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CHARACTERIZING A CLASS OF TOTALLY REAL SUBMANIFOLDS OF S^6 BY THEIR SECTIONAL CURVATURES

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Abstract. The first author introduced in a previous paper an important Riemannian invariant for a Riemannian manifold, namely take the scalar curvature function and subtract at each point the smallest sectional curvature at that point. He also proved a sharp inequality for this invariant for submanifolds of real space forms. In this paper we study totally real submanifolds in the nearly Kähler six-sphere that realize the equality in that inequality. In this way we characterize a class of totally real warped product immersions by one equality involving their sectional curvatures.

1. Introduction. In [C], the first author gives a general best possible inequality between the main intrinsic invariants of a submanifold M^n in a Riemannian space form $\tilde{M}^m(c)$, namely its sectional curvature function K and its scalar curvature function τ, and the main extrinsic invariant, namely its mean curvature function $\|H\|$, H being the mean curvature vector field of M in $\tilde{M}$. It is convenient to define a Riemannian invariant δ_M of M^n by

$$\delta_M(p)=\tau(p)-\inf K(p)\,,$$

where $\inf K$ is the function assigning to each $p\in M^n$ the infimum of $K(\pi)$, where π runs over all planes in T_pM and τ is defined by $\tau=\sum_{i<j}K(e_i\wedge e_j)$. The inequality can be written as follows.

$$\delta_M\leq\frac{n^2(n-2)}{2(n-1)}\|H\|^2+\frac{1}{2}(n+1)(n-2)c\,. \tag{1.1}$$

He then started to investigate those submanifolds, with dimension $n\geq 3$, for which the above inequality actually becomes an equality, i.e. submanifolds which satisfy

$$\delta_M=\frac{n^2(n-2)}{2(n-1)}\|H\|^2+\frac{1}{2}(n+1)(n-2)c\,. \tag{1.2}$$

For such submanifolds, a distribution can be defined by

$$\mathscr{D}(p)=\{X\in T_pM\,|\,(n-1)h(X,Y)=n\langle X,Y\rangle H,\ \forall Y\in T_pM\}\,.$$

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If the dimension of $\mathscr{D}(p)$ is constant, it is shown in [C] that the distribution $\mathscr{D}$ is completely integrable.

In this paper, we investigate 3-dimensional totally real submanifolds in the nearly Kähler 6-sphere $S^6(1)$. Since such a submanifold is always minimal (cf. [E3]), we get

$$\delta_M \leq 2 . \tag{1.3}$$

When M has constant scalar curvature (τ is constant), a complete classification of submanifolds satisfying the equality in (1.3) has been obtained in [CDVV1]. Here, we will investigate those totally real 3-dimensional submanifolds in $S^6(1)$ which satisfy:

(1) $\delta_M = 2$,

(2) the dimension of the distribution $\mathscr{D}$ is constant (and hence it is a completely integrable distribution),

(3) the distribution $\mathscr{D}^\perp$ is also integrable.

We will relate submanifolds satisfying the above conditions to minimal (non-totally geodesic) totally real immersions of surfaces N^2 into $S^6(1)$ whose ellipse of curvature is a circle. The ellipse of curvature of a surface at a point p is the set $\{h(u,u) \mid u \in T_pM, \|u\| = 1\}$ in the normal space, where h is the second fundamental form. It is shown in [BVW] that every such immersion is linearly full in a totally geodesic S^5. An alternative proof of this will be given in Section 5. Other characterizations will be given below. The Main Theorem we prove here is:

MAIN THEOREM. *Let $f : M^2 \to S^6(1)$ be a minimal (non-totally geodesic) totally real immersion in $S^6(1)$ whose ellipse of curvature is a circle. Then M^2 is linearly full in a totally geodesic S^5. Let N be a unit vector perpendicular to this S^5. Then*

$$x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times M^2 \to S^6(1),\ (t,p) \mapsto \sin(t)N + \cos(t)f(p) \tag{1.4}$$

is a totally real immersion which satisfies the equality in (1.3). Conversely, every totally real (non-totally geodesic) immersion of M^3 into $S^6(1)$ satisfying

(1) $\delta_M = 2$,

(2) the dimension of $\mathscr{D}$ is constant,

(3) $\mathscr{D}^\perp$ is an integrable distribution,

can be locally obtained in this way.

2. The nearly Kähler structure on $S^6(1)$.

We give a brief explanation of how the standard nearly Kähler structure on $S^6(1)$ arises in a natural manner from Cayley multiplication. For elementary facts about the Cayley numbers and their automorphism group G_2, we refer the reader to Section 4 of [W] and to [HL].

The multiplication on the Cayley numbers $\mathcal{O}$ may be used to define a vector crossproduct on the purely imaginary Cayley numbers $\boldsymbol{R}^7$ using the formula

$$u \times v = \frac{1}{2}(uv - vu) , \tag{2.1}$$

while the standard inner product on $\boldsymbol{R}^7$ is given by

$$\langle u, v \rangle = -\frac{1}{2}(uv + vu) . \tag{2.2}$$

It is now elementary to show that

$$u \times (v \times w) + (u \times v) \times w = 2\langle u, w \rangle v - \langle u, v \rangle w - \langle w, v \rangle u , \tag{2.3}$$

and that the triple scalar product $\langle u \times v, w \rangle$ is skew symmetric in u, v, w.

Conversely, Cayley multiplication of $\mathcal{O}$ is given in terms of the vector crossproduct and the inner product by

$$(r+u)(s+v) = rs - (u, v) + rv + su + (u \times v) , \qquad r, s \in \operatorname{Re} \mathcal{O} , \quad u, v \in \operatorname{Im} \mathcal{O} . \tag{2.4}$$

In view of (2.1), (2.2) and (2.4), it is clear that the group G_2 of automorphisms of $\mathcal{O}$ is precisely the group of isometries of $\boldsymbol{R}^7$ which preserve the vector crossproduct.

An ordered orthonormal basis $\{e_1, \ldots, e_7\}$ of $\boldsymbol{R}^7$ is said to be *canonical* if

$$e_3 = e_1 \times e_2 , \quad e_5 = e_1 \times e_4 , \quad e_6 = e_2 \times e_4 , \quad e_7 = e_3 \times e_4 . \tag{2.5}$$

For example, the standard basis of $\boldsymbol{R}^7$ is canonical. Moreover, if e_1, e_2, e_4 are mutually orthogonal unit vectors with e_4 orthogonal to $e_1 \times e_2$, then e_1, e_2, e_4 determine a unique canonical basis $\{e_1, \ldots, e_7\}$ and $(\boldsymbol{R}^7, \times)$ is generated by e_1, e_2, e_4 subject to the relations

$$e_i \times (e_j \times e_k) + (e_i \times e_j) \times e_k = 2\delta_{ik} e_j - \delta_{ij} e_k - \delta_{jk} e_i . \tag{2.6}$$

Given any two canonical bases $\{e_1, \ldots, e_7\}$ and $\{f_1, \ldots, f_7\}$ of $\boldsymbol{R}^7$, there is a unique element $g \in G_2$ such that $ge_i = f_i$; and thus g is uniquely determined by ge_1, ge_2, ge_4.

Let J be the automorphism of the tangent bundle $TS^6(1)$ of $S^6(1)$ defined by

$$Ju = x \times u , \qquad u \in T_x S^6(1) , \quad x \in S^6(1) .$$

It is clear that J is an almost complex structure on $S^6(1)$ and in fact J is a nearly Kähler structure on $S^6(1)$ in the sense that $(\tilde{\nabla}_u J)u = 0$, for any vector u tangent to $S^6(1)$, where $\tilde{\nabla}$ is the Levi-Civita connection of $S^6(1)$. We define by

$$G(X, Y) = (\tilde{\nabla}_X J)(Y) ,$$

the corresponding skew-symmetric (2,1)-tensor field. From [S], we know that this tensor field has the following properties:

$$G(X, JY) + JG(X, Y) = 0 , \tag{2.7}$$

$$(\tilde{\nabla} G)(X, Y, Z) = \langle Y, JZ \rangle X + \langle X, Z \rangle JY - \langle X, Y \rangle JZ , \tag{2.8}$$

(2.9) $$\langle G(X, Y), Z\rangle+\langle G(X, Z), Y\rangle=0\,,$$

(2.10) $$\langle G(X, Y), G(Z, W)\rangle=\langle X, Z\rangle\langle Y, W\rangle-\langle X, W\rangle\langle Z, Y\rangle +\langle JX, Z\rangle\langle Y, JW\rangle-\langle JX, W\rangle\langle Y, JZ\rangle\,,$$

(2.11) $$G(X, Y)=X\times Y+\langle X, JY\rangle x\,.$$

It is clear from the above that G_2 acts transitively on $S^6(1)$ and that the stabilizer of the point $(1, 0, \dots, 0)$ is $SU(3)$. It follows that G_2, a connected subgroup of $SO(7)$ of dimension 14, is the group of automorphisms of the nearly Kähler structure J.

3. Warped product immersions. Let $M_0, \dots, M_k$ be Riemannian manifolds, M their product $M_0\times\cdots\times M_k$, and let $\pi_i\colon M\to M_i$ denote the canonical projection. If $\rho_1, \dots, \rho_k\colon M_0\to \boldsymbol{R}_+$ are positive-valued functions, then

$$\langle X, Y\rangle:=\langle \pi_{0*}X, \pi_{0*}Y\rangle+\sum_{i=1}^{k}(\rho_i\circ\pi_0)^2\langle \pi_{i*}X, \pi_{i*}Y\rangle\,, \qquad X, Y\in\Gamma(TM)$$

defines a Riemannian metric on M. We call $(M; \langle\cdot,\cdot\rangle)$ *the warped product* $M_0\times_{\rho_1}M_1\times\cdots\times_{\rho_k}M_k$ of $M_0, \dots, M_k$, and $\rho_1, \dots, \rho_k$ *the warping functions.*

Let $f_i\colon N_i\to M_i$, $i=0, \dots, k$ be isometric immersions, and define $\sigma_i:=\rho_i\circ f_0\colon N_0\to \boldsymbol{R}_+$ for $i=1, \dots, k$. Then the map $f\colon N_0\times_{\sigma_1}N_1\times\cdots\times_{\sigma_k}N_k\to M_0\times_{\rho_1}M_1\times\cdots\times_{\rho_k}M_k$ given by $f(p_0, \dots, p_k):=(f_0(p_0), f_1(p_1), \dots, f_k(p_k))$ is an isometric immersion, and is called a *warped product immersion*.

The decomposition of an immersion into warped product immersions is in particular a very powerful tool when applied to immersions into Euclidean spaces, spheres or hyperbolic spaces. In this respect, the main result from [N] can be stated as follows. Let $f\colon N_0\times_{\sigma_1}N_1\times\cdots\times_{\sigma_k}N_k\to M(c)$ be an isometric immersion into a space of constant curvature c. If h is the second fundamental form of f and $h(X_i, X_j)=0$, for all vector fields X_i and X_j, tangent to N_i and N_j respectively, with $i\neq j$, then, locally, M is a warped product immersion. The problem of how $M(c)$ can be decomposed into a warped product is solved in [N], see also [DN] for the statement.

Using warped product immersions, we can give a class of examples of minimal submanifolds in a unit sphere which satisfy the equality (1.2). Let $S^{n-2}_+(1)=\{x\in\boldsymbol{R}^{n-1}\mid \|x\|=1 \text{ and } x_1>0\}$ be an open hemisphere and let $S^{m-n+2}(1)$ be the unit hypersphere of $\boldsymbol{R}^{m-n+3}$. Then $\psi\colon S^{n-2}_+(1)\times_{x_1}S^{m-n+2}(1)\to S^m(1)$, $(x, y)\mapsto(x_1y, x_2, \dots, x_{n-1})$ is an isometry onto an open dense subset of $S^m(1)$. This can be considered as a warped product decomposition of $S^m(1)$. Now if N^2 is any minimal surface in $S^{m-n+2}(1)$, immersed by f_1, then the immersion $f\colon S^{n-2}_+(1)\times_{x_1}N^2\to S^m(1)$, $(x, p)\mapsto \psi(x, f_1(p))$ is an isometric immersion satisfying the equality (1.2). This follows trivially since the dimension of the distribution $\mathscr{D}$ is $n-2$. It is easy to see that the immersion (1.4) is a special case of this family. We now focus on (1.4).

We consider a totally geodesic $S^5(1)$ in $S^6(1)$. Let N denote the unit vector orthogonal to the hyperplane containing $S^5(1)$. We parametrize the half circle $S^1_+(1)$ by $(-\pi/2, \pi/2) \to S^1_+(1)$, $t \mapsto (\cos(t), \sin(t))$. Then the isometry ψ of the previous paragraph can also be written as $S^1_+(1) \times_{\cos(t)} S^5(1) \to S^6(1)$, $(t, p) \mapsto \sin(t)N + \cos(t)p$. Let $f: M^2 \to S^5(1)$ be an immersion of a surface into $S^5(1)$. Then the associated warped product immersion is given by

$$x: (-\pi/2, \pi/2) \times_{\cos(t)} M^2 \to S^6(1), (t, p) \mapsto \sin(t)N + \cos(t)f(p). \tag{3.1}$$

We will determine in Section 5 for which immersions f, the warped product immersion x is totally real.

4. Totally real submanifolds in $S^6(1)$. A submanifold M in $S^6(1)$ is called totally real if for any vector field X, tangent to M, JX is a normal vector field.

The dimension of M can be 2 or 3. Totally real surfaces in $S^6(1)$ were first studied in [DOVV]. Totally real 3-dimensional submanifolds were first studied in Ejiri [E3], who proved that a 3-dimensional totally real submanifold of $S^6(1)$ is orientable and minimal. In both cases it can be proved that $G(X, Y)$ is orthogonal to M, for tangent vectors X and Y.

We denote the Levi-Civita connection of M by ∇. The formulas of Gauss and Weingarten are respectively given by

$$D_X Y = \nabla_X Y + h(X, Y), \tag{4.1}$$

$$D_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{4.2}$$

for tangent vector fields X and Y and normal vector fields ξ. The second fundamental form h is related to A_ξ by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

From (4.1) and (4.2), we find that

$$\nabla^\perp_X JY = J\nabla_X Y + G(X, Y) + (Jh(X, Y))^n, \tag{4.3}$$

$$A_{JY} X = -(Jh(X, Y))^t, \tag{4.4}$$

where $(Jh(X, Y))^n$ and $(Jh(X, Y))^t$ denote the normal and tangential parts of $Jh(X, Y)$. Obviously, if $\dim M = 3$, then $Jh(X, Y)$ is tangent.

The above formulas immediately imply that $\langle h(X, Y), JZ \rangle$ is totally symmetric. If we denote the curvature tensors of ∇ and $\nabla^\perp$ by R and $R^\perp$, respectively, then the equations of Gauss, Codazzi and Ricci are given by

$$R(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y + A_{h(Y,Z)} X - A_{h(X,Z)} Y, \tag{4.5}$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z), \tag{4.6}$$

$$\langle R^\perp(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta] X, Y \rangle, \tag{4.7}$$

where X, Y, Z (respectively, η and ξ) are tangent (respectively, normal) vector fields to M and ∇h is defined by

$$(\nabla h)(X, Y, Z)=\nabla_X^{\perp}h(Y, Z)-h(\nabla_X Y, Z)-h(Y, \nabla_X Z).$$

5. Totally real surfaces in $S^6(1)$. We now continue to investigate the immersion (3.1).

Let X denote a vector field tangent to M^2. Then

$$x_*(\partial/\partial t)=\cos(t)N-\sin(t)f(p), \quad x_*(X)=\cos(t)f_*(X),$$

and

$$Jx_*(\partial/\partial t)=N\times f(p), \quad Jx_*(X)=\cos(t)\sin(t)N\times f_*(X)+\sin^2(t)Jf_*(X).$$

From this it is easy to check that x is totally real if and only if

$$\langle N\times f(p), f_*(X)\rangle=0, \tag{5.1}$$

$$\langle N\times f_*(X), f_*(Y)\rangle=0, \tag{5.2}$$

$$\langle Jf_*(X), f_*(Y)\rangle=0, \tag{5.3}$$

for all tangent vector fields X and Y to M. Now (5.3) simply says f has to be totally real; from (2.11), we obtain that (5.2) is equivalent to $\langle G(f_*(X), f_*(Y)), N\rangle=0$; (5.1) is equivalent to $\langle Jf_*(X), N\rangle=0$. We have reduced the condition that x is totally real to conditions depending only on f. From now on, for simplicity, we identify M with $f(M)$, so we do not write f_* if there is no confusion.

Differentiating (5.1) gives us

$$\langle N\times Y, X\rangle+\langle N\times p, h(X, Y)\rangle=0. \tag{5.4}$$

Since the first term in (5.4) is skew symmetric and the second is symmetric, both terms have to be zero. Therefore (5.1) implies (5.2). Now take any orthonormal basis $\{e_1, e_2\}$ of T_pM. From the properties of G, we obtain that $G(e_1, e_2)$ is orthogonal to e_1, e_2, Je_1, Je_2 and p; (5.2) implies that $G(e_1, e_2)$ is also orthogonal to N. On the other hand, (5.1) implies that $N\times p$ is orthogonal to e_1 and e_2; clearly $N\times p$ is orthogonal to p and N, and a straightforward calculation shows that $N\times p$ is orthogonal to Je_1 and Je_2. Therefore $G(e_1, e_2)=\pm p\times N$. After changing the sign of e_1, if necessary, we can make sure that $e_1\times e_2=JN$. This implies that $e_1\times N=-Je_2$, $e_2\times N=Je_1$. Note that the normal space of M in $S^5(1)$ at p is spanned by Je_1, Je_2 and JN.

Differentiating (5.2), we obtain

$$\langle N\times h(Y, Z), X\rangle+\langle N\times Y, h(X, Z)\rangle=0, \tag{5.5}$$

for all tangent vector fields X, Y and Z to M. Putting $X=e_1$ and $Y=e_2$, we obtain that

$$0=\langle Je_2, h(e_2, Z)\rangle+\langle Je_1, h(e_1, Z)\rangle=\langle h(e_2, e_2), JZ\rangle+\langle h(e_1, e_1), JZ\rangle ,$$

such that the mean curvature vector H of M in $S^5(1)$ is orthogonal to Je_1 and Je_2; from (5.4) we obtain that H is orthogonal to JN. Therefore H can only be zero. Then (5.4) immediately implies that h is of the form

$$h(e_1, e_1)=\alpha Je_1+\beta Je_2\ , \quad h(e_1, e_2)=\beta Je_1-\alpha Je_2\ , \quad h(e_2, e_2)=-\alpha Je_1-\beta Je_2\ ,$$

such that the ellipse of curvature of M at p is a circle (possibly a point).

Conversely, let M^2 be a minimal, totally real surface of $S^6(1)$ whose ellipse of curvature is a circle. Then we can use exactly the same computations as in the proof of [DOVV, Lemma] to obtain that $h(X, Y)$ is contained in $J(TM)$. Let $\{E_1, E_2\}$ be any local orthonormal basis of TM. Then $G(E_1, E_2)$ does not depend on the choice of $\{E_1, E_2\}$ up to sign. Hence we can define a subbundle B of the normal bundle by $B(p)=J(T_pM)\oplus\langle G(E_1, E_2)\rangle$. From (4.3), (2.8) and the minimality of M, we obtain that B is $\nabla^{\perp}$-parallel. Hence by the Erbacher theorem, M lies in a 5-dimensional totally geodesic hypersphere of $S^6(1)$. Let N be a unit vector orthogonal to this $S^5(1)$. By construction, JX is tangent to $S^5(1)$ and hence orthogonal to N for all X tangent to M. Therefore (5.1) is satisfied; (5.2) follows from (5.1) and (5.3) is true since M is totally real. Even if M is totally geodesic, this 5-dimensional unit sphere is uniquely determined as follows: take any point p in M. Then $S^5(1)$ is the unique great hypersphere of $S^6(1)$ through p, tangent to the $T_pM\oplus B(p)$. If M is not totally geodesic, it follows again as in [DOVV] that M is not contained in a totally geodesic 4-sphere. Hence we have proved the following theorem.

THEOREM 5.1. (1) *Let f: $M^2\to S^6(1)$ be a minimal non-totally geodesic totally real immersion in $S^6(1)$ whose ellipse of curvature is a circle. Then M^2 is contained in a unique totally geodesic S^5 and the warped product immersion* (3.1) *is totally real.*

(2) *Let f and x as in Section 3. Then x is totally real if and only if f is totally real and $J(f_*X)$ is tangent to $S^5(1)$ for all X tangent to M.*

(3) *Let f and x as in Section 3. If x is totally real, then f is totally real, minimal and has ellipse of curvature a circle.*

Other examples of totally real 3-dimensional submanifolds in S^6 were constructed by Ejiri in [E1] in the following way. Let f: $M^2\to S^6$ be a linearly full superminimal (in the sense of [BVW]) almost complex immersion. Let U and V be local orthonormal vector fields, defined on a neighborhood W, which span the second normal bundle. Then for any real number γ $(0<\gamma<\pi)$ we can define the tube of radius γ in the direction of the second normal bundle by

$$F_\gamma\colon\ W\times S^1\to S^6,\ (x, \theta)\mapsto\cos\gamma f(x)+\sin\gamma(\cos\theta U+\sin\theta V)\ .$$

Then F_γ defines a totally real immersion if and only if either $\cos\gamma=0$ or $\tan^2\gamma=4/5$. A similar construction of totally real submanifolds of CP^3 can be found in [E2].

A straightforward computation shows that all tubes of radius $\pi/2$ satisfy the equality in (3.1). In particular, starting from the Veronese immersion f: $S^2(1/6)\to S^6$ one obtains a 3-dimensional totally real submanifold with constant scalar curvature, which corresponds to Example 3.1 of [CDVV1]. It is also possible to show that for this class of tubes $F_{\pi/2}$ the distribution $\mathscr{D}^\perp$ is never integrable. As for the second possibility ($\tan^2\gamma=4/5$), one can show that the equality is never realized.

We now elaborate some more on totally real minimal surfaces whose ellipse of curvature is a circle. For simplicity, assume that $N=e_4$. We denote by π the Hopf map from S^5 to CP^2 given by

$$\pi(x_1, x_2, x_3, 0, x_5, x_6, x_7)=[x_1+ix_5, x_2+ix_6, x_3+ix_7] .$$

Then the following theorem from [BVW] gives a relation between minimal totally real surfaces in $S^6(1)$ whose ellipse of curvature is a circle and minimal totally real surfaces in CP^2.

THEOREM 5.2 (cf. [BVW]). *If f: $M^2\to S^5\subset S^6(1)$ is a minimal totally real isometric immersion, not totally geodesic, whose ellipse of curvature is a circle, then $\pi(f)$: $M^2\to CP^2$ is a totally real, not totally geodesic, minimal isometric immersion of M^2 into CP^2. Conversely, if M^2 is simply connected and if ψ: $M^2\to CP^2$ is a totally real, not totally geodesic, weakly conformal harmonic map, then there is a minimal totally real immersion f: $M^2\to S^5$ whose ellipse of curvature is a circle such that $\psi=\pi(f)$.*

In this respect Theorem 5.1 should be compared with [BVW, Theorem 7.1]. In its turn, minimal totally real immersions of a surface in CP^2 can be characterized as follows:

THEOREM 5.3. *Let $(M^2, \langle\cdot, \cdot\rangle)$ be a simply connected surface with Gaussian curvature K satisfying $K<1$. Then the following two conditions are equivalent*:

(1) $\Delta\log(1-K)=6K$;

(2) *there exists a totally real minimal immersion f: $M^2\to CP^2(4)$.*

PROOF. The fact that (2) implies (1) follows from a straightforward computation, in view of the basic formulas for a totally real submanifold in $CP^2(4)$ from [CO].

Let us now prove the converse. We take isothermal coordinates on M^2. So, we have a local non-zero function E such that $\langle\partial/\partial u, \partial/\partial u\rangle=E^2=\langle\partial/\partial v, \partial/\partial v\rangle$ and $\langle\partial/\partial u, \partial/\partial v\rangle=0$. Then $K=-\Delta\log E$. We now define a function ϕ by

$$\phi^2=\frac{E^6}{2}(1-K) .$$

Then, by the assumption of the theorem, we get that

$$\Delta \log \phi = \frac{1}{2} \Delta \log \phi^2 = \frac{1}{2} \Delta \log \frac{E^6}{2}(1-K)$$

$$= 3\Delta \log E + \frac{1}{2} \Delta \log(1-K) = -3K + 3K = 0 .$$

Hence there exist a function

$$\psi(u, v) = F(u, v) - iG(u, v) ,$$

holomorphic in $z = u + iv$ such that $F^2 + G^2 = \phi^2$. Put

$$f(u, v) = \frac{F(u, v)}{E^2(u, v)} , \quad g(u, v) = \frac{G(u, v)}{E^2(u, v)} ,$$

and define α: $TM^2 \times TM^2 \to TM^2$ by

$$\alpha(\partial/\partial u, \partial/\partial u) = f(u, v)\partial/\partial u + g(u, v)\partial/\partial v ,$$

$$\alpha(\partial/\partial u, \partial/\partial v) = \alpha(\partial/\partial v, \partial/\partial u) = g(u, v)\partial/\partial u - f(u, v)\partial/\partial v ,$$

$$\alpha(\partial/\partial v, \partial/\partial v) = -f(u, v)\partial/\partial u - g(u, v)\partial/\partial v .$$

Then

$$(\nabla\alpha)(\partial/\partial u, \partial/\partial u, \partial/\partial v) = \left(g_u - 3f\frac{E_v}{E} - g\frac{E_u}{E}\right)\partial/\partial u - \left(f_u + 3g\frac{E_v}{E} - f\frac{E_u}{E}\right)\partial/\partial v ,$$

$$(\nabla\alpha)(\partial/\partial v, \partial/\partial u, \partial/\partial u) = \left(f_v - f\frac{E_v}{E} - 3g\frac{E_u}{E}\right)\partial/\partial u + \left(g_v - g\frac{E_v}{E} + 3f\frac{E_u}{E}\right)\partial/\partial v ,$$

$$(\nabla\alpha)(\partial/\partial v, \partial/\partial u, \partial/\partial v) = \left(g_v - 3f\frac{E_u}{E} - g\frac{E_v}{E}\right)\partial/\partial u + \left(-f_v + 3g\frac{E_u}{E} + f\frac{E_v}{E}\right)\partial/\partial v ,$$

$$(\nabla\alpha)(\partial/\partial u, \partial/\partial v, \partial/\partial v) = \left(-f_u + f\frac{E_u}{E} - 3g\frac{E_v}{E}\right)\partial/\partial u + \left(-g_u + 3f\frac{E_v}{E} + g\frac{E_u}{E}\right)\partial/\partial v ,$$

showing that $\nabla\alpha$ is totally symmetric if and only if

$$f_u + g_v = -2\left(f\frac{E_u}{E} + g\frac{E_v}{E}\right), \quad f_v - g_u = 2\left(g\frac{E_u}{E} - f\frac{E_v}{E}\right),$$

which by the definition of f and g is satisfied indeed. Since

$$\frac{E^2}{2}(1-K) = \frac{\phi}{E^4} , \quad \phi^2 = F^2 + K^2 = E^4(f^2 + g^2) ,$$

we get that

$$R(X, Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y + \alpha(\alpha(Y, Z), X) - \alpha(\alpha(X, Z), Y).$$

Applying the basic existence theorem (cf. [CDVV2]) then completes the proof of the theorem.

6. Proof of the main theorem.

We first recall some results of [CDVV1] for 3-dimensional totally real (and therefore minimal) submanifolds of $S^6(1)$.

LEMMA 6.1. *Let M be a 3-dimensional totally real submanifold of $S^6(1)$. Then*

$$\delta_M \leq 2.$$

Equality holds at a point p of M if there exists a tangent basis $\{e_1, e_2, e_3\}$ of T_pM such that

$$h(e_1, e_1) = \lambda Je_1, \quad h(e_1, e_3) = 0,$$
$$h(e_1, e_2) = -\lambda Je_2, \quad h(e_2, e_3) = 0,$$
$$h(e_2, e_2) = -\lambda Je_1, \quad h(e_3, e_3) = 0,$$

where λ is a positive number determined by the scalar curvature τ according to

$$2\lambda^2 = 3 - \tau(p).$$

So if we define a distribution $\mathscr{D}$ by

$$\mathscr{D}(p) = \{X \in T_pM \mid h(X, Y) = 0, \forall Y \in T_pM\},$$

we see that $\mathscr{D}(p)$ is either 3-dimensional, in which case p is a totally geodesic point, or 1-dimensional. From now on, we assume that the dimension of $\mathscr{D}(p)$ is constant on M. Then, exactly as in Lemma 5.3 of [CDVV1], we obtain:

LEMMA 6.2. *Let M^3 be a totally real submanifold of $S^6(1)$ satisfying the equality in* (1.3). *Assume also that the dimension of the distribution $\mathscr{D}$ is constantly equal to* 1 *and let $p \in M$. Then, there exists local orthonormal vector fields E_1, E_2, E_3 on a neighborhood of p such that*

$$h(E_1, E_1) = \lambda JE_1, \quad h(E_1, E_3) = 0,$$
$$h(E_1, E_2) = -\lambda JE_2, \quad h(E_2, E_3) = 0,$$
$$h(E_2, E_2) = -\lambda JE_1, \quad h(E_3, E_3) = 0,$$

where λ is a non-zero local function determined by the scalar curvature by $2\lambda^2 = 3 - \tau$.

Let us take the basis from the previous lemma. By (2.7) and (2.9), we see that $G(E_1, E_2)$ is in the direction of JE_3. From (2.10), we obtain that $G(E_1, E_2)$ is a unit vector. Replacing E_3 by $-E_3$ if necessary, we may assume

$$G(E_1, E_2) = JE_3, \quad G(E_2, E_3) = JE_1, \quad G(E_3, E_1) = JE_2.$$

From now on, assume that we take this choice of orthonormal basis.

Throughout this section, M^3 is assumed to be a (non-totally geodesic) totally real submanifold in $S^6(1)$ which at every point p of M satisfies the equality in (1.3). We also assume that

(1) the dimension of $\mathscr{D}$ is constant on M,

(2) the distribution $\mathscr{D}^{\perp}$ is integrable.

Since M is assumed to be non-totally geodesic, we have that $\dim\mathscr{D}=1$. Let $p\in M$. We introduce local functions γ_{ij}^k by

$$\gamma_{ij}^k=\langle\nabla_{E_i}E_j, E_k\rangle .$$

Since $\{E_1, E_2, E_3\}$ is an orthonormal basis, $\gamma_{ij}^k+\gamma_{ik}^j=0$. Then, we have the following lemma.

LEMMA 6.3. *We have*

$$\gamma_{33}^1=\gamma_{33}^2=0 , \tag{1}$$

$$\gamma_{11}^3=\gamma_{22}^3 , \tag{2}$$

$$\gamma_{12}^3=-\gamma_{21}^3 , \tag{3}$$

$$\gamma_{31}^2=-\frac{1}{3}(\gamma_{12}^3+1) . \tag{4}$$

Moreover, the function λ satisfies the following system of differential equations.

$$E_1(\lambda)=-3\lambda\gamma_{21}^2 , \tag{5}$$

$$E_2(\lambda)=3\lambda\gamma_{11}^2 , \tag{6}$$

$$E_3(\lambda)=-\lambda\gamma_{13}^1 . \tag{7}$$

PROOF. Since

$$(\nabla h)(E_1, E_3, E_3)=\nabla_{E_1}^{\perp}h(E_3, E_3)-2h(\nabla_{E_1}E_3, E_3)=0$$

and

$$(\nabla h)(E_3, E_1, E_3)=\nabla_{E_3}^{\perp}h(E_1, E_3)-h(\nabla_{E_3}E_1, E_3)-h(E_1, \nabla_{E_3}E_3)=-h(E_1, \nabla_{E_3}E_3) ,$$

Codazzi's equation yields $\nabla_{E_3}E_3=0$. Next we compute

$$\begin{aligned}(\nabla h)(E_3, E_1, E_1)&=\nabla_{E_3}^{\perp}h(E_1, E_1)-2h(\nabla_{E_3}E_1, E_1)\\&=E_3(\lambda)JE_1+\lambda J\nabla_{E_3}E_1+\lambda JE_2+2\langle\nabla_{E_3}E_1, E_2\rangle\lambda JE_2\end{aligned}$$

and

$$\begin{aligned}(\nabla h)(E_1, E_3, E_1)&=\nabla_{E_1}^{\perp}h(E_3, E_1)-h(\nabla_{E_1}E_3, E_1)-h(E_3, \nabla_{E_1}E_1)\\&=-\langle\nabla_{E_1}E_3, E_1\rangle\lambda JE_1+\langle\nabla_{E_1}E_3, E_2\rangle\lambda JE_2 .\end{aligned}$$

Since $\langle \nabla_{E_3}E_3, E_1\rangle = 0$, by comparing components, we get that

$$(6.1) \qquad E_3(\lambda) = \lambda\langle \nabla_{E_1}E_1, E_3\rangle = \lambda\gamma_{11}^3 , \quad 3\langle \nabla_{E_3}E_1, E_2\rangle = -\langle \nabla_{E_1}E_2, E_3\rangle - 1 .$$

This proves (4) and (7). Similarly, from $(\nabla h)(E_3, E_2, E_2) = (\nabla h)(E_2, E_3, E_2)$ we obtain

$$(6.2) \qquad \gamma_{22}^3 = \gamma_{11}^3 .$$

Finally, we have

$$\begin{aligned}(\nabla h)(E_2, E_1, E_1) &= \nabla_{E_2}^{\perp} h(E_1, E_1) - 2h(\nabla_{E_2}E_1, E_1) \\ &= E_2(\lambda)JE_1 + \lambda J\nabla_{E_2}E_1 - \lambda JE_3 + 2\langle \nabla_{E_2}E_1, E_2\rangle\lambda JE_2\end{aligned}$$

and

$$\begin{aligned}(\nabla h)(E_1, E_2, E_1) &= \nabla_{E_1}^{\perp} h(E_2, E_1) - h(\nabla_{E_1}E_2, E_1) - h(E_2, \nabla_{E_1}E_1) \\ &= -E_1(\lambda)JE_2 - \lambda J\nabla_{E_1}E_2 - \lambda JE_3 - \langle \nabla_{E_1}E_2, E_1\rangle\lambda JE_1 + \langle \nabla_{E_1}E_1, E_2\rangle\lambda JE_1 \\ &= -\lambda J\nabla_{E_1}E_2 - \lambda JE_3 + 2\lambda\langle \nabla_{E_1}E_1, E_2\rangle JE_1 .\end{aligned}$$

So, by comparing components, we get

$$\langle \nabla_{E_2}E_1 + \nabla_{E_1}E_2, E_3\rangle = 0 , \quad E_2(\lambda) = 3\lambda\langle \nabla_{E_1}E_1, E_2\rangle , \quad E_1(\lambda) = -3\lambda\langle \nabla_{E_2}E_2, E_1\rangle .$$

This completes the proof of the lemma.

In order to simplify the notation, we introduce local functions a, b, c and d by

$$a = \gamma_{11}^3 , \quad b = \gamma_{12}^3 , \quad c = \gamma_{11}^2 , \quad d = \gamma_{21}^2 .$$

Then Lemma 6.3 implies that

$$\begin{aligned}&\nabla_{E_1}E_1 = cE_2 + aE_3 , && \nabla_{E_1}E_2 = -cE_1 + bE_3 , && \nabla_{E_1}E_3 = -aE_1 - bE_2 , \\ &\nabla_{E_2}E_1 = dE_2 - bE_3 , && \nabla_{E_2}E_2 = -dE_1 + aE_3 , && \nabla_{E_2}E_3 = bE_1 - aE_2 , \\ &\nabla_{E_3}E_1 = -\frac{1}{3}(b+1)E_2 , && \nabla_{E_3}E_2 = \frac{1}{3}(b+1)E_1 , && \nabla_{E_3}E_3 = 0 ,\end{aligned}$$

and

$$E_1(\lambda) = -3\lambda d , \quad E_2(\lambda) = 3\lambda c , \quad E_3(\lambda) = \lambda a .$$

Let us now use the assumption that the distribution $\mathscr{D}^{\perp}$, which is locally spanned by the vector fields E_1 and E_2 is an integrable distribution. Then, the above formulas imply that $b = 0$. Then, we have:

LEMMA 6.4. *The local function a, under the assumptions made above, satisfies the following system of differential equations*:

$$E_1(a) = 0 , \quad E_2(a) = 0 , \quad E_3(a) = 1 + a^2 .$$

PROOF. Using the Gauss equation, we find that

$$0=\langle R(E_1, E_2)E_1, E_3\rangle=\langle\nabla_{E_1}\nabla_{E_2}E_1-\nabla_{E_2}\nabla_{E_1}E_1-\nabla_{\nabla_{E_1}E_2-\nabla_{E_2}E_1}E_1, E_3\rangle$$
$$=-ca-E_2(a)+ca=-E_2(a).$$

Similarly, it follows from the Gauss equation $0=\langle R(E_2, E_1)E_2, E_3\rangle$ that $E_1(a)=0$. Finally, in order to prove that $E_3(a)=1+a^2$, we use again the Gauss equation. We have

$$1=\langle R(E_1, E_3)E_3, E_1\rangle=\langle\nabla_{E_1}\nabla_{E_3}E_3-\nabla_{E_3}\nabla_{E_1}E_3-\nabla_{\nabla_{E_1}E_3-\nabla_{E_3}E_1}E_3, E_1\rangle=E_3(a)-a^2.$$

□

LEMMA 6.5. *Let M be as above and let $p\in M$. Then, in a neighborhood of the point p, M is warped product of an interval $(-\varepsilon, \varepsilon)$ and N^2, the leaf of the distribution $\mathscr{D}^{\perp}$ through p.*

PROOF. We check Hiepko's condition [H], using the formalism of [N, §3]. In particular, we have to check that $\mathscr{D}$ is totally geodesic and that $\mathscr{D}^{\perp}$ is spherical. Since $\nabla_{E_3}E_3=0$, the first assumption is trivially satisfied. For the second assertion, we first have for $i,j\in\{1, 2\}$ that

$$\langle\nabla_{E_i}E_j, E_3\rangle=\delta_{ij}aE_3,$$

which shows that $\mathscr{D}^{\perp}$ is totally umbilical in M with mean curvatuve vector $\eta=aE_3$. Since, by the previous lemma, $E_1(a)=E_2(a)=0$, the mean curvature vector is parallel. So, we get that $\mathscr{D}^{\perp}$ is spherical. □

The warping function can be determined from Lemma 6.4, but we do not need an explicit expression. Now we can finish the proof. Indeed, we know that M is locally a warped product and that the distributions on M, determined by the product structure, coincide with $\mathscr{D}$ and $\mathscr{D}^{\perp}$. Moreover, since $h(\mathscr{D}, \mathscr{D}^{\perp})=0$, we obtain that M^3 (locally) is immersed as a warped product; further, the first factor is totally geodesic, and therefore we can assume that the first factor of the corresponding warped product decomposition of $S^6(1)$ is 1-dimensional. Since the decomposition of $S^6(1)$ into a warped product whose first factor is 1-dimensional is unique up to isometries, we obtain that M^3 is immersed as described by (3.1).

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