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STABILITY PROPERTIES AND INTEGRABILITY OF THE RESOLVENT OF LINEAR VOLTERRA EQUATIONS

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Abstract. Integrability of the resolvent and the stability properties of the zero solution of linear Volterra integrodifferential systems are studied. In particular, it is shown that, the zero solution is uniformly stable if and only if the resolvent is integrable in some sense. It is also shown that, the zero solution is uniformly asymptotically stable if and only if the resolvent is integrable and an additional condition in terms of the resolvent and the kernel is satisfied. Finally, the integrability of the resolvent is obtained under an explicit condition.

1. Introduction. Let n be a positive integer. Consider the Volterra integrodifferential system

$$x'(t)=A(t)x(t)+\int_0^t B(t, s)x(s)ds \tag{1}$$

where A and B are square matrices of order n, $A(t)$ is continuous for $0\le t<\infty$, $B(t, s)$ is continuous for $0\le s\le t<\infty$, $B(t, s)=0$ for $s>t$, and $B(t, s)$ is integrable in the sense that

$$\sup_{t\ge 0}\int_0^t |B(t, s)|ds<\infty .$$

Here $|\cdot|$ denotes the matrix norm induced by a vector norm, also denoted $|\cdot|$, in R^n.

For any $t_0\ge 0$ and any continuous function ϕ: $[0, t_0]\to R^n$, there exists a unique solution $x(t)$ of (1) for $t\ge t_0$, where $x(t)=\phi(t)$ on $[0, t_0]$. If $x(t, t_0, \phi)$ denotes the solution of (1) for an initial pair (t_0, ϕ), then

$$\begin{aligned} x'(t+t_0, t_0, \phi)=A(t+t_0)x(t+t_0)&+\int_0^t B(t+t_0, s+t_0)x(s+t_0)ds \\ &+\int_0^{t_0} B(t+t_0, s)\phi(s)ds , \end{aligned} \tag{2}$$

for $t\ge 0$ where $x(t_0, t_0, \phi)=\phi(t_0)$. Using the variation of parameters formula, the solution of the initial value problem (2) is given by

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(3) $$x(t+t_0, t_0, \phi)=R(t+t_0, t_0)\phi(t_0)+\int_0^t R(t+t_0, s+t_0)\left[\int_0^{t_0} B(s+t_0, u)\phi(u)du\right]ds$$

where $R(t, s)$ is the solution of the resolvent equation

(4) $$\frac{\partial R(t, s)}{\partial s}=-R(t, s)A(s)-\int_s^t R(t, u)B(u, s)du\,, \qquad R(t, t)=I\,,$$

(see Grossman and Miller [7]). The function in (3) is the form of the solution of (1) that is needed in the sequel.

The resolvent $R(t, s)$ is said to be integrable if

(5) $$\sup_{t\geq 0}\int_0^t |R(t, s)|\,ds<\infty\,.$$

In this paper we study the integrability property (5), and the relationship between (5) and the stabilities of the zero solution $x(t)\equiv 0$ of (1). In particular, we show in Section 2, that the zero solution of (1) is uniformly stable (u.s.) if and only if (5) holds. We also show that, the zero solution of (1) is uniformly asymptotically stable (u.a.s.) if and only if (5) holds and an additional condition in terms of R and B, given in Theorem 1, is satisfied. We refer the readers to Burton [4, p. 34] or Miller [13] for definitions of these stabilities. If $A(t)\equiv A$, a constant matrix, and $B(t, s)=\tilde{B}(t-s)$, a convolution kernel, then the uniform asymptotic stability of the zero solution of (1) is equivalent to the integrability of the resolvent (Miller [13]). Some conditions for the uniform asymptotic stability of the zero solution of (1) when $A(t)\equiv A$, and $B(t, s)=\tilde{B}(t-s)$, are available in Brauer [2] and Jordan [10].

Burton [5], Grimmer and Seifert [6], and Mahfound [12] studied the uniform stability of the zero solution of (1) through the uniform boundedness of the solutions of a related nonhomogeneous equation. Burton considered the scalar case of (1), and Grimmer and Seifert assumed $A(t)\equiv A$, a constant stable matrix. Since the uniform stability of the zero solution of (1) is equivalent to the integrability of the resolvent (Theorem 1), the conditions given in [5], [6], [12] will provide the property (5). When (1) is scalar, a more explicit condition in terms of A and B is given by Kato [11, Th. 8(I)] for the uniform stability of the zero solution of (1) and hence for the integrability property (5). The condition by Kato states that if

(6) $$A(t)+\int_0^t |B(t, s)|\,ds\leq 0$$

for all $t\geq 0$ then the zero solution of (1) is u.s. One would like to obtain uniform stability or perhaps the uniform asymptotic stability of the zero solution of the system of equations (1) under conditions like (6). Since property (5) is equivalent to the uniform stability, one may try to obtain property (5) instead. In Theorem 2 of Section 3, we obtain (5) under a condition similar to (6) for the scalar case. We hope that Theorem

2 will provide a new approach by which one may attempt to prove the uniform stability of the zero solution of the system of equation (1) through the integrability of the resolvent.

2. Stability properties and integrability of the resolvent.

LEMMA 1. *Let* $|A(t)| \leq A^*$ for $t \geq 0$. *If* (5) *holds then* $|R(t, s)|$ *is uniformly bounded for* $0 \leq s \leq t < \infty$, *and* $|R(t, s)| \to 0$ *as* $t - s \to \infty$ *uniformly.*

PROOF. Let

$$\sup_{t \geq 0} \int_0^t |R(t, s)| ds \leq R^*, \quad \text{and} \quad \sup_{t \geq 0} \int_0^t |B(t, s)| ds \leq B^*.$$

As the solution of (4), one obtains

$$R(t, s) = I + \int_s^t R(t, u)A(u)du + \int_s^t \int_v^t R(t, u)B(u, v)dudv.$$

This implies that for $0 \leq s \leq t < \infty$,

$$\begin{aligned} |R(t, s)| &\leq 1 + A^* \int_0^t |R(t, u)| du + \int_0^t |R(t, u)| \int_0^u |B(u, v)| dvdu \\ &\leq 1 + A^*R^* + R^*B^*. \end{aligned}$$

Let $|R(u, s)| \leq \bar{R}$ for $0 \leq s \leq u < \infty$. Then

$$\left| \int_s^t R(t, u)R(u, s)du \right| \leq \int_s^t |R(t, u)| |R(u, s)| du \leq \int_0^t |R(t, u)| \bar{R}du \leq R^*\bar{R}.$$

One can easily verify that $R(t, u)R(u, s) = R(t, s)$. Then, the first expression of the above inequality is equal to $(t-s)|R(t, s)|$. This implies that $|R(t, s)| \to 0$ as $t - s \to \infty$ uniformly.

THEOREM 1. *Let* $|A(t)| \leq A^*$ *for* $t \geq 0$.

(i) *The zero solution of* (1) *is u.s. if and only if* (5) *holds.*

(ii) *The zero solution of* (1) *is u.a.s. if and only if* (5) *holds and for every* $\alpha \geq 0$,

$$\int_0^\alpha \left| \int_\alpha^{t+\alpha} R(t+\alpha, s)B(s, u)ds \right| du \to 0 \quad \text{as } t \to \infty \text{ uniformly}.$$

PROOF. (i) Suppose the zero solution of (1) is u.s. Then all solutions of

$$y'(t) = A(t)y(t) + \int_0^t B(t, s)y(s)ds + f(t) \tag{N}$$

are uniformly bounded for every bounded f (Burton [5], Mahfoud [12]). From the variation of parameters formula, the solution $y(t, 0, 0)$ of (N) is given by

$$y(t, 0, 0) = R(t, 0)0 + \int_0^t R(t, s)f(s)ds .$$

The rest of the proof follows from Perron's theorem (Hale [8, pp. 152–153]) for (5) to hold.

Conversely, suppose (5) holds. Then from (3), one gets

$$|x(t+t_0, t_0, \phi)| \leq |\phi|_{t_0}[\bar{R} + R^*B^*] ,$$

where $\bar{R}$, R^*, and B^* are the constants used in Lemma 1. Here $|\phi|_{t_0} = \sup_{0 \leq s \leq t_0} |\phi(s)|$. This proves that the zero solution of (1) is u.s.

(ii) Suppose the zero solution of (1) is u.a.s. Then the zero solution of (1) is u.s. and hence, from (i), (5) holds. Then by Lemma 1, $|R(t, s)| \to 0$ as $t-s \to \infty$ uniformly. Let (t_0, ϕ) be any initial pair with $|\phi|_{t_0} \leq 1$. From (3), one has

$$\begin{aligned}\left|\int_0^t R(t+t_0, s+t_0)\left[\int_0^{t_0} B(s+t_0, u)\phi(u)du\right]ds\right| &= |x(t+t_0, t_0, \phi) - R(t+t_0, t_0)\phi(t_0)| \\ &\leq |x(t+t_0, t_0, \phi)| + |R(t+t_0, t_0)|\,|\phi(t_0)| \to 0 \qquad \text{as} \quad t \to \infty \text{ uniformly} .\end{aligned}$$

Changing the order of integrations in the first expression of the above relation one obtains

$$\left|\int_0^{t_0} \int_{t_0}^{t+t_0} R(t+t_0, s)B(s, u)ds\phi(u)du\right| .$$

This means

$$\int_0^{t_0} \left|\int_{t_0}^{t+t_0} R(t+t_0, s)B(s, u)ds\right| du \to 0 \qquad \text{as} \quad t \to \infty \text{ uniformly} .$$

Conversely, since (5) holds, $|R(t, s)| \to 0$ as $t \to \infty$ uniformly by Lemma 1, and the zero solution of (1) is u.s. by (i). Now, from (3), we get

$$|x(t+t_0, t_0, \phi)| \leq |R(t+t_0, t_0)|\,|\phi(t_0)| + \int_0^{t_0} \left|\int_{t_0}^{t+t_0} R(t+t_0, s)B(s, u)ds\right| |\phi(u)|\,du$$

which tends to zero as $t \to \infty$ uniformly. This proves that the zero solution of (1) is u.a.s.

3. An integrability result for the resolvent. In this section, for the scalar equations (1)–(4), we shall obtain a condition on A and B under which the integrability property (5) holds. A Liapunov function is used in the analysis.

We shall first introduce the properties that we shall require in that Liapunov function. Let V: $R^2 \to R$ be a continuous function. For any continuous, real valued function, $x(s)$, define

$$V'(s, x(s)) = \liminf_{h \to 0^+} \left[\frac{1}{h}(V(s+h, x(s+h)) - V(s, x(s)) \right].$$

LEMMA 2. *If* $V'(s, x(s)) \geq 0$ *for all* s, *then*

$$\int_0^t V'(s, x(s))ds \leq V(t, x(t)) - V(0, x(0)), \qquad t \geq 0.$$

PROOF. Since $V'(s, x(s)) \geq 0$, we see that $V(s, x(s))$ is nondecreasing in s [14, p. 4]. Therefore, on $[0, t]$,

$$\frac{1}{h}[V(s+h, x(s+h)) - V(s, x(s))] \geq 0.$$

Lemma 2 now follows from Fatou's lemma.

THEOREM 2. *Let* (1)–(4) *be scalar equations. Suppose there exists a real* $\alpha > 0$ *such that*

$$A(t) + \int_0^t |B(t, s)| ds \leq -\alpha, \qquad t \geq 0. \tag{7}$$

Then (5) *holds.*

PROOF. Let $R(t, s)$ be the solution of (4). For each $t \geq 0$ denote $R(t, s)$ by $R_t(s)$. Then (4) becomes

$$R_t'(s) = -R_t(s)A(s) - \int_s^t R_t(u)B(u, s)du. \tag{8}$$

Define $V(s, R_t(s))$ by

$$V(s, R_t(s)) = |R_t(s)| + \int_s^t \int_0^s |R_t(u)| |B(u, m)| dmdu, \qquad 0 \leq s \leq t. \tag{9}$$

Notice that $V(s, R_t(s)) \geq 0$ and $V(t, R_t(t)) = 1$.

If $R_t(s) \neq 0$ then differentiating (9) and using (8), we get

$$\begin{aligned} V'(s, R_t(s)) &= \frac{R_t(s)}{|R_t(s)|} \left\{ -R_t(s)A(s) - \int_s^t R_t(u)B(u, s)du \right\} \\ &\quad + \int_s^t |R_t(u)| |B(u, s)| du - \int_0^s |R_t(s)| |B(s, m)| dm \\ &\geq -|R_t(s)| A(s) - \frac{|R_t(s)|}{|R_t(s)|} \int_s^t |R_t(u)| |B(u, s)| du \end{aligned}$$

$$+\int_s^t |R_t(u)|\,|B(u,s)|\,du-\int_0^s |R_t(s)|\,|B(s,m)|\,dm$$

$$=-|R_t(s)|\left\{A(s)+\int_0^s |B(s,m)|\,dm\right\}\ge \alpha\,|R_t(s)|>0\,.$$

Now, suppose $R_t(s)=0$. Let $I_t(s)$ be the second term of the right hand side of (9). Define

$$V'(s,R_t(s))=\liminf_{h\to 0^+}\frac{1}{h}[(|R_t(s+h)|+I_t(s+h))-(|R_t(s)|+I_t(s))]\,. \tag{10}$$

Since $|R_t(s)|=0$, we get from (10),

$$V'(s,R_t(s))\ge \liminf_{h\to 0^+}\frac{1}{h}|R_t(s+h)|+\liminf_{h\to 0^+}\frac{1}{h}[I_t(s+h)-I_t(s)]$$

$$\ge 0+\frac{d}{ds}I_t(s)=\int_s^t |R_t(u)|\,|B(u,s)|\,du\ge 0=\alpha\,|R_t(s)|\,.$$

So, in both cases ($R_t(s)=0$ and $R_t(s)\ne 0$), we obtain

$$V'(s,R_t(s))\ge \alpha\,|R_t(s)|\ge 0\,. \tag{11}$$

Therefore, $V(s,R_t(s))$ is nondecreasing in s. We already noticed that $V(s,R_t(s))\ge 0$ and $V(t,R_t(t))=1$. Thus, for $t\ge 0$, we have $0\le V(0,R_t(0))\le V(t,R_t(t))=1$. Now, from Lemma 2 and from (11), we get

$$\alpha\int_0^t |R_t(s)|\,ds\le \int_0^t V'(s,R_t(s))ds\le V(t,R_t(t))=1\,.$$

This implies that

$$\int_0^t |R(t,s)|\,ds\le \frac{1}{\alpha}\,,\qquad t\ge 0\,,$$

and the proof is complete.

REMARK. Suppose for some $T>0$,

$$A(t+T)=A(t)\,,\qquad -\infty<t<\infty\,, \tag{12}$$

$$B(t+T,s+T)=B(t,s)\,,\qquad -\infty<s\le t<\infty\,.$$

Then by [3] $R(t,s)$ satisfies

$$R(t+T,s+T)=R(t,s)\,,\qquad -\infty<s\le t<\infty\,. \tag{13}$$

For this case Becker, Burton, and Krisztin [1] presented some conditions which characterizes the integrability property (5).

Islam [9] proved that if (12) holds then for a T-periodic f,

$$x'(t)=A(t)x(t)+\int_{-\infty}^{t} B(t, s)x(s)ds+f(t),$$

has a unique T-periodic solution given by

$$x(t)=\int_{-\infty}^{t} R(t, s)f(s)ds.$$

Islam [9] assumed that the resolvent $R(t, s)$ is integrable in the following sense:

$$\sup_{-\infty<t<\infty}\int_{-\infty}^{t} |R(t, s)|\,ds<\infty. \tag{14}$$

We remark that if $R(t, s)$ satisfies (13), then (14) holds if and only if (5) holds.

References

[1] L. C. Becker, T. A. Burton and T. Krisztin, Floquet theory for a Volterra equation, J. London Math. Soc. 37 (1988), 141–147.

[2] F. Brauer, Asymptotic stability of a class of integro-differential equation, J. Differential Equations 28 (1978), 180–188.

[3] T. A. Burton, Periodicity and limiting equations in Volterra systems, Boll. Un. Mat. Ital. C 1 (1985), 31–39.

[4] T. A. Burton, Volterra Integral and Differential Equations, Academic Press, 1983.

[5] T. A. Burton, Perturbed Volterra equations, J. Differential Equations 43 (1982), 168–183.

[6] R. Grimmer and G. Seifert, Stability properties of Volterra integrodifferential equations, J. Differential Equations 19 (1975), 142–166.

[7] S. I. Grossman and R. K. Miller, Perturbation theory for Volterra integrodifferential systems, J. Differential Equations 8 (1970), 457–474.

[8] J. K. Hale, Ordinary Differential Equations, Wiley, New York, 1969.

[9] M. N. Islam, Bounded solutions and periodic solutions of Volterra equations with infinite delay, Appl. Anal. 40 (1991), 53–65.

[10] G. S. Jordan, Asymptotic stability of a class of integrodifferential systems, J. Differential Equations 31 (1979), 359–365.

[11] J. Kato, Liapunov functional vs Liapunov function, Proc. Internat. Symposium on Functional Differential Equations, World Scientific, Singapore, 1991.

[12] W. E. Mahfoud, Boundedness properties in Volterra integrodifferential systems, Proc. Amer. Math. Soc. 100 (1987), 37–45.

[13] R. K. Miller, Asymptotic stability properties of linear Volterra integrodifferential equations, J. Differential Equations 10 (1971), 485–506.

[14] T. Yoshizawa, Stability theory by Liapunov's second method, Math. Soc. Japan, Tokyo, 1966.

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