0$ and $\lim_{w\to 0} |\varphi(w)/w| = \infty$ which operates on an algebra of continuous functions. We may easily suppose that such functions are not so many unless the algebra contains every continuous functions. In the case of normal Banach function algebras we have shown, for example, the following (cf. [6]):

THEOREM A. *Let A be a normal Banach function algebra on a compact Hausdorff space X and φ a complex-valued function defined on the open unit disk such that $|(\varphi(w)-\varphi(0))/w| \to \infty$ as $w \to 0$. Suppose that φ operates on A. Then A consists of all complex-valued continuous functions on X.*

We also see that many non-Lipschitz functions cannot operate on a Banach space of continuous functions on a compact Hausdorff space unless it contains every continuous functions (cf. [1], [2], [4], [5], [11]). We also study in [7] the case where Banach space of continuous functions on a locally compact Hausdorff space.

PROOF OF PROPOSITION 2. We see that $\varphi(0)=0$ since $\varphi(0)\in A$ and $1\notin A$. We suppose $\limsup_{w\to 0}|\varphi(w)/w|=\infty$ and derive contradiction. Without loss of generality we may suppose that $\|\cdot\|_{\infty(\boldsymbol{Z})}\le\|\cdot\|_A$. So $\|\cdot\|_A$ is complete on $A\cap C_0(\boldsymbol{Z})$. By Lemma 1 we can choose two sequences $\{X_n^{(0)}\}$ and $\{X_n^{(1)}\}$ of compact subsets of $\boldsymbol{Z}$ such that $X_m^{(i)}\cap X_n^{(j)}=\emptyset$ if $(i,m)\ne(j,n)$ and that

$$M_n=\inf\{\|f\|_A : f\in A\cap C_0(\boldsymbol{Z}),\ f\,|\,X_n^{(1)}=1,\ f\,|\,X_n^{(0)}=0\}\ge n\,.$$

Indeed, we choose $\{X_n^{(0)}\}$ and $\{X_n^{(1)}\}$ as follows: Let z_0 and z_1 be a pair of different points in $\boldsymbol{Z}$. Put $X_1^{(0)}=\{z_0\}$ and $X_1^{(1)}=\{z_1\}$. Then we see that $X_1^{(0)}\cap X_1^{(1)}=\emptyset$ and

$$\inf\{\|f\|_A : f\in A\cap C_0(\boldsymbol{Z}),\ f\,|\,X_1^{(1)}=1,\ f\,|\,X_1^{(0)}=0\}\ge 1\,.$$

Suppose that $X_1^{(0)}, X_2^{(0)},\dots, X_n^{(0)}$ and $X_1^{(1)}, X_2^{(1)},\dots, X_n^{(1)}$ are so chosen that they are pairwise disjoint and

$$k\le\inf\{\|f\|_A : f\in A\cap C_0(\boldsymbol{Z}),\ f\,|\,X_k^{(1)}=1,\ f\,|\,X_k^{(0)}=0\}$$

for every k with $1\le k\le n$. Put $O=\bigcup_{k=1}^{n}(X_k^{(0)}\cup X_k^{(1)})$. Then O consists of a finite number of isolated points. Thus we see that $(A\,|(\boldsymbol{Z}\setminus O))\cap C_0(\boldsymbol{Z}\setminus O)$ is a complex Banach space with respect to the quotient norm $\|\cdot\|_{A|(\boldsymbol{Z}\setminus O)}$ defined by

$$\|f\|_{A|(\boldsymbol{Z}\setminus O)}=\inf\{\|\tilde f\|_A : \tilde f\in A,\ \tilde f\,|(\boldsymbol{Z}\setminus O)=f\}\,.$$

Then we see that $\|\cdot\|_{\infty(\boldsymbol{Z}\setminus O)}\le\|\cdot\|_{A|(\boldsymbol{Z}\setminus O)}$ and $(A\,|(\boldsymbol{Z}\setminus O))\cap C_0(\boldsymbol{Z}\setminus O)\ne C_0(\boldsymbol{Z}\setminus O)$. We also see that $\chi_{\{z\}}|(\boldsymbol{Z}\setminus O)\in(A\,|(\boldsymbol{Z}\setminus O))\cap C_0(\boldsymbol{Z}\setminus O)$ for $z\in\boldsymbol{Z}$. It follows by Lemma 1 that there is a pair of disjoint compact subsets $X_{n+1}^{(0)}$ and $X_{n+1}^{(1)}$ of $\boldsymbol{Z}\setminus O$ such that

$$\inf\{\|f\|_{A|(\boldsymbol{Z}\setminus O)} : f\in(A\,|(\boldsymbol{Z}\setminus O))\cap C_0(\boldsymbol{Z}\setminus O),\ f\,|\,X_{n+1}^{(0)}=0,\ f\,|\,X_{n+1}^{(1)}=1\}\ge n+1\,.$$

By simple calculation we see that

$$\begin{aligned}&\inf\{\|f\|_A : f\in A\cap C_0(\boldsymbol{Z}),\ f\,|\,X_{n+1}^{(0)}=0,\ f\,|\,X_{n+1}^{(1)}=1\}\\ &\quad=\inf\{\|f\|_{A|(\boldsymbol{Z}\setminus O)} : f\in(A\,|(\boldsymbol{Z}\setminus O))\cap C_0(\boldsymbol{Z}\setminus O),\ f\,|\,X_{n+1}^{(0)}=0,\ f\,|\,X_{n+1}^{(1)}=1\}\,.\end{aligned}$$

It follows by induction that we can choose two sequences $\{X_n^{(0)}\}$ and $\{X_n^{(1)}\}$ with the required properties. For each positive integer m put

$$\boldsymbol{Z}_m=\bigcup_{n=1}^{\infty}(X_{2^m n-2^{m-1}}^{(0)}\cup X_{2^m n-2^{m-1}}^{(1)})\,.$$

Then we see that

$$(A\,|\boldsymbol{Z}_m)\cap C_0(\boldsymbol{Z}_m)=(A\cap C_0(\boldsymbol{Z}))\,|\boldsymbol{Z}_m$$

and the two norms $\|\cdot\|_{A|\boldsymbol{Z}_m}$ and $\|\cdot\|_{A|\boldsymbol{Z}_m}^0$ are equivalent Banach norms on $(A\cap C_0(\boldsymbol{Z}))\,|\boldsymbol{Z}_m$, where

$$\|f\|_{A|\boldsymbol{Z}_m}=\inf\{\|\tilde f\|_A : \tilde f\in A,\ \tilde f\,|\boldsymbol{Z}_m=f\}\,,$$

$$\|f\|^0_{A|\boldsymbol{Z}_m}=\inf\{\|\tilde{f}\|_A:\tilde{f}\in A\cap C_0(\boldsymbol{Z}),\ \tilde{f}|\boldsymbol{Z}_m=f\}$$

for every $f\in(A\cap C_0(\boldsymbol{Z}))|\boldsymbol{Z}_m$. The inclusion $(A\cap C_0(\boldsymbol{Z}))|\boldsymbol{Z}_m\subset(A|\boldsymbol{Z}_m)\cap C_0(\boldsymbol{Z}_m)$ is trivial. We show the opposite inclusion. Let $g\in(A|\boldsymbol{Z}_m)\cap C_0(\boldsymbol{Z}_m)$. Then there is $\tilde{g}\in A$ such that $\tilde{g}|\boldsymbol{Z}_m=g$. Since $\boldsymbol{Z}_m$ is a countable set, we see that $\tilde{g}\chi_{\boldsymbol{Z}_m}\in A$ by the condition (4). Thus we have $\tilde{g}\chi_{\boldsymbol{Z}_m}\in A\cap C_0(\boldsymbol{Z})$ since $g\in C_0(\boldsymbol{Z}_m)$. It follows that $g=(\tilde{g}\chi_{\boldsymbol{Z}_m})|\boldsymbol{Z}_m\in(A\cap C_0(\boldsymbol{Z}))|\boldsymbol{Z}_m$. We conclude that $(A|\boldsymbol{Z}_m)\cap C_0(\boldsymbol{Z}_m)=(A\cap C_0(\boldsymbol{Z}))|\boldsymbol{Z}_m$. Since $\|\cdot\|_{\infty(\boldsymbol{Z})}\le\|\cdot\|_A$, we see that $\|\cdot\|_A$ is complete on $A\cap C_0(\boldsymbol{Z})$. Thus $\|\cdot\|^0_{A|\boldsymbol{Z}_m}$ is a Banach norm on $(A\cap C_0(\boldsymbol{Z}))|\boldsymbol{Z}_m$. We also see that $\|\cdot\|_{A|\boldsymbol{Z}_m}$ is a Banach norm on $(A|\boldsymbol{Z}_m)\cap C_0(\boldsymbol{Z}_m)$. It follows by the open mapping theorem that the two norms $\|\cdot\|^0_{A|\boldsymbol{Z}_m}$ and $\|\cdot\|_{A|\boldsymbol{Z}_m}$ are equivalent. In fact, there is a positive constant c_m such that the inequalities

$$\|f\|_{A|\boldsymbol{Z}_m}\le\|f\|^0_{A|\boldsymbol{Z}_m}\le c_m\|f\|_{A|\boldsymbol{Z}_m}$$

hold for every $f\in(A\cap C_0(\boldsymbol{Z}))|\boldsymbol{Z}_m$. For every positive integer n put

$$N_n^{(m)}=\sup\{\|f\|_{A|\boldsymbol{Z}_m}:f\in A|\boldsymbol{Z}_m,\ f^2=f,\ \sharp\{z\in\boldsymbol{Z}_m:f(z)=1\}\le n\}\ ,$$

where $\sharp$ denotes the cardinality. Suppose that $f\in A|\boldsymbol{Z}_m$ satisfies $f^2=f$ and $\sharp\{z\in\boldsymbol{Z}_m:f(z)=1\}\le n$. Then we have $f=\sum_{z\in\{z\in\boldsymbol{Z}_m:f(z)=1\}}\chi_{\{z\}}|\boldsymbol{Z}_m$, hence

$$\|f\|_{A|\boldsymbol{Z}_m}\le\sum\|\chi_{\{z\}}|\boldsymbol{Z}_m\|_{A|\boldsymbol{Z}_m}\le nL\ .$$

Thus we have $N_n^{(m)}\le nL$. We also see that the inequalities

$$N_n^{(m)}\le N_{n+1}^{(m)}\le N_n^{(m)}+L$$

hold. $N_n^{(m)}\le N_{n+1}^{(m)}$ is trivial by the definition of $N_n^{(m)}$. Suppose that f is a function in $A|\boldsymbol{Z}_m$ such that $f^2=f$ and $\sharp\{z\in\boldsymbol{Z}_m:f(z)=1\}\le n+1$. Choose a point $z\in\{z\in\boldsymbol{Z}_m:f(z)=1\}$ and put $g=f-\chi_{\{z\}}|\boldsymbol{Z}_m$. Then $g\in A|\boldsymbol{Z}_m$, $g=g^2$ and $\sharp\{z\in\boldsymbol{Z}_m:g(z)=1\}\le n$. We also see that

$$\|f\|_{A|\boldsymbol{Z}_m}\le\|g\|_{A|\boldsymbol{Z}_m}+\|\chi_{\{z\}}|\boldsymbol{Z}_m\|_{A|\boldsymbol{Z}_m}\le N_n^{(m)}+L\ .$$

Hence $N_{n+1}^{(m)}\le N_n^{(m)}+L$ holds. Next we show that $N_n^{(m)}\to\infty$ as $n\to\infty$. For each positive integer k put

$$n_k=\sharp X^{(1)}_{2^mk-2^{m-1}}\ .$$

Then

$$\begin{aligned}N^{(m)}_{n_k}&\ge\|\chi_{X^{(1)}_{2^mk-2^{m-1}}}|\boldsymbol{Z}_m\|_{A|\boldsymbol{Z}_m}\ge\frac{1}{c_m}\|\chi_{X^{(1)}_{2^mk-2^{m-1}}}|\boldsymbol{Z}_m\|^0_{A|\boldsymbol{Z}_m}\\&\ge\frac{1}{c_m}\inf\{\|f\|_A:f\in A\cap C_0(\boldsymbol{Z}),\ f|X^{(1)}_{2^mk-2^{m-1}}=1,\ f|X^{(0)}_{2^mk-2^{m-1}}=0\}\\&\ge\frac{1}{c_m}(2^mk-2^{m-1})\ ,\end{aligned}$$

so we have $N_{n_k}^{(m)} \to \infty$ as $k \to \infty$. Since $N_n^{(m)}$ is increasing we see that $N_n^{(m)} \to \infty$ as $n \to \infty$.

Let T_m be the linear operator on A defined by $T_m f = f\chi_{\boldsymbol{Z}_m}$ for $f \in A$. By the condition (4) T_m is well-defined and we see by the closed graph theorem that T_m is a bounded linear operator on A. We denote the operator norm of T_m by $\|T_m\|$. There exists a sequence $\{w_m\} \subset S$ such that $0 < |w_m| < 1/(2^{m+1} N_1^{(m)} \|T_m\|)$ and that $|w_m| \|T_m\| 2^{m+1} m < |\varphi(w_m)|$. There exists a positive real number $n(m)$ such that

$$2^{m+1}\|T_m\| N_{n(m)}^{(m)} < |w_m|^{-1} \leq 2^{m+1}\|T_m\| N_{n(m)+1}^{(m)}$$

since $N_n^{(m)} \to \infty$ as $n \to \infty$, hence

$$2^{m+1}\|T_m\| N_{n(m)}^{(m)} < |w_m|^{-1} \leq 2^{m+1}\|T_m\|(L+1) N_{n(m)}^{(m)}$$

since $N_{n(m)+1}^{(m)} \leq N_{n(m)}^{(m)} + L \leq (L+1) N_{n(m)}^{(m)}$. By the definition of $N_{n(m)}^{(m)}$ there exists $h_m \in A|\boldsymbol{Z}_m$ such that $h_m^2 = h_m$, $\sharp\{z \in \boldsymbol{Z}_m : h_m(z) = 1\} \leq n(m)$ and

$$N_{n(m)}^{(m)}/2 < \|h_m\|_{A|\boldsymbol{Z}_m} \leq N_{n(m)}^{(m)}\,,$$

so $\tilde{h}_m \in A$ exists such that $\tilde{h}_m|\boldsymbol{Z}_m = h_m$ and $\|\tilde{h}_m\|_A < 2N_{n(m)}^{(m)}$. Thus a function $w_m \tilde{h}_m \chi_{\boldsymbol{Z}_m}$ is in A and satisfies $\|w_m \tilde{h}_m \chi_{\boldsymbol{Z}_m}\|_A < 2^{-m}$, so that $h = \sum_{m=1}^{\infty} w_m \tilde{h}_m \chi_{\boldsymbol{Z}_m}$ converges in A and $h(\boldsymbol{Z}) \subset \{0\} \cup \{w_m\} \subset S$ since $h_m^2 = h_m$. Thus we conclude that the inequalities

$$\|\varphi \circ h\|_A \geq \|\varphi(w_m h_m)\|_{A|\boldsymbol{Z}_m} = |\varphi(w_m)|\,\|h_m\|_{A|\boldsymbol{Z}_m} > m/(2L+2)$$

hold for every m, which is a contradiction. □

3. Operating functions on multipliers. As a corollary of the results in the previous section we characterize the functions which operate on certain multipliers.

COROLLARY 3. *Let G be an infinite compact abelian group and $\hat{G}$ the dual group of G. Suppose that p is a real number such that $1 < p < 2$. Suppose also that S is a subset of $\boldsymbol{C}$ such that $0 \in S$ and 0 is an accumulation point of S. Then a complex-valued function φ defined on S operates on $M(p, 2)^\wedge$ if and only if φ satisfies that $\varphi(0) = 0$, φ is bounded on every bounded subset of S and that* $\limsup_{w \to 0} |\varphi(w)/w| < \infty$.

PROOF. Let $\boldsymbol{Z} = \hat{G}$ and $A = M(p, 2)^\wedge$. Then A is a Banach algebra such that $A \subset C^b(\boldsymbol{Z})$, hence $\|\cdot\|_{\infty(\boldsymbol{Z})} \leq \|\cdot\|_A$. We show that A satisfies the four conditions in Proposition 2. It is elementary to show that $1 \notin A$, $A \cap C_0(\boldsymbol{Z}) \neq C_0(\boldsymbol{Z})$, $\chi_{\{z\}} \in A$ and $\|\chi_{\{z\}}\|_A = 1$ for every $z \in \boldsymbol{Z}$. Thus we verify that $f\chi_X \in A$ for every $f \in A$ and for every countable subset X of $\boldsymbol{Z}$. Let $\operatorname{Trig} G$ denote the set of all trigonometric polynomials on G. Put

$$T_X F = \sum_{z \in X} f(z) \hat{F}(z) z$$

for $F \in \operatorname{Trig} G$. Then by the Plancherel formula we see that

$$\|T_X F\|_2^2 = \sum_{z \in X} |f(z)\hat{F}(z)|^2 .$$

On the other hand there exists a positive constant c such that

$$\left(\sum_{z \in \boldsymbol{Z}} |f(z)\hat{F}(z)|^2\right)^{1/2} \leq c\|F\|_p$$

for every $F \in \operatorname{Trig} G$ since $f \in A = M(p, 2)\hat{}$. It follows that the inequality

$$\|T_X F\|_2 \leq c\|F\|_p$$

holds for every $F \in \operatorname{Trig} G$. We conclude that T_X can be extended to an operator $\tilde{T}_X$ in $M(p, 2)$. By a simple calculation $(\tilde{T}_X)\hat{} = f\chi_X$, that is, $f\chi_X \in A$. Thus A satisfies the condition (4) in Proposition 2.

Let Ω_1 be the set of all complex-valued functions defined on S which operate on A and Ω_2 the set of all complex-valued functions φ defined on S such that $\varphi(0)=0$, $\limsup_{w \to 0} |\varphi(w)/w| < \infty$ and that φ is bounded on any bounded subset of S. By the Plancherel formula the inclusion $\Omega_1 \supset \Omega_2$ is easy to prove. We show the opposite inclusion. Suppose that $\varphi \in \Omega_1$. Then by Proposition 2 we see that $\varphi(0)=0$ and $\limsup_{w \to 0} |\varphi(w)/w| < \infty$. Suppose that φ is not bounded on a bounded subset S' of S. Then there exists a sequence $\{w_n\}$ of S' such that for a $w_0 \in \boldsymbol{C}$ we have $\sum_{n=1}^{\infty} |w_n - w_0| < \infty$ and $|\varphi(w_n)| \to \infty$ as $n \to \infty$. Let Y be an infinite $\Lambda(p')$ set, where p' is a real number such that $1/p + 1/p' = 1$. Then by a theorem of Hare [3, Corollary 1.9] there is $f \in A$ such that $f^2 = f$ and $Y = \{z \in \boldsymbol{Z} : f(z) = 1\}$. We may assume that Y is a countable set since a subset of a $\Lambda(p')$ set is again a $\Lambda(p')$ set. Put $Y = \{y_n\}$ and

$$g = w_0 f + \sum (w_n - w_0)\chi_{\{y_n\}} .$$

Then $g \in A$ since $\sum_{n=1}^{\infty} |w_n - w_0| < \infty$ and $\|\chi_{\{y_k\}}\|_A = 1$ for every k. We see that $g(\boldsymbol{Z}) \subset \{0\} \cup \{w_n\} \subset S$, hence $\varphi \circ g \in A$. Then $|\varphi \circ g(y_n)| = |\varphi(w_n)| \to \infty$ as $n \to \infty$, which is a contradiction. Thus φ is bounded on any bounded subset of S, and we conclude that $\Omega_1 \subset \Omega_2$. □

4. A generalization for abstract spaces. By using Proposition 2 we can prove results similar to Corollary 3 for a large class of Banach spaces which are continuously embedded in $C^b(\boldsymbol{Z})$. For a subset K of $\boldsymbol{C}$ we denote the interior of K by $\operatorname{int} K$.

LEMMA 4. *Let $\boldsymbol{Z}$ be a discrete space and A a complex Banach space continuously embedded in $C^b(\boldsymbol{Z})$ such that $\chi_{\{z\}} \in A$ for every $z \in \boldsymbol{Z}$. Let K be a subset of $\boldsymbol{C}$ such that $0 \in \operatorname{int} K$. Suppose that every complex-valued bounded function φ defined on K such that $\varphi(0)=0$ and $\limsup_{w \to 0} |\varphi(w)/w| < \infty$ operates on A. Then $f\chi_X \in A$ for every $f \in A$ and for every countable subset X of $\boldsymbol{Z}$.*

PROOF. Without loss of generality we may assume $\|\cdot\|_{\infty(\boldsymbol{Z})} \leq \|\cdot\|_A$. Let $f \in A$ and

X a countable subset of $\boldsymbol{Z}$. If X is finite, then $f\chi_X \in A$ since $\chi_{\{z\}} \in A$ for every $z \in \boldsymbol{Z}$ and hence $\sum_{z \in \boldsymbol{Z}} f\chi_{\{z\}} \in A$. Suppose that X is infinite. Put $X = \{x_n\}$. Since A is closed under the constant multiplication we may suppose that $\overline{f(\boldsymbol{Z})} \subset \operatorname{int} K$. Define a complex-valued function φ by

$$\varphi(w) = \begin{cases} w, & w \in f(X), \\ 0, & w \in K \setminus f(X). \end{cases}$$

Then $\varphi(0) = 0$ and φ is bounded since $f(X)$ is a bounded subset of $\boldsymbol{C}$. We also see that $\limsup_{w \to 0} |\varphi(w)/w| \le 1$, so that $\varphi \circ f \in A$ and $\varphi \circ f(\boldsymbol{Z}) \subset \{0\} \cup f(X)$. It follows that $\varphi \circ f(\boldsymbol{Z})$ is a countable set. Put

$$\varepsilon = \inf\{|w - w'| : w \in \overline{f(\boldsymbol{Z})},\ w' \in \boldsymbol{C} \setminus K\}.$$

Then $\varepsilon > 0$ since $\overline{f(\boldsymbol{Z})}$ is compact and $\overline{f(\boldsymbol{Z})} \subset \operatorname{int} K$. We define a sequence $\{a_n\}$ of non-negative real numbers by induction as follows: If $f(x_1) = 0$, then put $a_1 = 0$. If $f(x_1) \ne 0$, then choose a real number a_1 such that $0 < a_1 < \varepsilon/(4\|\chi_{\{x_1\}}\|_A)$ and $f(x_1) + a_1 f(x_1)/|f(x_1)| \in \operatorname{int} K \cap (f(X))^c$. Note that such an a_1 exists since $f(X)$ is a countable subset of $\boldsymbol{C}$. Suppose that $a_1, \dots, a_n$ have been chosen. If $f(x_{n+1}) = 0$, then put $a_{n+1} = 0$. If $f(x_{n+1}) \ne 0$, then choose a real number a_{n+1} such that $0 < a_{n+1} < \varepsilon/(2^{n+2}\|\chi_{\{x_{n+1}\}}\|_A)$ and

$$f(x_{n+1}) + a_{n+1} f(x_{n+1})/|f(x_{n+1})| \in \operatorname{int} K \cap \left\{ f(X) \cup \bigcup_{i=1}^{n} (f(x_i) + a_i M_i) \right\}^c,$$

where

$$M_i = \begin{cases} 0, & f(x_i) = 0, \\ f(x_i)/|f(x_i)|, & f(x_i) \ne 0. \end{cases}$$

Then $u = \sum_{n=1}^{\infty} a_n M_n \chi_{\{x_n\}}$ converges in A since $\|a_n M_n \chi_{\{x_n\}}\|_A < \varepsilon/2^{n+1}$. Since $\|\cdot\|_{\infty(\boldsymbol{Z})} \le \|\cdot\|_A$ we see that $u(z) = \sum_{n=1}^{\infty} a_n M_n \chi_{\{x_n\}}(z)$ for every $z \in \boldsymbol{Z}$, so that $u(z) = 0$ if $z \in \boldsymbol{Z} \setminus X$ and $u(x_k) = a_k M_k$ for every positive integer k. Put $g = \varphi \circ f + u$. Then $g \in A$ and $g(\boldsymbol{Z}) \subset K$. Define a complex-valued function ϕ on K by

$$\phi(w) = \begin{cases} f(x_n), & w = g(x_n) \text{ and } g(x_n) \ne 0, \\ 0, & \text{otherwise}. \end{cases}$$

We see by simple calculation that ϕ is well-defined. We also see that $|\phi(w)| \le |w|$ since $|f(x_n)| \le |g(x_n)|$. It follows that $f\chi_X = \phi \circ g \in A$. $\square$

THEOREM 5. *Let $\boldsymbol{Z}$ be a discrete space and A a complex Banach space continuously embedded in $C^b(\boldsymbol{Z})$ such that $A \cap C_0(\boldsymbol{Z}) \ne C_0(\boldsymbol{Z})$, $\chi_{\{z\}} \in A$ for every $z \in \boldsymbol{Z}$ and that $\sup\{\|\chi_{\{z\}}\|_A : z \in \boldsymbol{Z}\} = L < \infty$. Let K be a subset of $\boldsymbol{C}$ such that $0 \in \operatorname{int} K$.*

(1) *Suppose that $A \subset C_0(\boldsymbol{Z})$. Then the following four conditions are equivalent:*

(i) *Every complex-valued bounded continuous function* φ *defined on* K *such that* $\varphi(0)=0$ *and* $\limsup_{w\to 0}|\varphi(w)/w|<\infty$ *operates on* A.

(ii) *Every complex-valued function* φ *defined on* K *such that* $\varphi(0)=0$ *and* $\limsup_{w\to 0}|\varphi(w)/w|<\infty$ *operates on* A.

(iii) *The complex-valued function* φ *defined on* K *operates on* A *if and only if* φ *satisfies* $\varphi(0)=0$ *and* $\limsup_{w\to 0}|\varphi(w)/w|<\infty$.

(iv) *Suppose that* $f\in A$ *and* $g\in C_0(\boldsymbol{Z})$ *such that* $|g|\leq|f|$ *on* $\boldsymbol{Z}$. *Then* $g\in A$.

(2) *Suppose that* $A\not\subset C_0(\boldsymbol{Z})$ *and* $1\notin A$. *Then the following two conditions are equivalent*:

(i) *Every complex-valued function* φ *defined on* K *such that* $\varphi(0)=0$, φ *is bounded on every bounded subset of* K *and that* $\limsup_{w\to 0}|\varphi(w)/w|<\infty$ *operates on* A.

(ii) *The complex-valued function* φ *defined on* K *operates on* A *if and only if* φ *satisfies that* $\varphi(0)=0$, φ *is bounded on every bounded subset of* K *and that* $\limsup_{w\to 0}|\varphi(w)/w|<\infty$.

PROOF. Case (1). Obviously, (iii) implies (ii) and (ii) implies (i). We show that (i) implies (iv), (iv) implies (ii) and (ii) implies (iii). Suppose that (i) is satisfied. Suppose that $f\in A$ and $g\in C_0(\boldsymbol{Z})$ satisfy the inequality $|g|\leq|f|$ on $\boldsymbol{Z}$. Without loss of generality we may assume that $\overline{f(\boldsymbol{Z})}\subset\operatorname{int}K$. Since $f\in C_0(\boldsymbol{Z})$ the set $E=\{z\in\boldsymbol{Z}: f(z)\neq 0\}$ is a countable subset of $\boldsymbol{Z}$. In the same way as in the proof of Lemma 4 we see that there exists $u\in A$ such that $(f+u)(\boldsymbol{Z})\subset K$, $u=0$ on $\boldsymbol{Z}\setminus E$, $\arg u(z)=\arg f(z)$ for every $z\in E$ and that $\{z\in\boldsymbol{Z}: f(z)+u(z)=f(z_0)+u(z_0)\}=\{z_0\}$ for every $z_0\in E$, where arg denotes the argument of complex numbers. Choose a *continuous* function φ on K such that

$$\varphi(w)=\begin{cases} g(z), & w=(f+u)(z) \quad \text{for some} \quad z\in E,\\ 0, & w=0.\end{cases}$$

Note that such a function φ exists since 0 is the only accumulation point of $(f+u)(E)$. Then φ is bounded and $|\varphi(w)|\leq|w|$ on K. Thus we have $\varphi\circ(f+u)\in A$ and $\varphi\circ(f+u)=g$ on $\boldsymbol{Z}$. We conclude that (iv) holds.

Suppose that (iv) holds. Suppose that a function φ on K such that $\varphi(0)=0$ satisfies the inequality $\limsup_{w\to 0}|\varphi(w)/w|<\infty$. Let f be a function in A such that $f(\boldsymbol{Z})\subset K$. Since $f(\boldsymbol{Z})$ is a bounded set in $\boldsymbol{C}$ there is a positive constant L such that $|\varphi(w)|\leq L|w|$ for every $w\in f(\boldsymbol{Z})$. Thus we see that $(\varphi\circ f)/L\in C_0(\boldsymbol{Z})$ and $|(\varphi\circ f)/L|\leq|f|$ on $\boldsymbol{Z}$. It follows that $(\varphi\circ f)/L$ is in A, hence $\varphi\circ f\in A$. We conclude that φ operates on A.

Suppose that (ii) holds. Let Ω_1 be the set of all complex-valued functions defined on K which operate on A and Ω_2 the set of all complex-valued functions φ defined on K such that $\varphi(0)=0$ and $\limsup_{w\to 0}|\varphi(w)/w|<\infty$. By (ii) $\Omega_1\supset\Omega_2$ is trivial. Suppose that $\varphi\in\Omega_1$. Then $\varphi(0)=0$ since $1\notin A$. We see by Lemma 4 and Proposition 2 that $\limsup_{w\to 0}|\varphi(w)/w|<\infty$. Thus $f\in\Omega_2$. It follows that $\Omega_1=\Omega_2$. Thus (iii) holds.

Case (2). (ii) clearly implies (i). We show that (i) implies (ii). Let Ω_1 be the set of all complex-valued functions defined on K which operate on A and Ω_2 the set of all complex-valued functions on K such that $\varphi(0)=0$, φ is bounded on any bounded subset of K and that $\limsup_{w\to 0}|\varphi(w)/w|<\infty$. By (i) the inclusion $\Omega_1\supset\Omega_2$ is trivial. We show that $\Omega_1\subset\Omega_2$.

First we show that there is $f\in A$ such that $f^2=f$ and $\{z\in \boldsymbol{Z}: f(z)=1\}$ is an infinite countable subset of $\boldsymbol{Z}$. Since $A\not\subset C_0(\boldsymbol{Z})$ there are $g\in A$ and a countable subset $\{z_n\}$ of $\boldsymbol{Z}$ and $p\in \boldsymbol{C}\setminus\{0\}$ such that $g(z_n)\to p$ as $n\to\infty$. We may assume that $\overline{g(\boldsymbol{Z})}\subset \operatorname{int} K$ and there is a positive ε such that $g(\{z_n\})\subset\{w\in \boldsymbol{C}: |w-p|\le\varepsilon\}\subset \operatorname{int} K\setminus\{0\}$. Define a function φ on K by

$$\varphi(w)=\begin{cases}1, & |w-p|\le\varepsilon,\\ 0, & w\in K\setminus\{w\in \boldsymbol{C}: |w-p|\le\varepsilon\}.\end{cases}$$

Then we see that $\varphi(0)=0$, φ is bounded on K and $\limsup_{w\to 0}|\varphi(w)/w|=0$. It follows by (i) that φ operates on A. In particular, $\varphi\circ g\in A$. We see that $(\varphi\circ g)^2=\varphi\circ g$ on $\boldsymbol{Z}$ and $\{z\in \boldsymbol{Z}: \varphi\circ g(z)=1\}\supset\{z_n\}$. We define a sequence $\{a_n\}$ of positive numbers by induction as follows: Put $a_1=1$. Suppose that $a_1,\dots,a_n$ have been chosen. Choose a_{n+1} so that $0<a_{n+1}<1/(2^{n+1}\|\chi_{\{z_n\}}\|_A)$. Then $\sum_{n=1}^{\infty}a_n\chi_{\{z_n\}}$ converges in A. Choose a positive δ such that $\{w\in \boldsymbol{C}: |w|\le 2\delta\}\subset \operatorname{int} K$ and put $h=(\varphi\circ g+\sum_{n=1}^{\infty}a_n\chi_{\{z_n\}})\delta$. Then $h\in A$ and $\overline{h(\boldsymbol{Z})}\subset \operatorname{int} K$ since $\|\varphi\circ g\|_{\infty(\boldsymbol{Z})}=1$ and $\|\sum_{n=1}^{\infty}a_n\chi_{\{z_n\}}\|_{\infty(\boldsymbol{Z})}=1$. Define a function ϕ on K by

$$\phi(w)=\begin{cases}1, & w\in\{(1+a_n)\delta\},\\ 0, & \text{otherwise}.\end{cases}$$

Then $\phi(0)=0$, ϕ is bounded on K and $\limsup_{w\to 0}|\phi(w)/w|=0$, hence ϕ operates on A. In particular, $\phi\circ h\in A$. We see that $(\phi\circ h)^2=\phi\circ h$ and $\{z\in \boldsymbol{Z}: \phi\circ h(z)=1\}=\{z_n\}$.

Let φ be a function in Ω_1. Then by Lemma 4 and Proposition 2 we see that $\limsup_{w\to 0}|\varphi(w)/w|<\infty$. Since $1\notin A$ we have $\varphi(0)=0$. We show that φ is bounded on any bounded set. Suppose not. Then there exist $w_0\in \boldsymbol{C}$ and a sequence $\{w_n\}$ in K such that $|w_n-w_0|\le 1/\|\chi_{\{z_n\}}\|_A 2^n$ and $|\varphi(w_n)|\to\infty$ as $n\to\infty$. Put

$$g=w_0\phi\circ h+\sum_{n=1}^{\infty}(w_n-w_0)\chi_{\{z_n\}}.$$

Then $g\in A$ and $g(\boldsymbol{Z})=\{0\}\cup\{w_n\}\subset K$, hence $\varphi\circ g\in A$. On the other hand, $|\varphi\circ g(z_n)|=|\varphi(w_n)|\to\infty$ as $n\to\infty$, which is a contradiction since $A\subset C^b(\boldsymbol{Z})$. Thus φ is bounded on any bounded subset of K. It follows that $\varphi\in\Omega_2$, thus $\Omega_1=\Omega_2$, and (iii) holds. □

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