Tôhoku Math. J.
47 (1995), 55–80

POSITIVE SOLUTIONS WITH WEAK ISOLATED SINGULARITIES TO SOME SEMILINEAR ELLIPTIC EQUATIONS

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(Received October 5, 1993)

Abstract. We are concerned with the existence of positive solutions with prescribed weak isolated singularities to some semilinear elliptic equations. The existence property differs with the behavior of the nonlinear term. Under the positivity assumption and a growth condition on the nonlinear term, we obtain not only solutions with a finite number of singularities but also those with infinitely many singularities. We show also that, for some nonlinear terms which changes sign, there is no solution with prescribed singular behavior.

1. Introduction. In this paper we are concerned with the problem of finding solutions with isolated singularities to some semilinear elliptic partial differential equations. Choosing a finite set of points $\{a_j\}_{j=1}^m$ or a sequence $\{a_j\}_{j=1}^\infty$ without accumulation points in $\boldsymbol{R}^N$ and a bounded set of positive numbers $\{\kappa_j\}_{j=1}^m$ or $\{\kappa_j\}_{j=1}^\infty$ arbitrarily, we consider the following problems:

$$(\mathrm{P}_m)\qquad \begin{cases} -\Delta u+f(u)=0 \quad \text{and} \quad u>0 \quad \text{in } \boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m\,, \\ u(x)\sim\kappa_j E(x-a_j) \quad \text{as } x\to a_j \quad \text{for } j=1,2,\dots,m\,, \\ u(x)\to 0 \quad \text{as } |x|\to+\infty\,, \end{cases}$$

and

$$(\mathrm{P}_\infty)\qquad \begin{cases} -\Delta u+f(u)=0 \quad \text{and} \quad u>0 \quad \text{in } \boldsymbol{R}^N\setminus\{a_j\}_{j=1}^\infty\,, \\ u(x)\sim\kappa_j E(x-a_j) \quad \text{as } x\to a_j \quad \text{for } j=1,2,\dots\,. \end{cases}$$

Here $\Delta:=\sum_{i=1}^N(\partial/\partial x_i)^2$ is the Laplacian in $\boldsymbol{R}^N$ with $N\geq 2$ and E is the fundamental solution for $-\Delta$ in $\boldsymbol{R}^N$, that is,

$$(1.1)\qquad E(x)=E(|x|):=\begin{cases} \dfrac{1}{(N-2)N\omega_N}\dfrac{1}{|x|^{N-2}} & \text{if } N\geq 3 \\ \dfrac{1}{2\pi}\log\dfrac{1}{|x|} & \text{if } N=2 \end{cases} \qquad \text{for } x\in\boldsymbol{R}^N\setminus\{0\}\,,$$

where ω_N denotes the volume of the unit ball in $\boldsymbol{R}^N$. We assume that $f:[0,+\infty)\to\boldsymbol{R}$ is (locally) Lipschitz continuous and $f(0)=0$. We call the number κ_j the *intensity* of

1991 *Mathematics Subject Classification*. Primary 35J60.

singularity at a_j. When $m=1$, we assume $a_1=0$ and denote κ_1 simply by κ.

When we call u a solution to (P_m) or (P_∞), we assume that $u\in C^2(\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m)$ or $u\in C^2(\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^\infty)$ and u satisfies (P_m) or (P_∞), respectively. Under a suitable growth condition at infinity on f, if u is a solution to (P_m) or (P_∞), then u satisfies the following equation in the sense of distribution:

$$-\Delta u+f(u)=\sum_{j=1}^{m}\kappa_j\delta_{a_j} \quad \text{or} \quad \sum_{j=1}^{\infty}\kappa_j\delta_{a_j} \qquad \text{in} \quad \mathscr{D}'(\boldsymbol{R}^N)\,, \tag{1.2}$$

where δ_a denotes the Dirac delta function supported at a (see Lemma 2).

In general, isolated singularities do not necessarily have the order of the fundamental solution, even if they are isotropic (i.e., they have the same sign near the singularities). For example, in the case where

$$f(s)=s^p\,, \qquad p>1\,, \tag{1.3}$$

Véron [16] and Brézis-Véron [3] classified positive isolated singularities as follows (we agree that $N/(N-2)=+\infty$ if $N=2$ throughout the paper):

(i) If $N\geq 2$ and $1<p<N/(N-2)$, then a positive isolated singularity at the origin is either removable or satisfies one of the following:

$$u(x)\sim l_{p,N}|x|^{-2/(p-1)} \qquad \text{as} \quad x\to 0\,, \tag{1.4}$$

$$u(x)\sim\kappa E(x) \qquad \text{as} \quad x\to 0\,, \tag{1.5}$$

where $l_{p,N}:=[2(2p-N(p-1))/(p-1)^2]^{1/(p-1)}$ and κ is a positive constant (for each $\kappa>0$ there is a solution of type (1.5)).

(ii) If $N\geq 3$ and $p\geq N/(N-2)$, then any positive isolated singularity is removable. A singularity of type (1.4) is called a *strong singularity*, while that of type (1.5) is called a *weak singularity* (see also Vazquez-Véron [14, 15]).

The nonlinearity in which we are interested in this paper includes the following cases:

$$f(s)=1-e^{-s}\,, \tag{1.6}$$

$$f(s)=e^{-s}(1-e^{-s})\,, \tag{1.7}$$

$$f(s)=\frac{c+1}{c-1}e^{-s}-\frac{2}{c-1}e^{-cs}-1\,, \qquad c>0\,, \quad c\neq 1\,. \tag{1.8}$$

Problem (P_m) with f defined by (1.7) and $N=2$ appears in relativistic Chern-Simons gauge theories (see [7]). In the case (1.8), (P_1) is related to a problem of chemical process (see [5, § 16]). Note that in the cases (1.6)–(1.8) above, f satisfies

$$|f(s)|=O(s) \qquad \text{as} \quad s\to+\infty\,. \tag{1.9}$$

It is known that under the assumption (1.9) each positive isolated singularity is weak (see e.g. Serrin [10, Theorems 1 and 3]).

In the case (1.6), (P_m) with $N=2$ appears in nonrelativistic Chern-Simons gauge theories as an equation of stationary topological vortex soliton, in which $\kappa_j/(4\pi)$ is a positive integer for $j=1, 2, \ldots, m$ (see [4]). Construction of m-vortex solutions to the first order Ginzburg-Landau equation reduces also to the same equation, and Taubes [12] proved the existence and uniqueness of solutions via a variational method and studied their properties. More generally, in the case where f is a maximal monotone graph with $0 \in f(0)$ whose domain is $\boldsymbol{R}$, Bénilan-Brézis [1] and Vazquez [13] studied a necessary and sufficient condition for the existence of a solution (in an appropriate sense) to $-\Delta u + f(u) \ni \nu$ for a bounded Radon measure ν on $\boldsymbol{R}^N$. Their results are based on the case where $\nu \in L^1(\boldsymbol{R}^N)$ (see [2]) and a regularization method.

The purpose of this paper is to construct a solution to (P_m) or (P_∞) when f is not necessarily nondecreasing. In particular, for certain nonlinearities f including (1.6) we shall obtain a solution to (P_∞) where the measure $\sum_{j=1}^{\infty} \kappa_j \delta_{a_j}$ in (1.2) is not necessarily a bounded Radon measure (see Theorem 2). The only assumption on the sequence of points $\{a_j\}_{j=1}^{\infty}$ in $\boldsymbol{R}^N$ is that $\inf_{j \neq j'} |a_j - a_{j'}| > 0$ and no symmetry condition is required, though we impose some restriction on the intensity $\{\kappa_j\}_{j=1}^{\infty}$.

When f is nonnegative (and satisfies some technical assumptions), we can construct a solution to (P_m) for any intensity $\{\kappa_j\}_{j=1}^{m}$ (see Theorems 1 and 1′). However, the situation is different when f changes sign. Roughly speaking, when f is positive near the origin and tends to $-\infty$ as $s \to +\infty$ with appropriate order, the existence of a solution to (P_1) depends on the intensity of singularity (see Theorems 3 and 4).

The paper is organized as follows. Precise statements of our results are given in Section 2. In Sections 3 and 4, we prove the basic facts (Lemma 2 and Proposition 1) which will be used repeatedly throughout the paper, and we give the proofs of Theorems 1 and 1′. Problem (P_∞) is considered in Section 5 and Theorem 2 will be proved. Finally, we deal with the case where f changes sign and prove the existence and nonexistence results depending on the intensity of singularity in Sections 6 and 7, respectively.

The author thanks Professors Takeshi Kotake and Izumi Takagi for their helpful suggestions and advice. Thanks are also due to Professor Jun-ichi Ezawa and Mr. Masahiro Hotta for their bringing the author's attention to the problem of topological vortex solitons.

2. Statements of results. Throughout the paper, we assume that $f : [0, +\infty) \to \boldsymbol{R}$ is a locally Lipschitz continuous function which satisfies

$$\text{(F)} \qquad f(0) = 0 \quad \text{and} \quad f(s) - f(s') \leq \mu^2 (s - s') \qquad \text{for} \quad s \geq s' \geq 0$$

with a positive constant μ. Hence f is bounded from above by a linear function. Note that (F) is satisfied in the cases (1.6)–(1.8), but not in the case (1.3).

We begin with the case where f is nonnegative. First we state the existence of solutions with a finite number of singular points. Roughly speaking, under additional technical assumptions on f, the positivity of f implies the existence of solutions to (P_m) for arbitrary $\{a_j\}_{j=1}^m$ and $\{\kappa_j\}_{j=1}^m$.

THEOREM 1. *Let $N \geq 3$ and suppose that f satisfies, in addition to* (F),

$$(F_1) \qquad f(s) \geq 0 \quad \text{for } s \geq 0 .$$

Then for any $\{a_j\}_{j=1}^m \subset \boldsymbol{R}^N$ and $\{\kappa_j\}_{j=1}^m \subset \boldsymbol{R}_+ = (0, +\infty)$, (P_m) has a solution $u \in C^2(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^m)$.

When $N=2$, in order to construct a solution to (P_m) for arbitrary $\{a_j\}_{j=1}^m$ and $\{\kappa_j\}_{j=1}^m$, we need a slightly stronger condition on f.

THEOREM 1′. *Let $N=2$ and suppose that f satisfies* (F).

(i) *If $m=1$ and*

$$(F_1') \qquad f(s) \geq 0 \quad \text{for } s \geq 0 \quad \text{and} \quad f(s) > 0 \quad \text{for } 0 < s \ll 1 ,$$

then for any $\kappa > 0$, (P_1) has a radial solution $u \in C^2(\boldsymbol{R}^2 \setminus \{0\})$.

(ii) *If $m \geq 2$ and*

$$(F_1'') \qquad f(s) > 0 \quad \text{for } s > 0 \quad \text{and} \quad \liminf_{s \to +\infty} f(s) > 0 ,$$

then for any $\{a_j\}_{j=1}^m \subset \boldsymbol{R}^2$ and $\{\kappa_j\}_{j=1}^m \subset \boldsymbol{R}_+$, (P_m) has a solution $u \in C^2(\boldsymbol{R}^2 \setminus \{a_j\}_{j=1}^m)$.

The nonlinear term (1.6) satisfies all the assumptions of Theorems 1 and 1′. On the other hand, (1.7) satisfies the assumptions of Theorem 1 and (F_1') of Theorem 1′, but not (F_1''). Thus for f defined by (1.7), the problem of constructing solutions with a plural number of singularities of arbitrary intensity remains open, when $N=2$.

Next we state the existence of solutions with infinitely many singularities. We assume that f is positive, and the only assumption on $\{a_j\}_{j=1}^\infty$ is that the distance between any two points is uniformly bounded away from zero.

THEOREM 2. *Let $N \geq 2$ and suppose that f satisfies, in addition to* (F),

$$(F_2) \quad f(s) > 0 \quad \text{for } s > 0 , \quad \liminf_{s \to 0} f(s)/s > 0 \quad \text{and} \quad \gamma := \liminf_{s \to +\infty} f(s) \in (0, +\infty] .$$

Assume that $\{a_j\}_{j=1}^\infty \subset \boldsymbol{R}^N$ and $\{\kappa_j\}_{j=1}^\infty \subset \boldsymbol{R}_+$ satisfy

$$(A) \qquad |a_j - a_{j'}| \geq \alpha \quad \text{if } j \neq j' \quad \text{and} \quad \tilde{\kappa} := \limsup_{j \to \infty} \kappa_j \in [0, +\infty)$$

for some positive constant α. Then (P_∞) has a solution $u \in C^2(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^\infty)$ provided that $C_N \tilde{\kappa}/\alpha^N < \gamma$, where C_N is a positive constant depending only on N.

By Theorems 1 and 1′, (P_m) has a solution for any intensity if f is positive and satisfies certain technical assumptions. Then it is natural to ask what happens when f

is not necessarily positive. In [5, §16], Gel'fand showed that, in the case where $N=1$ and f is defined by (1.8), there is a $\kappa_*>0$ such that a problem corresponding to (P_1) has a solution if $0<\kappa\leq\kappa_*$, while there is no solution if $\kappa>\kappa_*$. The following two theorems are motivated by this observation. We assume a growth condition at infinity on f:

$$(F_0) \qquad |f(s)|=O(s^p) \quad \text{as} \quad s\to+\infty,$$

where $(1<)p<N/(N-2)$. Note that (F_0) is satisfied in cases of Theorems 1, 1′ and 2.

THEOREM 3. *Let $N\geq 2$ and suppose that f satisfies* (F), (F_0) *and*

$$(F_3) \qquad \begin{cases} f(s)\geq 0 \quad \text{for } 0\leq s\ll 1 & \text{if } N\geq 3, \\ \liminf_{s\to 0} f(s)/s>0 & \text{if } N=2. \end{cases}$$

Then, for any $\{a_j\}_{j=1}^m\subset \boldsymbol{R}^N$, (P_m) has a solution $u\in C^2(\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m)$ provided that $\kappa_j>0$ is small for each $j=1,2,\dots,m$.

While the existence result above is valid for f defined by (1.8), the following non-existence theorem does not cover the case (1.8).

THEOREM 4. *Let $N\geq 2$ and suppose that f satisfies* (F), (F_0) *and*

$$(F_4) \qquad \limsup_{s\to+\infty} f(s)/s<0.$$

Then (P_1) has no solution in $C^2(\boldsymbol{R}^N\setminus\{0\})$ if $\kappa>0$ is large.

To prove the preceding theorems we shall make use of Proposition 1 below which is obtained by way of the well-known *monotone iteration schemes* (cf. [9]). Given $\{a_j\}_{j=1}^m$ and $\{\kappa_j\}_{j=1}^m$, we set

$$(2.1) \qquad h(x):=\sum_{j=1}^m \kappa_j E_\mu(x-a_j) \quad \text{for} \quad x\in\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m,$$

where $\mu>0$ is given by (F) and E_μ is the fundamental solution for $-\Delta+\mu^2$ in $\boldsymbol{R}^N$ (see (4.1) for the definition of E_μ).

PROPOSITION 1. *Let $N\geq 2$ and suppose that f satisfies* (F) *and* (F_0). *Assume that there exists a pair of functions $(\tilde{u},\beta)$ with $\tilde{u}\in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)\cap C(\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m)$ and $\beta\in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ satisfying*

$$(\tilde{\mathrm{P}}_m)\qquad \begin{cases} -\Delta\tilde{u}+f(\tilde{u})=\beta+\sum_{j=1}^{m}\kappa_j\delta_{a_j} \quad \textit{in } \mathscr{D}'(\boldsymbol{R}^N)\,, \\ \tilde{u}>0 \quad \textit{in } \boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m \quad \textit{and} \quad \beta\geq 0 \quad \textit{a.e. in } \boldsymbol{R}^N\,, \\ \tilde{u}(x)\sim\kappa_j E(x-a_j) \quad \textit{as } x\to a_j \quad \textit{for } j=1,2,\dots,m\,, \\ \tilde{u}(x)\to 0 \quad \textit{and} \quad \beta(x)\to 0 \quad \textit{as } |x|\to+\infty\,. \end{cases}$$

Then we have the following (i) *and* (ii):

(i) $$h\leq\tilde{u} \quad \textit{in} \quad \boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m\,,$$

(ii) *Problem* (P_m) *has a solution* $u\in C^2(\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m)$ *satisfying*

$$h\leq u\leq\tilde{u} \quad \textit{in} \quad \boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m\,. \tag{2.2}$$

Since $-\Delta\tilde{u}+f(\tilde{u})\geq\sum_{j=1}^m\kappa_j\delta_{a_j}$ in the sense of distribution, we call $\tilde{u}$ a *supersolution* to (P_m). On the other hand, h satisfies $-\Delta h+f(h)\leq\sum_{j=1}^m\kappa_j\delta_{a_j}$ and hence is called a *subsolution* to (P_m).

REMARK 1. If the problem (P_m) for a nonlinearity $\tilde{f}$ has a solution $\tilde{u}$, then $\tilde{u}$ is a supersolution to (P_m) for any nonlinearity f satisfying $\tilde{f}\leq f$ as well as (F) and (F_0). To see this it is sufficient to take $\beta=f(\tilde{u})-\tilde{f}(\tilde{u})$.

In view of Proposition 1, it is essential to find a supersolution in order to construct a solution to (P_m). In the case of Theorems 1 and 1′, finding a supersolution is relatively easy. To prove Theorem 2 we shall show that for each $m=1,2,\dots$, (P_m) has a solution u_m which is bounded from above by a function independent of m, and then obtain a solution u to (P_∞) as the limit of u_m as $m\to\infty$. We shall construct an upper bound by using the multiple convolutions of the fundamental solutions for $-\Delta+\mu^2$ ($\mu>0$). To prove Theorem 3, we first observed that (P_1) has a solution with small intensity for some special nonlinearities, which is shown by elementary calculation and Proposition 1. By superimposing translations of this particular solution we can construct a supersolution for the general case. Similarly, in view of Remark 1, the proof of Theorem 4 is reduced to showing that the conclusion holds true when f is a piecewise linear function. In this special case, all the radial solutions to the differential equation near the singularity are given in terms of the Bessel functions and the modified Bessel functions.

3. Preliminary observation. We begin by showing that any solution to (P_m) or (P_∞) satisfies the equation (1.2) in the sense of distribution. First we introduce a few notations. For $r>0$ and $x\in\boldsymbol{R}^N$, $B_r(x)$ denotes the open ball of radius r centered at x, and B_r stands for $B_r(0)$. The gradient of u is denoted by ∇u, and the matrix of second order derivatives of u by D^2u.

Clearly, it is sufficient to discuss the local case, and hence we consider the following situation:

$$(3.1)\qquad \begin{cases} -\Delta u+f(u)=0 & \text{in } \overline{B_R}\setminus\{0\}, \\ u(x)\sim\kappa E(x) & \text{as } x\to 0. \end{cases}$$

We claim that a solution (3.1) satisfies

$$(3.2)\qquad -\Delta u+f(u)=\kappa\delta_0 \quad \text{in } \mathscr{D}'(B_R).$$

In order to deduce (3.2), we need an estimate for ∇u.

LEMMA 1. *Let $N\geq 2$ and suppose that f satisfies* (F_0). *If $u\in C^2(\overline{B_R}\setminus\{0\})$ solves* (3.1), *then it holds that*

$$(3.3)\qquad |\nabla u(x)|=O(|x|^{-1}E(x)) \quad \textit{as} \quad x\to 0.$$

PROOF. We use the following three inequalities:

$$(3.4)\qquad |v(x)|\leq Cr^{-N/q}(\|v\|_{L^q(B_r(x))}+r\|\nabla v\|_{L^q(B_r(x))}),$$

$$(3.5)\qquad \|\nabla v\|_{L^q(B_r(x))}\leq r\|D^2v\|_{L^q(B_{2r}(x))}+Cr^{-1}\|v\|_{L^q(B_{2r}(x))},$$

$$(3.6)\qquad \|D^2v\|_{L^q(B_r(x))}\leq C(\|\Delta v\|_{L^q(B_{2r}(x))}+r^{-2}\|v\|_{L^q(B_{2r}(x))}),$$

where $q>N$ and C is a positive constant depending only on N and q (see [6, Theorems 7.10, 9.9 and 9.11]). Substituting $v=\nabla u$ into (3.4), $v=u$ into (3.5) and (3.6), we have

$$|\nabla u(x)|\leq C'r^{1-N/q}(\|\Delta u\|_{L^q(B_{4r}(x))}+r^{-2}\|u\|_{L^q(B_{4r}(x))})$$

for some constant $C'>0$, provided that $B_{4r}(x)\Subset B_R\setminus\{0\}$ (i.e., $B_{4r}(x)$ is relatively compact in $B_R\setminus\{0\}$). We set $x=\varepsilon\omega$ with $\omega\in S^{N-1}$ and $4r=\varepsilon/2$ with small $\varepsilon>0$, where S^{N-1} is the unit sphere in $\boldsymbol{R}^N$. Then we have $B_{\varepsilon/2}(\varepsilon\omega)\Subset B_R\setminus\{0\}$ and

$$(3.7)\qquad |\nabla u(\varepsilon\omega)|\leq C''\varepsilon^{1-N/q}(\|f(u)\|_{L^q(B_{\varepsilon/2}(\varepsilon\omega))}+\varepsilon^{-2}\|u\|_{L^q(B_{\varepsilon/2}(\varepsilon\omega))})$$

for some constant $C''>0$.

When $N\geq 3$, since $|u(\varepsilon\omega)|\leq c/\varepsilon^{N-2}$ for some $c>0$, it follows from (F_0) that $|f(u(\varepsilon\omega))|\leq c'/\varepsilon^N$ for some $c'>0$. Hence.

$$(3.8)\qquad \|u\|_{L^q(B_{\varepsilon/2}(\varepsilon\omega))}\leq c''\varepsilon^{2-N/q'} \quad \text{and} \quad \|f(u)\|_{L^q(B_{\varepsilon/2}(\varepsilon\omega))}\leq c''\varepsilon^{-N/q'}$$

for some $c''>0$, where q' is the conjugate exponent of q, i.e., $1/q+1/q'=1$. By substituting (3.8) into (3.7) we obtain (3.3).

When $N=2$, since $|u(\varepsilon\omega)|\leq c\log(1/\varepsilon)$, we have $|f(u(\varepsilon\omega))|\leq c'(\log(1/\varepsilon))^p$ and

$$(3.9)\qquad \|u\|_{L^q(B_{\varepsilon/2}(\varepsilon\omega))}\leq c''\varepsilon^{2/q}\log(2/\varepsilon) \quad \text{and} \quad \|f(u)\|_{L^q(B_{\varepsilon/2}(\varepsilon\omega))}\leq c''\varepsilon^{2/q}(\log(2/\varepsilon))^p$$

for some $c', c''>0$. Therefore, (3.3) follows from (3.7) and (3.9). q.e.d.

For $v\in C(\overline{B_R}\setminus\{0\})$ and $r\in(0, R]$, we denote the average of v on rS^{N-1} by $\bar{v}(r)$, i.e.,

$$\bar{v}(r):=\frac{1}{N\omega_N}\int_{S^{N-1}} v(r\omega)d\sigma(\omega)\,, \tag{3.10}$$

where $d\sigma$ is the surface measure. The following lemma ensures that a solution to (P_m) or (P_∞) satisfies (1.2).

LEMMA 2. *Under the assumption of Lemma* 1, *if* $u\in C^2(\overline{B_R}\setminus\{0\})$ *solves* (3.1), *then* (3.2) *holds.*

PROOF. We first note that $f(u)\in L^1(B_R)$ by (3.1) and (F_0). By averaging the equation (3.1) on rS^{N-1}, we have

$$\frac{1}{r^{N-1}}\frac{d}{dr}\left[r^{N-1}\frac{d\bar{u}}{dr}\right]=\overline{f(u)} \qquad \text{for} \quad 0<r<R\,. \tag{3.11}$$

Multiplying both sides of (3.11) by r^{N-1} and integrating the resulting equation over (r, R) yield

$$R^{N-1}\frac{d\bar{u}}{dr}(R)-r^{N-1}\frac{d\bar{u}}{dr}(r)=\int_r^R \overline{f(u)}(s)s^{N-1}ds=\frac{1}{N\omega_N}\int_{B_R\setminus B_r} f(u(x))dx\,.$$

Since $f(u)\in L^1(B_R)$, there is a constant $\hat{\kappa}$ such that

$$-r^{N-1}\frac{d\bar{u}}{dr}(r)\to\hat{\kappa} \qquad \text{as} \quad r\to 0\,. \tag{3.12}$$

Now we fix $\varphi\in C_0^\infty(B_R)$ and take $\varepsilon\in(0, R)$ arbitrarily. By Green's formula we have

$$\int_{B_R\setminus B_\varepsilon}(-u\Delta\varphi+f(u)\varphi)dx=\int_{\partial B_\varepsilon}\left(u\frac{\partial\varphi}{\partial r}-\varphi\frac{\partial u}{\partial r}\right)d\sigma+\int_{B_R\setminus B_\varepsilon}(-\Delta u+f(u))\varphi dx$$

$$=\varepsilon^{N-1}\int_{S^{N-1}}\left[u(\varepsilon\omega)\frac{\partial\varphi}{\partial r}(\varepsilon\omega)-(\varphi(\varepsilon\omega)-\varphi(0))\frac{\partial u}{\partial r}(\varepsilon\omega)\right]d\sigma(\omega)-\varphi(0)N\omega_N\varepsilon^{N-1}\frac{d\bar{u}}{dr}(\varepsilon)\,.$$

Since $\sup_{\omega\in S^{N-1}}|\varphi(\varepsilon\omega)-\varphi(0)|\le c\varepsilon$ for some constant $c>0$, there holds

$$\varepsilon^{N-1}\int_{S^{N-1}}\left[u(\varepsilon\omega)\frac{\partial\varphi}{\partial r}(\varepsilon\omega)-(\varphi(\varepsilon\omega)-\varphi(0))\frac{\partial u}{\partial r}(\varepsilon\omega)\right]d\sigma(\omega)\to 0 \qquad \text{as} \quad \varepsilon\to 0$$

by virtue of Lemma 1. Therefore, we see that

$$\int_{B_R}(-u\Delta\varphi+f(u)\varphi)dx=N\omega_N\hat{\kappa}\varphi(0) \tag{3.13}$$

for any $\varphi\in C_0^\infty(B_R)$.

Now we show that $N\omega_N\hat{\kappa}=\kappa$. For any $\varepsilon>0$ we have from (3.12)

$$\hat{\kappa}-\varepsilon<-r^{N-1}\frac{d\bar{u}}{dr}(r)<\hat{\kappa}+\varepsilon \qquad \text{for} \quad 0<r<\eta$$

with some $\eta\in(0, \min\{1, R\})$. Thus it follows that

$$N\omega_N(\hat{\kappa}-\varepsilon)\left(1-\frac{E(\eta)}{E(r)}\right)\leq\frac{\bar{u}(r)-\bar{u}(\eta)}{E(r)}\leq N\omega_N(\hat{\kappa}+\varepsilon)\left(1-\frac{E(\eta)}{E(r)}\right) \qquad \text{for} \quad 0<r<\eta\,.$$

Letting $r\to 0$, we have $N\omega_N(\hat{\kappa}-\varepsilon)\leq\kappa\leq N\omega_N(\hat{\kappa}+\varepsilon)$ for any $\varepsilon>0$ so that $N\omega_N\hat{\kappa}=\kappa$. Therefore, (3.13) implies (3.2). q.e.d.

4. Solutions with a finite number of singularities. The first goal of this section is to prove Proposition 1 which is our main tool to establish all the theorems in this paper. Theorems 1 and 1′ are verified by a simple application of Proposition 1. We begin by introducing a few notations. For $N\geq 2$ and $\mu>0$, let E_μ denote the fundamental solution for $-\Delta+\mu^2$ in $\boldsymbol{R}^N$, that is,

$$(4.1) \qquad E_\mu(x)=E_\mu(|x|):=\frac{1}{2\pi}\left[\frac{\mu}{2\pi|x|}\right]^\nu K_\nu(\mu|x|) \qquad \text{for} \quad x\in\boldsymbol{R}^N\setminus\{0\}\,,$$

where $\nu=(N-2)/2$ and K_ν is the modified Bessel function of order ν (for Bessel functions, we use here and hereafter the notations in [8]). We list here some of the basic facts about E_μ:

$$(4.2) \qquad \begin{cases} -\Delta E_\mu+\mu^2 E_\mu=\delta_0 \ \text{in} \ \mathscr{D}'(\boldsymbol{R}^N)\,, \quad E_\mu(x)=\mu^{N-2}E_1(\mu x)>0 \ \text{for} \ x\in\boldsymbol{R}^N\setminus\{0\}\,, \\ E_\mu(x)\sim E(x) \ \text{as} \ x\to 0 \ \text{and} \ E_\mu(x)\sim c_\mu|x|^{-(N-1)/2}e^{-\mu|x|} \ \text{as} \ |x|\to+\infty\,, \end{cases}$$

where $c_\mu=(4\pi)^{-1}[\mu/(2\pi)]^{(N-1)/2}$. In particular, $E\in L^q_{\mathrm{loc}}(\boldsymbol{R}^N)$ and $E_\mu\in L^q(\boldsymbol{R}^N)$ for $1\leq q<N/(N-2)$.

Before starting the proof of Proposition 1 we recall fundamental facts in the theory of linear partial differential equations. We denote the class of tempered distributions by $\mathscr{S}'(\boldsymbol{R}^N)$ and follow the notations of Sobolev spaces and Hölder spaces in [6]. In the following lemma, the uniqueness assertion is verified by making use of the Fourier transform. For the proof of the second assertion, see e.g. [11, Chapter V, Theorem 3].

LEMMA 3. *Let $N\geq 2$ and $\mu>0$. Then for all $\varphi\in\mathscr{S}'(\boldsymbol{R}^N)$, the equation*

$$-\Delta v+\mu^2 v=\varphi \qquad \text{in} \quad \mathscr{D}'(\boldsymbol{R}^N)$$

has a unique solution in $\mathscr{S}'(\boldsymbol{R}^N)$, which is given by $v=E_\mu*\varphi$ (where the symbol $*$ denotes the convolution). Moreover, the mapping $\varphi\mapsto E_\mu*\varphi$ from $L^p(\boldsymbol{R}^N)$ into $W^{2,p}(\boldsymbol{R}^N)$ is continuous for $1<p<+\infty$.

PROOF OF PROPOSITION 1. We construct a solution by way of the well-known monotone iteration schemes and break up the proof into three steps.

Step 1: Reduction. From $(\tilde{P}_m)$ it follows that $\tilde{u}\in L^q_{loc}(\boldsymbol{R}^N)$ with $1<q<N/(N-2)$. Since $\tilde{u}$ and β vanish at infinity we see that $\tilde{u}$, $\beta\in\mathscr{S}'(\boldsymbol{R}^N)$. Set

$$g(s):=\mu^2 s-f(s) \qquad \text{for} \quad s\geq 0\,. \tag{4.3}$$

Then by assumptions (F) and (F_0), g is nondecreasing and

$$0=g(0)\leq g(s)\leq \tau(s+s^p) \qquad \text{for} \quad s\geq 0 \tag{4.4}$$

holds for some constant $\tau>0$. Thus we have $g(\tilde{u})\in L^1_{loc}(\boldsymbol{R}^N)$ and $g(\tilde{u})$ vanishes at infinity, hence $g(\tilde{u})\in\mathscr{S}'(\boldsymbol{R}^N)$. By $(\tilde{P}_m)$, (2.1) and (4.2),

$$\tilde{w}:=\tilde{u}-h \tag{4.5}$$

satisfies

$$-\Delta\tilde{w}+\mu^2\tilde{w}=g(\tilde{u})+\beta \qquad \text{in} \quad \mathscr{D}'(\boldsymbol{R}^N)\,.$$

Since $\tilde{w}\in\mathscr{S}'(\boldsymbol{R}^N)$, we have from Lemma 3

$$\tilde{w}=E_\mu*[g(\tilde{u})+\beta]\geq 0\,, \tag{4.6}$$

which proves assertion (i). Clearly, if $w\in C^2(\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m)$ satisfies

$$-\Delta w+\mu^2 w=g(w+h) \quad \text{and} \quad 0\leq w\leq\tilde{w} \quad \text{in } \boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m\,, \tag{4.7}$$

then

$$u:=w+h \tag{4.8}$$

is a solution to (P_m) with property (2.2). Thus, problem (P_m) is reduced to finding a function w satisfying (4.7).

Step 2: Iteration. In order to solve (4.7), we define a sequence of functions $\{w_k\}_{k=0}^\infty$ on $\boldsymbol{R}^N$ by

$$w_0:=0 \quad \text{and} \quad w_k:=E_\mu*[g(w_{k-1}+h)] \qquad \text{for } k=1,2,\dots. \tag{4.9}$$

We shall show by induction that

$$w_k\in W^{2,q/p}(\boldsymbol{R}^N) \quad \text{and} \quad 0\leq w_{k-1}\leq w_k\leq\tilde{w} \tag{4.10}$$

for $k=1,2,\dots$, where $p<q<N/(N-2)$.

When $k=1$, we see from (4.4) and $h\in L^{q/p}(\boldsymbol{R}^N)\cap L^q(\boldsymbol{R}^N)$ that $g(h)\in L^{q/p}(\boldsymbol{R}^N)$. Thus $w_1\in W^{2,q/p}(\boldsymbol{R}^N)$ and $w_1\geq 0$. Since g is nondecreasing, it follows from (4.6) that

$$\tilde{w}-w_1=E_\mu*[g(\tilde{u})+\beta-g(h)]=E_\mu*[g(\tilde{w}+h)-g(h)+\beta]\geq 0\,.$$

Therefore (4.10) holds true for $k=1$.

We now assume that (4.10) is true for $k-1$ with $k\geq 2$, that is, $w_{k-1}\in W^{2,q/p}(\boldsymbol{R}^N)$ and $0\leq w_{k-2}\leq w_{k-1}\leq\tilde{w}$. By Sobolev's inequality it holds $w_{k-1}\in L^t(\boldsymbol{R}^N)$ with $1/t=p/q-2/N$. Since $q/p<q<N/(N-2)<t$, we have $w_{k-1}\in L^q(\boldsymbol{R}^N)$ and hence $w_{k-1}+h\in$

$L^{q/p}(\boldsymbol{R}^N)\cap L^q(\boldsymbol{R}^N)$ and $g(w_{k-1}+h)\in L^{q/p}(\boldsymbol{R}^N)$. Then by Lemma 3 we have $w_k=E_\mu*[g(w_{k-1}+h)]\in W^{2,q/p}(\boldsymbol{R}^N)$ and

$$w_k-w_{k-1}=E_\mu*[g(w_{k-1}+h)-g(w_{k-2}+h)]\geq 0\,,$$
$$\tilde{w}-w_k=E_\mu*[g(\tilde{w}+h)-g(w_{k-1}+h)+\beta]\geq 0\,.$$

Therefore, (4.10) holds true for k and we have proved that (4.10) is true for all $k=1,2,\ldots$.

Step 3: Convergence. By (4.9) and Lemma 3 we have

$$-\Delta w_k+\mu^2 w_k=g(w_{k-1}+h) \quad \text{in } \mathscr{D}'(\boldsymbol{R}^N) \tag{4.11}$$

for $k=1,2,\ldots$. Now we fix any open set Ω with $\Omega\Subset\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m$ and choose two open sets Ω' and Ω'' so that $\Omega\Subset\Omega'\Subset\Omega''\Subset\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m$. From (4.10) we have

$$0=w_0\leq w_1\leq\cdots\leq\tilde{w}\leq\tilde{u}\in L^\infty_{\mathrm{loc}}(\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m)\,,$$

hence $\{w_k\}_{k=0}^\infty$ and $\{g(w_{k-1}+h)\}_{k=1}^\infty$ are uniformly bounded in Ω''. Since Ω'' is a bounded set, $\{g(w_{k-1}+h)\}_{k=1}^\infty$ is bounded in $L^{\hat{p}}(\Omega'')$ for each $\hat{p}\in(1,+\infty)$. By (4.11) and the interior L^p-estimate, it follows that $\{w_k\}_{k=1}^\infty$ is bounded in $W^{2,\hat{p}}(\Omega')$. With $\hat{p}$ sufficiently large, $\{w_k\}_{k=1}^\infty$ is bounded in $C^\alpha(\overline{\Omega'})$ for some $\alpha\in(0,1)$ in virtue of Sobolev's inequality. Since g is (locally) Lipschitz continuous, $\{g(w_{k-1}+h)\}_{k=2}^\infty$ is bounded in $C^\alpha(\overline{\Omega'})$. By the Schauder interior estimate, $\{w_k\}_{k=2}^\infty$ is bounded in $C^{2,\alpha}(\overline{\Omega})$. Therefore, $\{w_k\}_{k=2}^\infty$ has a subsequence which is convergent in $C^2(\overline{\Omega})$.

When we choose a sequence $\{\Omega_n\}_{n=1}^\infty$ of open sets with $\Omega_1\Subset\Omega_2\Subset\cdots\to\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m$, by the diagonal process for $\{\Omega_n\}_{n=1}^\infty$ we can select a subsequence $\{w_{k_i}\}_{i=1}^\infty$ of $\{w_k\}_{k=0}^\infty$ and $w\in C^2(\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m)$ such that

$$w_{k_i}\to w \quad \text{as } i\to\infty \quad \text{in } C^2(\overline{\Omega_n})$$

for each $n\in\boldsymbol{N}$. By the monotonicity of $\{w_k\}_{k=0}^\infty$ we see that $w_k\to w$ as $k\to\infty$ in $C(\overline{\Omega_n})$. Therefore, we obtain (4.7). q.e.d.

REMARK 2. In the case $m=1$, we see from (4.9) that each w_k is radial. Therefore, the solution obtained above is also radial, even if $\tilde{u}$ is not radial.

Now we prove Theorem 1 by applying Proposition 1.

PROOF OF THEOREM 1. We set

$$\tilde{u}:=\sum_{j=1}^m \kappa_j E(\cdot-a_j) \quad \text{and} \quad \beta:=f(\tilde{u})\,. \tag{4.12}$$

Then $\tilde{u}$ is a supersolution to (P_m), that is, $\tilde{u}$ and β satisfy $(\tilde{\mathrm{P}}_m)$, and Theorem 1 follows immediately from Proposition 1. q.e.d.

To prove Theorem 1′ we use the following lemma. We shall give the proof in the Appendix since it is lengthy.

LEMMA 4. *Let $N=2$ and suppose that f is a Lipschitz continuous function on $[0, +\infty)$ such that*

$$f(0)=0\,, \quad f(s)>0 \quad \text{for } 0<s<b \quad \text{and} \quad f(s)=0 \quad \text{for } s\geq b \tag{4.13}$$

with a constant $b>0$. Then (P_1) *has a radial solution $u\in C^2(\boldsymbol{R}^2\setminus\{0\})$ for each $\kappa>0$.*

PROOF OF THEOREM 1′. (i) By (F_1') we can choose a Lipschitz continuous function $\tilde{f}$ on $[0, +\infty)$ which satisfies (4.13) and $\tilde{f}\leq f$. Then by Lemma 4 and Lemma 2 there exists $\tilde{u}\in C^2(\boldsymbol{R}^2\setminus\{0\})$ such that

$$\begin{cases} -\Delta\tilde{u}+\tilde{f}(\tilde{u})=\kappa\delta_0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}^2)\,, \quad \tilde{u}>0 \quad \text{in } \boldsymbol{R}^2\setminus\{0\}\,, \\ \tilde{u}(x)\sim\kappa E(x) \quad \text{as } x\to 0 \quad \text{and} \quad \tilde{u}(x)\to 0 \quad \text{as } |x|\to+\infty\,. \end{cases}$$

This $\tilde{u}$ is certainly a supersolution to (P_1) with $\beta:=f(\tilde{u})-\tilde{f}(\tilde{u})$. Therefore, (P_1) has a radial solution u by Proposition 1 and Remark 2.

(ii) In view of (F_1''), there is a nondecreasing Lipschitz continuous function $\tilde{f}_m$ on $[0, +\infty)$ satisfying $0<m\tilde{f}_m(s)\leq f(s)$ for $s>0$. For each $j=1, 2, \ldots, m$, there exists $\tilde{u}_j\in C^2(\boldsymbol{R}^2\setminus\{a_j\})$ such that

$$\begin{cases} -\Delta\tilde{u}_j+\tilde{f}_m(\tilde{u}_j)=\kappa_j\delta_{a_j} \quad \text{in } \mathscr{D}'(\boldsymbol{R}^2)\,, \quad \tilde{u}_j>0 \quad \text{in } \boldsymbol{R}^2\setminus\{a_j\}\,, \\ \tilde{u}_j(x)\sim\kappa_j E(x-a_j) \quad \text{as } x\to a_j \quad \text{and} \quad \tilde{u}_j(x)\to 0 \quad \text{as } |x|\to+\infty\,, \end{cases}$$

by (i) and Lemma 2. It is easy to see that

$$\tilde{u}:=\sum_{j=1}^{m}\tilde{u}_j \quad \text{and} \quad \beta:=f(\tilde{u})-\sum_{j=1}^{m}\tilde{f}_m(\tilde{u}_j) \tag{4.14}$$

satisfy the assumptions of Proposition 1 and therefore (P_m) has a solution u. q.e.d.

5. Solutions with infinitely many singularities. In this section we shall prove Theorem 2. For this purpose we shall construct a solution u_m to (P_m) for $m=1, 2, \ldots$, with an upper bound $\tilde{u}$ independent of m. For $\mu>0$ and positive integer n, we denote the n-time convolution of E_μ by $E_\mu^{\langle n\rangle}$, that is,

$$E_\mu^{\langle 0\rangle}:=\delta_0 \quad \text{and} \quad E_\mu^{\langle n\rangle}:=E_\mu^{\langle n-1\rangle}*E_\mu \qquad \text{for} \quad n=1, 2, \ldots. \tag{5.1}$$

Then we can calculate $E_\mu^{\langle n\rangle}$ as

$$E_\mu^{\langle n\rangle}(x)=E_\mu^{\langle n\rangle}(|x|)=\frac{1}{(2\pi)^{\nu+1}(n-1)!\,2^{n-1}}\left(\frac{\mu}{|x|}\right)^{\nu+1-n}K_{\nu+1-n}(\mu|x|) \tag{5.2}$$

for $n\geq 1$ (where $\nu=(N-2)/2$) by using the Fourier transform. Thus we have

$$E_\mu^{\langle n\rangle}(r)=\begin{cases} o(E(r)) & \text{as } r\to 0 \quad \text{if } n\geq 2\,, \\ o(E_{\mu_0}(r)) & \text{as } r\to+\infty \quad \text{if } n\geq 1\,, \end{cases} \tag{5.3}$$

for $0<\mu_0<\mu$, and hence

$$E_\mu^{\langle n\rangle} \le c_{\mu,\mu_0,n} E_{\mu_0} \tag{5.4}$$

for some constant $c_{\mu,\mu_0,n} > 0$ if $n \ge 1$.

LEMMA 5. (i) *For $\mu > 0$ and $1 \le q < N/(N-2)$, we have $E_\mu \in L^q(\boldsymbol{R}^N)$ and*

$$\|E_\mu\|_{L^1(\boldsymbol{R}^N)} = \mu^{-2} \quad \text{and} \quad \|E_\mu\|_{L^q(\boldsymbol{R}^N)} = \mu^{-(2-N/q')} \|E_1\|_{L^q(\boldsymbol{R}^N)}, \tag{5.5}$$

where q' is the conjugate exponent of q.

(ii) *For $0 < \eta < \mu$ and $n \ge 1$, it holds that*

$$E_\mu^{\langle n\rangle} * E_\eta = \frac{1}{(\mu^2-\eta^2)^n}\left(E_\eta - \sum_{i=1}^n (\mu^2-\eta^2)^{i-1} E_\mu^{\langle i\rangle}\right) \le \frac{1}{(\mu^2-\eta^2)^n} E_\eta . \tag{5.6}$$

PROOF. (i) If $v \in \mathscr{S}'(\boldsymbol{R}^N)$ satisfies

$$-\Delta v + \mu^2 v = 1 \quad \text{in} \quad \mathscr{D}'(\boldsymbol{R}^N), \tag{5.7}$$

then it follows from Lemma 3 that $v(x) = (E_\mu * 1)(x) = \|E_\mu\|_{L^1(\boldsymbol{R}^N)}$. On the other hand, $\tilde{v}(x) := \mu^{-2}$ is a solution to (5.7) which belongs to $\mathscr{S}'(\boldsymbol{R}^N)$. Therefore, the first equality of (5.5) holds true due to Lemma 3. The second equality of (5.5) follows from (4.2).

(ii) Note that $(-\Delta + \mu^2) E_\mu^{\langle i\rangle} * E_\eta = E_\mu^{\langle i-1\rangle} * E_\eta$ and $(-\Delta + \eta^2) E_\mu^{\langle i\rangle} * E_\eta = E_\mu^{\langle i\rangle}$ for $i \ge 1$. Therefore, we have

$$(\mu^2-\eta^2)^i E_\mu^{\langle i\rangle} * E_\eta = (\mu^2-\eta^2)^{i-1} E_\mu^{\langle i-1\rangle} * E_\eta - (\mu^2-\eta^2)^{i-1} E_\mu^{\langle i\rangle} \tag{5.8}$$

and get (5.6) by adding (5.8) for $i = 1, 2, \ldots, n$. q.e.d.

Now we take an integer $n > \nu$ so as to satisfy $E_\mu^{\langle n\rangle} \in L^{q'}(\boldsymbol{R}^N)$ for some $q' > N/2$. Note that $E_\mu^{\langle n\rangle} * E_\eta \in L^\infty(\boldsymbol{R}^N)$ for $\eta > 0$. For given $\{a_j\}_{j=1}^\infty$ and $\{\kappa_j\}_{j=1}^\infty$, we set

$$h := \sum_{j=1}^\infty \kappa_j E_\mu(\cdot - a_j) \quad \text{and} \quad H := \sum_{i=1}^n \mu^{2(i-1)} \sum_{j=1}^\infty \kappa_j E_\mu^{\langle i\rangle}(\cdot - a_j) . \tag{5.9}$$

LEMMA 6. *Let $N \ge 2$ and $\{a_j\}_{j=1}^\infty$ be a sequence in $\boldsymbol{R}^N$ without accumulation points. If*

$$\tilde{h} := \sum_{j=1}^\infty \kappa_j E_{\mu_0}(\cdot - a_j) \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N) \tag{5.10}$$

for some $\mu_0 \in (0, \mu)$, then $h \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N) \cap L^\infty_{\mathrm{loc}}(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^\infty)$ and

$$\begin{cases} H \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N) \cap L^\infty_{\mathrm{loc}}(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^\infty) , \\ H(x) \sim \kappa_j E(x - a_j) \quad \text{as } x \to a_j \quad \text{for } j = 1, 2, \ldots . \end{cases} \tag{5.11}$$

PROOF. Since $\tilde{h} \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ it follows from (4.2) that

$$-\Delta \tilde{h} + \mu_0^2 \tilde{h} = \sum_{j=1}^\infty \kappa_j \delta_{a_j} \quad \text{in} \quad \mathscr{D}'(\boldsymbol{R}^N) .$$

For any open set Ω with $\Omega \Subset \boldsymbol{R}^N \setminus \{a_j\}_{j=1}^{\infty}$, we have $\tilde{h} \in L^{\infty}(\Omega)$ by the hypoellipticity of the operator $-\Delta + \mu_0^2$, so that $\tilde{h} \in L_{\mathrm{loc}}^{\infty}(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^{\infty})$. The same argument holds for

$$\tilde{h}_{(m)} := \tilde{h} - \kappa_m E_{\mu_0}(\cdot - a_m) \tag{5.12}$$

and hence $\tilde{h}_{(m)} \in L_{\mathrm{loc}}^{\infty}(\boldsymbol{R}^N \setminus \{a_j\}_{j \neq m})$. By (5.4) we have

$$H_{(m)} := H - \kappa_m \sum_{i=1}^{n} \mu^{2(i-1)} E_{\mu}^{\langle i \rangle}(\cdot - a_m) \in L_{\mathrm{loc}}^{1}(\boldsymbol{R}^N) \cap L_{\mathrm{loc}}^{\infty}(\boldsymbol{R}^N \setminus \{a_j\}_{j \neq m}), \tag{5.13}$$

and (5.11) follows from (5.3) since $\{a_j\}_{j=1}^{\infty}$ does not accumulate at a_m. q.e.d.

By assumption (F_2) we can choose a nondecreasing function $\tilde{f}$ on $[0, +\infty)$ such that

$$0 \leq \tilde{f}(s) \leq f(s) \quad \text{for } s \geq 0, \quad \liminf_{s \to 0} \tilde{f}(s)/s > 0 \quad \text{and} \quad \lim_{s \to +\infty} \tilde{f}(s) = \gamma. \tag{5.14}$$

We define $l(\eta) > 0$ for small $\eta > 0$ by

$$l(\eta) := \sup\{l > 0 \mid \eta^2 s \leq \tilde{f}(s) \text{ for } 0 \leq s \leq l\}. \tag{5.15}$$

Then there holds

$$\begin{aligned} &\eta^2 s \leq \tilde{f}(s) \quad \text{for } 0 \leq s \leq l(\eta), \\ &l(\eta) \to +\infty \quad \text{and} \quad \eta^2 l(\eta) = \tilde{f}(l(\eta)) \to \gamma \quad \text{as } \eta \to 0. \end{aligned} \tag{5.16}$$

To prove Theorem 2, the following is a key lemma.

LEMMA 7. *Let $N \geq 2$ and suppose that f satisfies* (F) *and* (F_2). *Under the notations above, if* (5.10) *holds for some $\mu_0 \in (0, \mu)$ and*

$$Z := \mu^{2n} \sum_{j=1}^{\infty} \kappa_j E_{\mu}^{\langle n \rangle} * E_{\eta}(\cdot - a_j) \leq l(\eta) \tag{5.17}$$

for some $\eta > 0$, then (P_{∞}) *has a solution $u \in C^2(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^{\infty})$.*

PROOF. Set

$$\tilde{u} := H + Z. \tag{5.18}$$

For $m = 1, 2, \ldots$, let h_m, H_m, Z_m and $\tilde{u}_m$ be defined by

$$\begin{cases} h_m := \sum_{j=1}^{m} \kappa_j E_{\mu}(\cdot - a_j), \quad H_m := \sum_{i=1}^{n} \mu^{2(i-1)} \sum_{j=1}^{m} \kappa_j E_{\mu}^{\langle i \rangle}(\cdot - a_j), \\ Z_m := \mu^{2n} \sum_{j=1}^{m} \kappa_j E_{\mu}^{\langle n \rangle} * E_{\eta}(\cdot - a_j), \quad \tilde{u}_m := H_m + Z_m. \end{cases} \tag{4.19}$$

By Lemma 6 and (5.17) we obtain

$$\begin{cases} \tilde{u} \in L^1_{\text{loc}}(\boldsymbol{R}^N) \cap L^\infty_{\text{loc}}(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^\infty), \\ \tilde{u}(x) \sim \kappa_j E(x-a_j) \quad \text{as } x \to a_j \quad \text{for } j=1, 2, \dots . \end{cases}$$

By the definition of $\tilde{u}_m$ there holds

$$-\Delta \tilde{u}_m + f(\tilde{u}_m) = \beta_m + \sum_{j=1}^m \kappa_j \delta_{a_j} \qquad \text{in } \mathscr{D}'(\boldsymbol{R}^N),$$

where $\beta_m := f(\tilde{u}_m) - \eta^2 Z_m$. Since $0 \le Z_m \le Z \le l(\eta)$, we have $\eta^2 Z_m \le \tilde{f}(Z_m)$ by (5.6). From the fact $\tilde{f} \le f$ and the monotonicity of $\tilde{f}$ we see that

$$\beta_m \ge \tilde{f}(\tilde{u}_m) - \eta^2 Z_m \ge \tilde{f}(Z_m) - \eta^2 Z_m \ge 0,$$

and hence $\tilde{u}_m$ is a supersolution to (P_m). Therefore, by Proposition 1 and Lemma 2 there exists $u_m \in C^2(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^m)$ satisfying

(5.20) $$-\Delta u_m + f(u_m) = \sum_{j=1}^m \kappa_j \delta_{a_j} \quad \text{in } \mathscr{D}'(\boldsymbol{R}^N) \quad \text{and} \quad h_m \le u_m \le \tilde{u}_m \le \tilde{u}.$$

Since $\tilde{u} \in L^\infty_{\text{loc}}(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^\infty)$, $\{u_m\}_{m=1}^\infty$ is locally uniformly bounded in $\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^\infty$. By a compactness argument similar to that in the proof of Proposition 1, we can select a subsequence $\{u_{m_i}\}_{i=1}^\infty$ which converges locally in $C^2(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^\infty)$. Then clearly, the limit u belongs to $C^2(\boldsymbol{R}^N \setminus \{a_j\}_{j=1}^\infty)$ and is a solution to (P_∞) satisfying

(5.21) $$h \le u \le \tilde{u} \qquad \text{in } \boldsymbol{R}^N \setminus \{a_j\}_{j=1}^\infty.$$

q.e.d.

From (A) and (4.2) it is clear that $\tilde{h} \in L^1_{\text{loc}}(\boldsymbol{R}^N)$ for some $\mu_0 \in (0, \mu)$. Thus we have only to show (5.17) for some $\eta > 0$ for the proof of Theorem 2. By assumption (A) there exists a positive constant C_N depending only on N such that

(5.22) $$\#\{j \in \boldsymbol{N} \,|\, |x - a_j| \le k\alpha\} \le C_N \omega_N k^N \qquad \text{for all } x \in \boldsymbol{R}^N \text{ and } k \in \boldsymbol{N},$$

where $\#S$ denotes the number of elements of a set S. In order to estimate Z, we shall make use of the following lemma with $F = E_\mu^{\langle n \rangle} * E_\eta$.

LEMMA 8. *Let $N \ge 2$ and $F(x) = F(|x|)$ be a nonnegative radial function on $\boldsymbol{R}^N$ which is decreasing in $r = |x|$. Then, under assumption* (A), *there holds*

(5.23) $$\sum_{k=\tilde{k}}^\infty F(k\alpha) \#\{j \in \boldsymbol{N} \,|\, k\alpha < |x - a_j| \le (k+1)\alpha\}$$
$$\le C_N \omega_N \left[\tilde{k}^N F(\tilde{k}\alpha) + \sum_{k=\tilde{k}}^\infty ((k+1)^N - k^N) F(k\alpha) \right]$$

for any $\tilde{k} \in \boldsymbol{N}$ and $x \in \boldsymbol{R}^N$.

PROOF. This assertion follows from

$$\sum_{k=\tilde{k}}^{\infty} F(k\alpha)\#\{j\in N \mid k\alpha<|x-a_j|\le(k+1)\alpha\}$$

$$=\sum_{k=\tilde{k}}^{\infty}[F(k\alpha)-F((k+1)\alpha)]\#\{j\in N \mid \tilde{k}\alpha<|x-a_j|\le(k+1)\alpha\}$$

and the assumptions above. q.e.d.

PROOF OF THEOREM 2. We now show that (5.17) holds for some $\eta>0$. Note that

$$\|E_\mu^{\langle n\rangle}*E_\eta\|_{L^\infty(\boldsymbol{R}^N)}\le\|E_\mu^{\langle n\rangle}\|_{L^{q'}(\boldsymbol{R}^N)}\|E_1\|_{L^q(\boldsymbol{R}^N)}\eta^{-(2-N/q')}=o(\eta^{-2})\qquad\text{as}\quad \eta\to 0\,. \tag{5.24}$$

By assumption (A), for any $\varepsilon>0$ there exist positive integers $j(\varepsilon)$ and $k(\varepsilon)$ such that

$$\kappa_j\le\tilde{\kappa}+\varepsilon\quad\text{for } j>j(\varepsilon)\quad\text{and}\quad\frac{(k+1)^N-k^N}{k^N-(k-1)^N}\le 1+\varepsilon\quad\text{for } k\ge k(\varepsilon)\,. \tag{5.25}$$

Choose $\varepsilon>0$ with $C_N(\tilde{\kappa}+\varepsilon)(1+\varepsilon)/\alpha^N<\gamma$ and set $K(\varepsilon):=\sum_{j=1}^{j(\varepsilon)}\kappa_j$.

Then for all $x\in\boldsymbol{R}^N$, it follows from (5.22)–(5.25) and (5.5)–(5.6) that

$$\begin{aligned}
&\mu^{2n}\sum_{j=1}^{\infty}\kappa_j E_\mu^{\langle n\rangle}*E_\eta(x-a_j)\\
&\le\mu^{2n}\sum_{j=1}^{j(\varepsilon)}\kappa_j E_\mu^{\langle n\rangle}*E_\eta(x-a_j)+\mu^{2n}(\tilde{\kappa}+\varepsilon)\sum_{|x-a_j|\le k(\varepsilon)\alpha}E_\mu^{\langle n\rangle}*E_\eta(x-a_j)\\
&\quad+\mu^{2n}(\tilde{\kappa}+\varepsilon)\sum_{k=k(\varepsilon)}^{\infty}\sum_{k\alpha<|x-a_j|\le(k+1)\alpha}E_\mu^{\langle n\rangle}*E_\eta(x-a_j)\\
&\le\mu^{2n}\left[\sum_{j=1}^{j(\varepsilon)}\kappa_j+(\tilde{\kappa}+\varepsilon)\sum_{|x-a_j|\le k(\varepsilon)\alpha}1\right]\|E_\mu^{\langle n\rangle}*E_\eta\|_{L^\infty(\boldsymbol{R}^N)}\\
&\quad+\mu^{2n}(\tilde{\kappa}+\varepsilon)\sum_{k=k(\varepsilon)}^{\infty}E_\mu^{\langle n\rangle}*E_\eta(k\alpha)\#\{j\in N \mid k\alpha<|x-a_j|\le(k+1)\alpha\}\\
&\le\mu^{2n}[K(\varepsilon)+(\tilde{\kappa}+\varepsilon)C_N\omega_N k(\varepsilon)^N]\|E_\mu^{\langle n\rangle}*E_\eta\|_{L^\infty(\boldsymbol{R}^N)}\\
&\quad+\mu^{2n}(\tilde{\kappa}+\varepsilon)C_N\omega_N\left[k(\varepsilon)^N E_\mu^{\langle n\rangle}*E_\eta(k(\varepsilon)\alpha)+\sum_{k=k(\varepsilon)}^{\infty}((k+1)^N-k^N)E_\mu^{\langle n\rangle}*E_\eta(k\alpha)\right]\\
&\le\mu^{2n}[K(\varepsilon)+2(\tilde{\kappa}+\varepsilon)C_N\omega_N k(\varepsilon)^N]\|E_\mu^{\langle n\rangle}*E_\eta\|_{L^\infty(\boldsymbol{R}^N)}\\
&\quad+\mu^{2n}(\tilde{\kappa}+\varepsilon)C_N\omega_N\sum_{k=k(\varepsilon)}^{\infty}\frac{(k+1)^N-k^N}{\omega_N[(k\alpha)^N-((k-1)\alpha)^N]}\int_{B_{k\alpha}\setminus B_{(k-1)\alpha}}E_\mu^{\langle n\rangle}*E_\eta(k\alpha)dy\\
&\le\mu^{2n}[K(\varepsilon)+2(\tilde{\kappa}+\varepsilon)C_N\omega_N k(\varepsilon)^N]\|E_\mu^{\langle n\rangle}*E_\eta\|_{L^\infty(\boldsymbol{R}^N)}
\end{aligned}$$

$$+\mu^{2n}(\tilde{\kappa}+\varepsilon)\frac{C_N}{\alpha^N}\sum_{k=k(\varepsilon)}^{\infty}(1+\varepsilon)\int_{B_{k\alpha}\setminus B_{(k-1)\alpha}}\frac{1}{(\mu^2-\eta^2)^n}E_\eta(y)dy$$

$$\leq \mu^{2n}[K(\varepsilon)+2(\tilde{\kappa}+\varepsilon)C_N\omega_N k(\varepsilon)^N]\|E_\mu^{\langle n\rangle}*E_\eta\|_{L^\infty(\boldsymbol{R}^N)}+C_N\frac{(\tilde{\kappa}+\varepsilon)(1+\varepsilon)}{\alpha^N}\frac{\mu^{2n}}{(\mu^2-\eta^2)^n}\eta^{-2}$$

$$\sim C_N\frac{(\tilde{\kappa}+\varepsilon)(1+\varepsilon)}{\alpha^N}\eta^{-2}\quad \text{as } \eta\to 0\,.$$

Since $C_N(\tilde{\kappa}+\varepsilon)(1+\varepsilon)/\alpha^N<\gamma$, (5.17) follows from (5.16) for sufficiently small $\eta>0$.

q.e.d.

REMARK 3. If $h\geq c$ in $\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^{\infty}$ holds for some constant $c>0$, then we may drop the second condition of (F_2). Indeed, we can deform the part below c of f so as to satisfy (F_2) and construct a solution satisfying (5.21).

6. Existence of solutions with singularities of small intensity. In the remainder of this paper we do not assume that f is nonnegative. Then whether (P_m) has a solution or not may depend on the size of the intensity at singularities. This fact is quite different from the case where f is nonnegative. In this section we shall prove Theorem 3. For this purpose we deal with the case where f is defined by

$$f(s)=\begin{cases}-\tau[(s-b)^+]^p & \text{if } N\geq 3\,,\\ \mu^2 s-\tau s^p & \text{if } N=2\,.\end{cases}\tag{6.1}$$

Here τ, b and μ are arbitrary positive constants, p satisfies $1<p<N/(N-2)$ ($2<p<3$ if $N=3$) and $s^+:=\max\{0,s\}$ for $s\in\boldsymbol{R}$.

LEMMA 9. *Let $N\geq 2$ and suppose that f is given by* (6.1). *Then* (P_1) *has a radial solution $u\in C^2(\boldsymbol{R}^N\setminus\{0\})$ if the intensity $\kappa>0$ is small.*

PROOF. (i) *Case $N\geq 3$.* We first note that $\max\{1, 2/(N-2)\}<p<N/(N-2)$. Set

$$\tilde{u}(x)=\tilde{u}(|x|):=\begin{cases}\dfrac{b}{2}\left[\left(\dfrac{\tilde{r}}{|x|}\right)^{N-2}+\theta\left(\dfrac{\tilde{r}}{|x|}\right)^{\sigma-2}+1-\theta\right] & \text{for } 0<|x|\leq\tilde{r}\,,\\ b\left(\dfrac{\tilde{r}}{|x|}\right)^{N-2} & \text{for } |x|>\tilde{r}\,,\end{cases}\tag{6.2}$$

where $\sigma:=p(N-2)\in(2,N)$, $\theta:=(N-2)/(\sigma-2)$ and $\tilde{r}:=[2\kappa/(b(N-2)N\omega_N)]^{1/(N-2)}$. Then we obtain $\tilde{u}\in C^1(\boldsymbol{R}^N\setminus\{0\})$ and

$$-\Delta\tilde{u}+f(\tilde{u})=\beta+\kappa\delta_0\quad \text{in } \mathscr{D}'(\boldsymbol{R}^N)\tag{6.3}$$

with

(6.4) $\beta(x)=\beta(|x|)$

$$:=\begin{cases} \dfrac{b\rho}{2\tilde{r}^2}\left(\dfrac{\tilde{r}}{|x|}\right)^{\sigma}-\tau\left[\dfrac{b}{2}\left(\left(\dfrac{\tilde{r}}{|x|}\right)^{N-2}+\theta\left(\dfrac{\tilde{r}}{|x|}\right)^{\rho-2}-1-\theta\right)\right]^p & \text{for } 0<|x|\le\tilde{r}\,, \\ 0 & \text{for } |x|>\tilde{r}\,, \end{cases}$$

where $\rho:=(N-2)(N-\sigma)$. If we can show that

(6.5) $$\beta(r)\ge 0 \qquad \text{for} \quad 0<r\le\tilde{r}\,,$$

then we easily see that $\tilde{u}$ and β satisfy the assumptions of Proposition 1 with $m=1$. Thus we are going to show that (6.5) holds true if $\kappa>0$ is small, i.e., if $\tilde{r}>0$ is small.

Note that

(6.6) $$\beta(r)\ge\tilde{\beta}(r) \qquad \text{for} \quad 0<r\le\tilde{r}\,,$$

where

(6.7) $$\tilde{\beta}(r):=\frac{b}{2}\left(\frac{\rho}{\tilde{r}^2}-\tau b^{p-1}\right)\left(\frac{\tilde{r}}{r}\right)^{\sigma}-\frac{\tau}{2}(b\theta)^p\left(\frac{\tilde{r}}{r}\right)^{p(\sigma-2)}+\tau\left[\frac{b}{2}(1+\theta)\right]^p \quad \text{for } r>0\,.$$

If $\rho/\tilde{r}^2>\tau b^{p-1}$, then $\tilde{\beta}(r)$ is positive and decreasing in r for $0<r\ll 1$. Since

$$\tilde{\beta}'(r)=\frac{1}{2r}\left(\frac{\tilde{r}}{r}\right)^{p(\sigma-2)}\left[\tau p(\sigma-2)(b\theta)^p-b\sigma\left(\frac{\rho}{\tilde{r}^2}-\tau b^{p-1}\right)\left(\frac{\tilde{r}}{r}\right)^{p(N-\sigma)}\right],$$

$r_0>0$ satisfies $\tilde{\beta}'(r_0)=0$ if and only if

(6.8) $$\left(\frac{r_0}{\tilde{r}}\right)^{p(N-\sigma)}=\frac{1}{\theta^{p-1}}\left(\frac{\rho}{\tau b^{p-1}\tilde{r}^2}-1\right).$$

Furthermore, if

(6.9) $$r_0\ge\tilde{r} \quad \text{and} \quad \tilde{\beta}(\tilde{r})\ge 0\,,$$

then by (6.6) we can conclude that $\tilde{\beta}(r)\ge 0$ for $0<r\le\tilde{r}$ and (6.5) holds true. From (6.8) and (6.7) it follows that (6.9) holds true if and only if

$$\tilde{r}^{-2}\ge\frac{\tau b^{p-1}}{\rho}\left[1+\max\left\{\theta^{p-1},\,\theta^p-\left(\frac{1+\theta}{2}\right)^p\right\}\right].$$

Thus we obtain (6.9) if $\tilde{r}>0$ is small enough, i.e., if $\kappa>0$ is small enough. For such $\kappa>0$, $\tilde{u}$ is a supersolution to (P_1) and we obtain a radial solution to (P_1).

(ii) *Case* $N=2$. Let $c>0$ and set

(6.10) $$\tilde{u}(x)=\tilde{u}(|x|):=\kappa E_\mu(x)+ce^{-\mu|x|} \qquad \text{for} \quad x\in\boldsymbol{R}^2\setminus\{0\}\,.$$

Then $u\in C^2(\boldsymbol{R}^2\setminus\{0\})$ and we have (6.3) with

(6.11) $\beta(x)=\beta(|x|):=c\mu|x|^{-1}e^{-\mu|x|}-\tau[\kappa E_\mu(x)+ce^{-\mu|x|}]^p$ for $x\in \boldsymbol{R}^2\setminus\{0\}$.

If we choose $c>0$ and $\kappa>0$ small enough, then $\beta\geq 0$ and $\tilde{u}$ is certainly a supersolution to (P_1). Therefore, the assertion follows from Proposition 1. q.e.d.

PROOF OF THEOREM 3. (i) *Case* $N\geq 3$. Note that

$$\left[\sum_{j=1}^m s_j\right]^p \leq 2^{(m-1)(p-1)} \sum_{j=1}^m s_j^p \qquad \text{for} \quad s_1, s_2, \ldots, s_m \geq 0 \tag{6.12}$$

if $p>1$. Without loss of generality, we can assume that $\max\{1, 2/(N-2)\}<p<N/(N-2)$. By assumptions (F), (F_0) and (F_3) we can choose positive constants τ and b such that

$$\tilde{f}(s):=-\tau[(s-b)^+]^p \leq f(s) \qquad \text{for} \quad s\geq 0 . \tag{6.13}$$

When we set

$$\tilde{f}_m(s):=-2^{(m-1)(p-1)}\tau[(s-b/m)^+]^p \qquad \text{for} \quad s\geq 0 , \tag{6.14}$$

it follows from (6.12) that

$$\tilde{f}\left(\sum_{j=1}^m s_j\right) \geq \sum_{j=1}^m \tilde{f}_m(s_j) \qquad \text{for} \quad s_1, s_2, \ldots, s_m \geq 0 .$$

For $j=1, 2, \ldots, m$, by Lemma 9 and Lemma 2 there exists $\tilde{u}_j\in C^2(\boldsymbol{R}^N\setminus\{a_j\})$ such that

$$\begin{cases} -\Delta\tilde{u}_j+\tilde{f}_m(\tilde{u}_j)=\kappa_j\delta_{a_j} \quad \text{in } \mathscr{D}'(\boldsymbol{R}^N) , \quad \tilde{u}_j>0 \quad \text{in } \boldsymbol{R}^N\setminus\{a_j\} , \\ \tilde{u}_j(x)\sim\kappa_j E(x-a_j) \quad \text{as } x\to a_j \quad \text{and} \quad \tilde{u}_j(x)\to 0 \quad \text{as } |x|\to+\infty , \end{cases}$$

provided that $\kappa_j>0$ is small. If we set

$$\tilde{u}:=\sum_{j=1}^m \tilde{u}_j \quad \text{and} \quad \beta:=f(\tilde{u})-\sum_{j=1}^m \tilde{f}_m(\tilde{u}_j) , \tag{6.15}$$

then we can easily see that $\tilde{u}$ and β satisfy the assumptions of Proposition 1. Therefore, (P_m) has a solution $u\in C^2(\boldsymbol{R}^N\setminus\{a_j\}_{j=1}^m)$ if $\kappa_j>0$ is small for $j=1, 2, \ldots, m$.

(ii) *Case* $N=2$. By (F) and (F_3), we can choose positive constants η and τ such that

$$\tilde{f}(s):=\eta^2 s-\tau s^p \leq f(s) \qquad \text{for} \quad s\geq 0 . \tag{6.16}$$

If we set

$$\tilde{f}_m(s):=\eta^2 s-2^{(m-1)(p-1)}\tau s^p \qquad \text{for} \quad s\geq 0 , \tag{6.17}$$

then the remainder of the proof is exactly the same as in the case $N\geq 3$. q.e.d.

7. Nonexistence of solutions with a singularity of large intensity. In this section we shall prove Theorem 4. For this purpose we first consider radial solutions to (P_1) in the case where f is defined by

$$(7.1)\qquad f(s)=\min\{\mu^2 s,\ -\lambda^2(s-b)\}=\begin{cases}\mu^2 s & \text{for}\quad 0\le s\le\tilde{b}\,,\\ -\lambda^2(s-b) & \text{for}\quad s\ge\tilde{b}\,.\end{cases}$$

Here μ, λ and b are arbitrary positive constants and $\tilde{b}:=b\lambda^2/(\lambda^2+\mu^2)$. In order to deal with this case, we introduce the following functions:

$$(7.2)\qquad \begin{cases}\tilde{E}_\mu(x)=\tilde{E}_\mu(|x|):=\dfrac{1}{2\pi}\left[\dfrac{\mu}{2\pi|x|}\right]^\nu I_\nu(\mu|x|)\\ \tilde{Y}_\lambda(x)=\tilde{Y}_\lambda(|x|):=\dfrac{1}{4}\left[\dfrac{\lambda}{2\pi|x|}\right]^\nu J_\nu(\lambda|x|) \qquad \text{for}\quad x\in\boldsymbol{R}^N\setminus\{0\}\,,\\ Y_\lambda(x)=Y_\lambda(|x|):=-\dfrac{1}{4}\left[\dfrac{\lambda}{2\pi|x|}\right]^\nu N_\nu(\lambda|x|)\end{cases}$$

where I_ν is the modified Bessel function of order $\nu=(N-2)/2$, J_ν and N_ν are the Bessel functions of order ν. Then we have $\tilde{E}_\mu$, $\tilde{Y}_\lambda\in C^\infty(\boldsymbol{R}^N)$, while $Y_\lambda\in C^\infty(\boldsymbol{R}^N\setminus\{0\})$ and

$$Y_\lambda(x)\sim E(x)\quad\text{as } x\to 0\quad\text{and}\quad -\Delta Y_\lambda-\lambda^2 Y_\lambda=\delta_0\quad\text{in } \mathscr{D}'(\boldsymbol{R}^N)\,.$$

We note that $\{\tilde{E}_\mu, E_\mu\}$ and $\{\tilde{Y}_\lambda, Y_\lambda\}$ are independent solutions to

$$\frac{1}{r^{N-1}}\frac{d}{dr}\left[r^{N-1}\frac{du}{dr}\right]=\mu^2 u\quad\text{and}\quad \frac{1}{r^{N-1}}\frac{d}{dr}\left[r^{N-1}\frac{du}{dr}\right]=-\lambda^2 u\quad\text{for } r>0\,,$$

respectively. We also note that $\tilde{E}_\mu(r)$ is positive, increasing in r and divergent at infinity, while $\tilde{Y}_\lambda(r)$ and $Y_\lambda(r)$ are oscillating in r, and the first zero of Y_λ is smaller than that of $\tilde{Y}_\lambda$. We now recall some properties of the Bessel functions and the modified Bessel functions (see [8]):

$$(7.3)\qquad N'_\nu(s)J_\nu(s)-J'_\nu(s)N_\nu(s)=2/(\pi s)\quad\text{and}\quad K'_\nu(s)I_\nu(s)-I'_\nu(s)K_\nu(s)=-1/s\,,$$

$$(7.4)\qquad \begin{cases}K_\nu(s)\sim-(s/\nu)K'_\nu(s)\sim(\Gamma(\nu)/2)(2/s)^\nu & \text{if}\quad \nu>0\\ K_0(s)\sim\log(1/s)\ \text{and}\ K'_0(s)\sim-1/s & \text{if}\quad \nu=0\end{cases}\qquad \text{as } s\to 0\,.$$

LEMMA 10. *Let $N\ge 2$ and suppose that f is given by* (7.1). *If the intensity $\kappa>0$ is large, then* (P_1) *has no radial solution in $C^2(\boldsymbol{R}^N\setminus\{0\})$.*

PROOF. We choose $\kappa>0$ large enough to satisfy

$$(7.5)\qquad \kappa Y_\lambda(z_\lambda)<-(b-\tilde{b})\,,$$

where z_λ is the first zero of $\tilde{Y}_\lambda$. If u is a radial solution to (P_1) (except the condition at infinity), then u needs to satisfy

$$(7.6)\qquad u(r)=\kappa Y_\lambda(r)+\alpha\tilde{Y}_\lambda(r)+b\qquad\text{for}\quad 0<r<\hat{r}\,,$$

where $\alpha \in \boldsymbol{R}$ is a constant and $\hat{r}$ is the first zero of $\kappa Y_\lambda + \alpha \tilde{Y}_\lambda + b - \tilde{b}$. By (7.5) we have $0 < \hat{r} < z_\lambda$ and hence $u(r) < \tilde{b}$ for $\hat{r} < r < \hat{r} + \varepsilon$ ($\varepsilon > 0$). Thus we obtain

$$u(r) = \gamma E_\mu(r) + \tilde{\gamma} \tilde{E}_\mu(r) \qquad \text{for} \quad \hat{r} < r < \hat{r} + \varepsilon \tag{7.7}$$

with some constants $\gamma, \tilde{\gamma} \in \boldsymbol{R}$. If $\tilde{\gamma} < 0$, then it follows that $\gamma > 0$ and $u(r)$ is decreasing in r for $r \geq \hat{r}$. Furthermore, we have $u(r_0) = 0$ for some $r_0 > \hat{r}$, and u does not satisfy the positivity condition of (P_1).

Thus we only have to show that $\tilde{\gamma} < 0$ for any $\alpha \in \boldsymbol{R}$ if $\kappa > 0$ is large enough. Since

$$u(\hat{r} - 0) = u(\hat{r} + 0) = \tilde{b} \quad \text{and} \quad \frac{du}{dr}(\hat{r} - 0) = \frac{du}{dr}(\hat{r} + 0),$$

(7.6) and (7.7) imply that

$$\begin{cases} \kappa Y_\lambda(\hat{r}) + \alpha \tilde{Y}_\lambda(\hat{r}) = -(b - \tilde{b}), \\ \gamma E_\mu(\hat{r}) + \tilde{\gamma} \tilde{E}_\mu(\hat{r}) = \tilde{b}, \\ \kappa Y'_\lambda(\hat{r}) + \alpha \tilde{Y}'_\lambda(\hat{r}) = \gamma E'_\mu(\hat{r}) + \tilde{\gamma} \tilde{E}'_\mu(\hat{r}). \end{cases}$$

By the definition of the functions above, we have

$$\lambda^\nu(\kappa N_\nu(\lambda \hat{r}) - \alpha J_\nu(\lambda \hat{r})) = 4(b - \tilde{b})(2\pi \hat{r})^\nu, \tag{7.8}$$

$$\mu^\nu(\gamma K_\nu(\mu \hat{r}) + \tilde{\gamma} I_\nu(\mu \hat{r})) = 2\pi \tilde{b}(2\pi \hat{r})^\nu, \tag{7.9}$$

and

$$\begin{aligned} &-\pi \lambda^\nu[\lambda \hat{r}(\kappa N'_\nu(\lambda \hat{r}) - \alpha J'_\nu(\lambda \hat{r})) - \nu(\kappa N_\nu(\lambda \hat{r}) - \alpha J_\nu(\lambda \hat{r}))] \\ &\quad = 2\mu^\nu[\mu \hat{r}(\gamma K'_\nu(\mu \hat{r}) + \tilde{\gamma} I'_\nu(\mu \hat{r})) - \nu(\gamma K_\nu(\mu \hat{r}) + \tilde{\gamma} I_\nu(\mu \hat{r}))]. \end{aligned} \tag{7.10}$$

Since $\hat{r} < z_\lambda$, it holds $J_\nu(\lambda \hat{r}) > 0$. Thus (7.8) yields

$$\alpha = \frac{1}{\lambda^\nu J_\nu(\lambda \hat{r})}(\lambda^\nu \kappa N_\nu(\lambda \hat{r}) - 4(b - \tilde{b})(2\pi \hat{r})^\nu). \tag{7.11}$$

By substituting (7.9) and (7.11) into (7.10) we have

$$\gamma K'_\nu(\mu \hat{r}) + \tilde{\gamma} I'_\nu(\mu \hat{r}) = -\frac{1}{\mu^{\nu+1} \hat{r} J_\nu(\lambda \hat{r})}[\lambda^\nu \kappa - 2\pi(2\pi \hat{r})^\nu(b\nu J_\nu(\lambda \hat{r}) + (b - \tilde{b})\lambda \hat{r} J'_\nu(\lambda \hat{r}))].$$

Combining with (7.9) it follows from (7.3) that

$$\tilde{\gamma} = -\frac{K_\nu(\mu \hat{r})}{\mu^\nu J_\nu(\lambda \hat{r})}(\lambda^\nu \kappa - 2\pi(2\pi \hat{r})^\nu Q(\hat{r})), \tag{7.12}$$

where

$$Q(r) := (b - \tilde{b})\lambda r J'_\nu(\lambda r) + \left(b\nu - \tilde{b}\frac{\mu r K'_\nu(\mu r)}{K_\nu(\mu r)}\right) J_\nu(\lambda r). \tag{7.13}$$

By (7.4) we see that $Q(r)$ is bounded from above for $0<r<z_\lambda$. Therefore, we obtain $\tilde{\gamma}<0$ for any $\hat{r}\in(0, z_\lambda)$ (hence for any $\alpha\in\boldsymbol{R}$) if $\kappa>0$ is large enough. q.e.d.

PROOF OF THEOREM 4. From (F) and (F_4) we can choose positive constants λ and b such that

$$\hat{f}(s):=\min\{\mu^2 s, -\lambda^2(s-b)\}\geq f(s) \qquad \text{for} \quad s\geq 0\,. \tag{7.14}$$

Then, by Lemma 10 the following problem has no radial solution $u\in C^2(\boldsymbol{R}^N\setminus\{0\})$ provided that $\kappa>0$ is large:

$$\begin{cases} -\Delta u+\hat{f}(u)=\kappa\delta_0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}^N)\,, \quad u>0 \quad \text{in } \boldsymbol{R}^N\setminus\{0\}\,, \\ u(x)\sim\kappa E(x) \quad \text{as } x\to 0 \quad \text{and} \quad u(x)\to 0 \quad \text{as } |x|\to+\infty\,. \end{cases} \tag{7.15}$$

If (P_1) has a solution $\tilde{u}\in C^2(\boldsymbol{R}^N\setminus\{0\})$ for such $\kappa>0$, then $\tilde{u}$ is a supersolution to (7.15) with $\beta:=\hat{f}(\tilde{u})-f(\tilde{u})$. Therefore ($P_1$) has a radial solution by virtue of Proposition 1, which is a contradiction. q.e.d.

REMARK 4. Theorem 4 claims that (P_1) has neither radial solution nor nonradial solution for large intensity $\kappa>0$.

Appendix: Proof of Lemma 4. For a nonlinearity f satisfying (4.13), by taking $\mu>0$ large if necessary, we may assume that

$$|f(s)-f(s')|\leq\mu^2|s-s'| \qquad \text{for} \quad s, s'\geq 0\,. \tag{A.1}$$

When we consider the radial case, we obtain the following ordinary differential equation (recall that $N=2$):

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{du}{dr}\right]=f(u) \qquad \text{for} \quad r>0\,, \tag{A.2}$$

$$\begin{cases} u(r)>0 & \text{for} \quad r>0\,, \\ u(r)\sim\kappa E(r) & \text{as} \quad r\to 0\,, \\ u(r)\to 0 & \text{as} \quad r\to+\infty\,. \end{cases} \tag{A.3}$$

Equation (A.2) is reduced to the system of first order equations

$$\begin{cases} \dfrac{du}{dr}=v\,, \\ \dfrac{dv}{dr}=f(u)-\dfrac{v}{r}\,. \end{cases} \tag{A.4}$$

Extend f on $\boldsymbol{R}$ by putting $f(s):=0$ for $s<0$ and set

$$F(s):=\int_0^s f(t)dt \quad \text{for } s\in\boldsymbol{R} \quad \text{and} \quad T(s,t):=\frac{1}{2}t^2-F(s) \quad \text{for } (s,t)\in\boldsymbol{R}^2\,. \tag{A.5}$$

Then, for any solution (u, v) to (A.4) we have

$$\frac{d}{dr}[T(u, v)] = -\frac{1}{r}v^2 \leq 0 , \tag{A.6}$$

that is, $T(u, v)$ is decreasing along a solution to (A.4).

LEMMA A.1. *Suppose that* (u, v) *is a solution to* (A.4) *which satisfies*

$$u(r_0) = b \quad \textit{and} \quad v(r_0) < 0 \tag{A.7}$$

for some $r_0 > 0$. *Then one of the following holds*:

$$\begin{cases} \text{(i)} \quad u(r_1) = 0 , & v(r_1) < 0 \quad \textit{for some } r_1 > r_0 , \\ \text{(ii)} \quad 0 < u(r) < b , & v(r) < 0 \quad \textit{for any } r > r_0 \\ \qquad \textit{and} \quad (u(r), v(r)) \to (0, 0) & \textit{as } r \to +\infty , \\ \text{(iii)} \quad 0 < u(r_1) < b , & v(r_1) = 0 \quad \textit{for some } r_1 > r_0 . \end{cases} \tag{A.8}$$

Furthermore, if

$$T(u(\hat{r}), v(\hat{r})) < 0 \quad \textit{and} \quad u(\hat{r}) > 0 \tag{A.9}$$

for some $\hat{r} > 0$, *then* (iii) *holds.*

PROOF. By (A.4) and the fact $f \geq 0$, u is decreasing and v is increasing when $v < 0$. Hence for (u, v) satisfying (A.7), one of the following holds:

$$\begin{cases} \text{(i)}' \quad u(r_1) = 0 , & v(r_1) \leq 0 \quad \text{for some } r_1 > r_0 , \\ \text{(ii)}' \quad 0 < u(r) < b , & v(r) < 0 \quad \text{for any } r > r_0 , \\ \text{(iii)}' \quad 0 \leq u(r_1) < b , & v(r_1) = 0 \quad \text{for some } r_1 > r_0 . \end{cases} \tag{A.10}$$

In the cases (i)′ and (iii)′, if

$$u(r_1) = v(r_1) = 0 , \tag{A.11}$$

then $(u, v) \equiv (0, 0)$ is the only solution to (A.4) with the initial value (A.11) at $r = r_1$ by the uniqueness theorem for the initial value problem. This contradicts (A.7), and hence (i)′ and (iii)′ imply (i) and (iii) in (A.8), respectively.

In the case (ii)′ there exist the limits u^* of $u(r)$ and v^* of $v(r)$ as $r \to +\infty$ satisfying $0 \leq u^* < b$ and $v^* \leq 0$. If $v^* < 0$, then we have

$$u^* - u(r_0) = \int_{r_0}^{+\infty} v(r) dr \leq \int_{r_0}^{+\infty} v^* dr = -\infty ,$$

which is a contradiction, hence $v^* = 0$. If $u^* > 0$, then it follows

$$-v(r_0) = v^* - v(r_0) = \int_{r_0}^{+\infty} \left[f(u(r)) - \frac{v(r)}{r} \right] dr \geq \int_{r_0}^{+\infty} f(u(r)) dr = +\infty$$

since $f(u^*)>0$, which is a contradiction. Therefore $u^*=0$ and hence (ii) holds.

Finally, if (A.9) holds, then the possibility of (i) and (ii) is ruled out by (A.6) and the fact $T(0, v)=v^2/2\geq 0$, and hence (iii) holds. q.e.d.

Now we fix $\kappa>0$ arbitrarily. Then by (4.13), any solution u to (A.2) satisfies

$$u(r)=\frac{\kappa}{2\pi}\log\frac{1}{r}+\alpha \qquad \text{for} \quad 0<r<\hat{r}_\alpha:=\exp\left[-2\pi\frac{b-\alpha}{\kappa}\right] \tag{A.12}$$

with some constant $\alpha\in\boldsymbol{R}$. We denote the solution to (A.4) satisfying (A.12) by (u_α, v_α) for $\alpha\in\boldsymbol{R}$. Then $\{(u_\alpha, v_\alpha)\}_{\alpha\in\boldsymbol{R}}$ is continuous in α with respect to the topology of locally uniform convergence on $(0, +\infty)$. If we set

$$\begin{cases} A^{(\mathrm{i})}:=\{\alpha\in\boldsymbol{R}\,|\,(u_\alpha, v_\alpha) \text{ satisfies (A.8) (i)}\}\,, \\ A^{(\mathrm{ii})}:=\{\alpha\in\boldsymbol{R}\,|\,(u_\alpha, v_\alpha) \text{ satisfies (A.8) (ii)}\}\,, \\ A^{(\mathrm{iii})}:=\{\alpha\in\boldsymbol{R}\,|\,(u_\alpha, v_\alpha) \text{ satisfies (A.8) (iii)}\}\,, \end{cases} \tag{A.13}$$

then $A^{(\mathrm{i})}$ and $A^{(\mathrm{iii})}$ are open in $\boldsymbol{R}$ and

$$\boldsymbol{R}=A^{(\mathrm{i})}\cup A^{(\mathrm{ii})}\cup A^{(\mathrm{iii})} \text{ (disjoint union)}\,. \tag{A.14}$$

The following lemma implies that $A^{(\mathrm{ii})}$ is nonempty, and Lemma 4 holds true.

LEMMA A.2. *$A^{(\mathrm{i})}$ and $A^{(\mathrm{iii})}$ are nonempty for any $\kappa>0$.*

PROOF. We first show that $A^{(\mathrm{iii})}$ is nonempty. From (A.12) we obtain

$$T(u_\alpha(\hat{r}_\alpha), v_\alpha(\hat{r}_\alpha))=\frac{1}{2}\left(\frac{\kappa}{2\pi}\right)^2\exp\left[4\pi\frac{b-\alpha}{\kappa}\right]-F(b)$$

and $T(u_\alpha(\hat{r}_\alpha), v_\alpha(\hat{r}_\alpha))<0$ for $\alpha\gg 1$ since $F(b)>0$. Therefore, $A^{(\mathrm{iii})}$ is nonempty by Lemma A1.

Next we show that $A^{(\mathrm{i})}$ is nonempty. For $R>0$ and $q\in(1, +\infty)$, we set

$$\begin{aligned} L^q(B_R)_{\mathrm{r}}:&=\{\varphi\in L^q(B_R)\,|\,\varphi \text{ is radial}\} \\ &=\left\{\varphi:(0, R)\to\boldsymbol{R}\,\middle|\,\varphi \text{ is measurable and } \|\varphi\|_{L^q(B_R)}:=\left[2\pi\int_0^R|\varphi(r)|^q r\,dr\right]^{1/q}<+\infty\right\}, \end{aligned} \tag{A.15}$$

and define a mapping Φ on $L^q(B_R)_{\mathrm{r}}$ by

$$\Phi[\varphi](r):=-\int_r^R s^{-1}\int_0^s f(\varphi(t)+\kappa E(t))t\,dt\,ds-\kappa E(R) \qquad \text{for} \quad 0<r<R\,, \tag{A.16}$$

for $\varphi\in L^q(B_R)_{\mathrm{r}}$. Then we obtain from (A.1)

$$\begin{cases} \|\Phi[\varphi]\|_{L^q(B_R)}\leq\hat{c}_q(\mu R)^2\|\varphi+\kappa E\|_{L^q(B_R)}+\kappa(\pi R^2)^{1/q}|E(R)|\,, \\ \|\Phi[\varphi]-\Phi[\psi]\|_{L^q(B_R)}\leq\hat{c}_q(\mu R)^2\|\varphi-\psi\|_{L^q(B_R)} \end{cases}$$

for $\varphi, \psi \in L^q(B_R)_r$, where $\hat{c}_q = (1/4)[(q')^{q+1}B(q+1, q')]^{1/q}$ ($B(\cdot, \cdot)$ is the beta function). Therefore Φ: $L^q(B_R)_r \to L^q(B_R)_r$ is a contraction mapping if $\hat{c}_q(\mu R)^2 < 1$, and hence there exists a unique $w \in L^q(B_R)_r$ such that

(A.17) $$w(r) = -\int_r^R s^{-1} \int_0^s f(w(t) + \kappa E(t))t\,dt\,ds - \kappa E(R) \quad \text{for} \quad 0 < r < R .$$

We can easily see that

(A.18) $$u := w + \kappa E$$

satisfies $u \in C(0, R] \cap C^2(0, R)$ and

$$\begin{cases} \dfrac{1}{r}\dfrac{d}{dr}\left[r\dfrac{du}{dr}\right] = f(u) \quad \text{and} \quad u(r) > 0 \quad \text{for } 0 < r < R , \\ u(r) \sim \kappa E(r) \quad \text{as } r \to 0 \quad \text{and} \quad u(R) = 0 . \end{cases}$$

Therefore $u = u_\alpha$ for some $\alpha \in \boldsymbol{R}$, and $\alpha \in \boldsymbol{A}^{(i)}$. q.e.d.

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