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THE SET OF SOLUTIONS FOR CERTAIN SEMILINEAR HEAT EQUATIONS

Dedicated to Professor Takeshi Kotake on his sixtieth birthday

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Abstract. We show that bounded solutions of an initial(-boundary) value problem for certain semilinear heat equations always come from the corresponding ordinary differential equations. As a consequence we immediately get a theorem of Kneser's type which was showed by Ballotti and Kikuchi in different methods.

1. Introduction. Kneser's theorem [5] is a famous result on ordinary differential equations concerning the structure of the set of all solutions for an initial value problem. Recently Ballotti [1] and Kikuchi [4] established a theorem of Kneser's type for the initial(-boundary) value problem of a semilinear heat equation

$$\begin{cases} u_t = \Delta u + \sqrt{u} & \text{in} \quad \Omega \times (0, T)\,, \\ u|_{t=0} = 0 & \text{in} \quad \Omega \times \{0\}\,, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0, T) \quad \text{if } \partial\Omega \neq \emptyset\,. \end{cases}$$

Here Ω is a bounded domain with smooth boundary, or $\boldsymbol{R}^n$. When $\Omega \neq \boldsymbol{R}^n$, ν denotes a unit outer normal vector of $\partial\Omega$. Let $L^p(\Omega)$ be the usual Lebesgue space, and $BC(\bar{\Omega})$ the set of bounded continuous functions on $\bar{\Omega}$. Their result is as follows:

THEOREM 1.1 (Kneser's theorem, cf. [1], [4]). *Let $X = L^p(\Omega)$ $(1 < p < \infty)$ when $\Omega \neq \boldsymbol{R}^n$, and $X = BC(\boldsymbol{R}^n)$ when $\Omega = \boldsymbol{R}^n$. Then the set of (mild) solutions in $C([0, T]; X)$ is compact and connected in the class. Hence the cross-section of (mild) solutions in $C([0, T]; X)$ is compact and connected in X.*

Their proofs are based on arguments on evolution equations or partial differential equations. The theorem for partial differential equations, however, easily follows from the corresponding theorem for ordinary differential equations, if we can prove that all solutions belonging to $C([0, T]; X)$ are independent of the space variables. In this article we show that it is in fact the case for $X = BC(\bar{\Omega})$, which is the same setting as in [4] when $\Omega = \boldsymbol{R}^n$. Our method is applicable to the following problem:

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(1.1) \qquad \begin{cases} u_t = \Delta u + f(u) & \text{in} \quad \Omega \times (0, T_{\max}) , \\ u|_{t=0} = 0 & \text{in} \quad \Omega \times \{0\} , \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0, T_{\max}) \quad \text{if } \partial\Omega \neq \varnothing . \end{cases}
$$

Here $T_{\max} = T_{\max}(u)$ is the life span of u, i.e., $(0, T_{\max})$ is the maximal interval of existence for u. We assume that f is a continuous function on $[0, \infty)$ satisfying

$$
(1.2) \qquad f(0) = 0 ,
$$

$$
(1.3) \qquad f(u) > 0 \quad \text{for} \quad u > 0 ,
$$

$$
(1.4) \qquad g(u) = \int_0^u \frac{dv}{f(v)} < \infty ,
$$

$$
(1.5) \qquad f(u) \text{ is non-decreasing} .
$$

A typical example is $f(u) = u^\alpha$ $(0 < \alpha < 1)$. Let

$$
T_\infty = \int_0^\infty \frac{dv}{f(v)} \in (0, \infty] .
$$

It is easy to see that the function g has the non-decreasing inverse function g^{-1} defined on $[0, T_\infty)$. We define $g^{-1}(t) \equiv 0$ for $t < 0$, and then $g^{-1} \in C^1(-\infty, T_\infty)$.

Our theorem is as follows:

THEOREM 1.2. *Any solution to* (1.1) *which is a $BC(\bar{\Omega})$-valued continuous function comes from the initial value problem of the ordinary differential equation*

$$
\begin{cases} \dfrac{du}{dt} = f(u) & \text{on} \quad (0, T_{\max}) , \\ u|_{t=0} = 0 . \end{cases}
$$

In other words, there exists $\tau \in [0, \infty]$ such that

$$
u(x, t) = g^{-1}(t - \tau) ,
$$

where we interpret $u \equiv 0$ when $\tau = \infty$.

The theorem of Kneser's type immediately follows from this results.

Our result does not exclude that of Ballotti or Kikuchi. Indeed, Ballotti's result is merely an example of his main result for a more general setting. Moreover, the author does not know whether any mild solution in $C([0, T]; L^p(\Omega))$ is independent of the space variables. Kikuchi's method is possibly applicable to more general $f(u)$ with or without modification. Actually he said that "for a continuous function $f(u)$ we can similarly treat the general equation $\cdots$ with the suitable initial and boundary conditions", though he did not mention the assumption precisely.

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2. The proof. Let $U(x, y, t)$ be the fundamental solution to the initial(-boundary) value problem of the heat equation (with the Neumann boundary condition when $\partial\Omega \neq \emptyset$). The properties

$$U(x, y, t) > 0 \quad \text{for} \quad (x, y, t) \in \bar{\Omega} \times \bar{\Omega} \times (0, \infty)\,, \tag{2.1}$$

$$\int_{\Omega} U(x, y, t)dy \equiv 1 \tag{2.2}$$

are well-known. A solution $u \in C([0, T]; BC(\bar{\Omega}))$ has an expression

$$u(x, t) = \int_0^t \int_{\Omega} U(x, y, t-\tau) f(u(y, \tau)) dy d\tau \qquad \text{for} \quad (x, t) \in \bar{\Omega} \times (0, T) \tag{2.3}$$

(see [2, Chapter 1] and [3, Chapter 2, §§8–9]).

LEMMA 2.1. *We have*

$$0 \leq u(x, t) \leq g^{-1}(t)\,.$$

PROOF. It is convenient to define $f(u) \equiv 0$ for $u < 0$. Since f is non-negative, the lower estimate comes from (2.1) and (2.3). The upper one is derived from (2.2) and the Bihari inequality [6, Theorem 1.9.2], [7, Lemma 1.5]. More precisely, see [4, Lemma 1], where the assertion is shown for $f(u) = \sqrt{u}$. The proof is similar in the general case. □

We define a subset P of $[0, T_{\max})$ and a non-negative number t_0 by

$$\begin{cases} P = \{t \in [0, T_{\max}) \,|\, u(x, t) > 0 \text{ for some } x \in \bar{\Omega}\}\,, \\ t_0 = \inf P\,. \end{cases}$$

They are well-defined unless $u \equiv 0$. In what follows we assume $u \not\equiv 0$. By the continuity of u, we have $u(x, t_0) \equiv 0$ as a function of x.

LEMMA 2.2. *Let $t_1 \in P$. Then $u(x, t) > 0$ for all $x \in \bar{\Omega}$ and all $t > t_1$.*

PROOF. Since $f(u(x, t_1)) > 0$ for some $x \in \bar{\Omega}$, the assertion is an easy consequence of (2.1) and (2.3). □

COROLLARY. *We have*

$$\begin{cases} u(x, t) \equiv 0 & \textit{for} \quad 0 \leq t \leq t_0\,, \\ u(x, t) > 0 & \textit{for} \quad t \geq t_0\,. \end{cases}$$

By virtue of this fact one can reduce the proof of Theorem 1.2 to the next theorem.

THEOREM 2.1. *$u(x, t)=g^{-1}(t)$ is a unique solution of*

$$(2.4)\qquad \begin{cases} u_t=\Delta u+f(u) & \text{in } \Omega\times(0, T_{\max}), \\ u>0 & \text{on } \bar{\Omega}\times(0, T_{\max}), \\ u|_{t=0}=0 & \text{in } \Omega\times\{0\}, \\ \dfrac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega\times(0, T_{\max}) \text{ if } \partial\Omega\neq\varnothing, \end{cases}$$

and $T_{\max}=T_\infty$.

To show this we need to prove the following technical lemma:

LEMMA 2.3. *Let f be a function satisfying* (1.2)–(1.5), *and define $\{f_k\}_{k\geq 2}$ by*

$$f_k(u)=\begin{cases} 0 & \text{for } u=0, \\ \dfrac{k}{u}\displaystyle\int_{(1-1/k)u}^{u} f(v)dv & \text{for } u>0. \end{cases}$$

Then $f_k\in C[0, \infty)\cap C^1(0, \infty)$ and satisfies

$$(2.5)\qquad f_k(u)>0 \quad \text{for } u>0,$$

$$(2.6)\qquad f_k(u)\leq f(u) \quad \text{and} \quad \lim_{k\to\infty} f_k(u)=f(u) \quad \text{for } u\geq 0,$$

$$(2.7)\qquad g_k(u)=\int_0^u \frac{dv}{f_k(v)}<\infty \quad \text{and} \quad \lim_{k\to\infty} g_k(u)=g(u) \quad \text{for } u\geq 0,$$

$$(2.8)\qquad f_k'(u)\geq 0 \quad \text{for } u>0.$$

PROOF. It is easy to see that $f_k\in C^1(0, \infty)$. First we show its continuity at $u=0$. It follows from (1.3) and (1.5) that for $u>0$

$$0<f_k(u)\leq\frac{k}{u}\int_{(1-1/k)u}^{u} f(u)dv=f(u)\to 0 \qquad \text{as} \quad u\downarrow 0.$$

Hence $f_k\in C[0, \infty)\cap C^1(0, \infty)$, (2.5) and the former part of (2.6) are proved. Since $f_k(u)$ is the mean of f on the interval $[(1-1/k)u, u]$ and since f is continuous, we have the latter part of (2.6). (1.5) gives for $u>0$

$$f_k(u)\geq\frac{k}{u}\int_{(1-1/k)u}^{u} f\left(\left(1-\frac{1}{k}\right)u\right)dv=f\left(\left(1-\frac{1}{k}\right)u\right)\geq f\left(\frac{u}{2}\right).$$

Combining this with (1.4) and (2.5), we get

$$0<\frac{1}{f_k(u)}\leq\frac{1}{f\left(\dfrac{u}{2}\right)}\in L^1_{\mathrm{loc}}[0, \infty).$$

Hence $g_k(u)<\infty$. Moreover, taking (2.6) into consideration we obtain $\lim_{k\to\infty} g_k(u)=g(u)$ by the dominated convergence theorem. Thus (2.7) is shown. It remains to prove (2.8). We have already proved the differentiability of f_k on $(0, \infty)$. To deduce the proof from its non-decreasing property, we rewrite the expression for f_k as

$$f_k(u)=k\int_{1-1/k}^{1} f(uw)dw .$$

We easily see from this and (1.5) that f_k is non-decreasing. $\square$

PROOF OF THEOREM 2.1. From the first equation of (2.4) we get

$$u_t\geq \Delta u+f_k(u),$$

where f_k is the function in Lemma 2.3. We divide both sides of this and the fourth equation of (2.4) by $f_k(u)>0$. The division is possible because of the second condition $u>0$ of (2.4). The result is

$$\begin{cases} g_k(u)_t\geq \Delta g_k(u)+\dfrac{f'_k(u)}{f_k^2(u)}|\nabla u|^2+1\geq \Delta g_k(u)+1 & \text{in} \quad \Omega\times(0, T_{\max}), \\ g_k(u)|_{t=0}=0 & \text{in} \quad \Omega\times\{0\}, \\ \dfrac{\partial g_k(u)}{\partial \nu}=0 & \text{on} \quad \partial\Omega\times(0, T_{\max}) \quad \text{if } \partial\Omega\neq\emptyset . \end{cases}$$

It follows from the comparison theorem that $g_k(u)\geq t$. Passing to the limit as $k\to\infty$, we get $g(u)\geq t$. Taking Lemma 2.1 into consideration, we find $u(x, t)\equiv g^{-1}(t)$. $\square$

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