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HARMONIC MAPS AND ASSOCIATED MAPS FROM SIMPLY CONNECTED RIEMANN SURFACES INTO THE 3-DIMENSIONAL SPACE FORMS

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Abstract. In this paper we introduce equations whose solutions are considered to be a generalization of simply connected, minimal surfaces or constant mean curvature surfaces in 3-dimensional space forms and prove that there exists a natural bijective correspondence among the sets of solutions of the equations under certain conditions.

Introduction. In 1970 Lawson [L, Theorem 8] gave an S^1-family of isometric immersions with constant mean curvature (CMC for short) $\sqrt{H^2-c}$ from a simply connected Riemann surface M into the simply connected 3-dimensional space form $\mathfrak{M}^3(c)$ of curvature c, where $H, c \in \boldsymbol{R}$ such that $H^2-c \geq 0$. His result implies that there exists a natural S^1-equivariant bijective correspondence between the space of isometric immersions with CMC$=\sqrt{H^2-c}$ of M in $\mathfrak{M}^3(c)$ and the space of isometric immersions with CMC$=\sqrt{H^2-c'}$ of M in $\mathfrak{M}^3(c')$, where $H, c, c' \in \boldsymbol{R}$ such that H^2-c, $H^2-c' \geq 0$ and $c \neq c'$.

In this paper we shall generalize his result. Before we state our main theorem we shall explain some basic notation.

We fix a Riemann surface M. Let $\Omega^1(M, \mathfrak{so}(3))$ be the space of $\mathfrak{so}(3)$-valued 1-forms on M and $\Omega^{1,0}(M, \boldsymbol{C}^3)$ the space of $\boldsymbol{C}^3$-valued $(1,0)$-forms on M. Set

$$L(c)=\begin{cases} 1 & \text{if} \quad c=0\,, \\ \dfrac{1}{\sqrt{|c|}} & \text{if} \quad c\neq 0 \end{cases}$$

and

$$\operatorname{sign}(c)=\begin{cases} 1 & \text{if} \quad c>0\,, \\ 0 & \text{if} \quad c=0\,, \\ -1 & \text{if} \quad c<0 \end{cases}$$

for $c \in \boldsymbol{R}$.

Given H, $c \in \boldsymbol{R}$, we shall consider equations $(*_{H,c})$ on $A \in \Omega^1(M, \mathfrak{so}(3))$ and $a \in \Omega^{1,0}(M, \boldsymbol{C}^3)$ defined by

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$$(*_{H,c})\qquad \begin{cases} da+A\wedge a=\sqrt{-1}HL(c)a\times\bar{a} \\ dA+\dfrac{1}{2}[A\wedge A]-\operatorname{sign}(c)(\bar{a}\wedge{}^t a+a\wedge{}^t\bar{a})=0\,, \end{cases}$$

where $\times$ is the complex linear extension of the exterior product on $\boldsymbol{R}^3$.

We set

$$\mathscr{A}(H,c)=\{(A,a);\ A\in\Omega^1(M,\mathfrak{so}(3))\text{ and }a\in\Omega^{1,0}(M,\boldsymbol{C}^3)\text{ satisfy }(*_{H,c})\}$$

and

$$\mathscr{K}=C^\infty(M,SO(3))\,.$$

The group $\mathscr{K}$ acts on $\mathscr{A}(H,c)$ from the right by

$$(A,a)k=(k^{-1}Ak+k^{-1}dk,\ k^{-1}a)\,,$$

where $k\in\mathscr{K}$ and $(A,a)\in\mathscr{A}(H,c)$.

Let p and q be $\boldsymbol{C}^3$-valued 1-forms on M and $\{z\}$ a holomorphic local coordinate on M. We define

$$\begin{aligned} p\otimes q=&\left\langle p\left(\frac{\partial}{\partial z}\right),q\left(\frac{\partial}{\partial z}\right)\right\rangle dz\otimes dz+\left\langle p\left(\frac{\partial}{\partial z}\right),q\left(\frac{\partial}{\partial \bar{z}}\right)\right\rangle dz\otimes d\bar{z} \\ &+\left\langle p\left(\frac{\partial}{\partial \bar{z}}\right),q\left(\frac{\partial}{\partial z}\right)\right\rangle d\bar{z}\otimes dz+\left\langle p\left(\frac{\partial}{\partial \bar{z}}\right),q\left(\frac{\partial}{\partial \bar{z}}\right)\right\rangle d\bar{z}\otimes d\bar{z}\,, \end{aligned}$$

where $\langle\ ,\ \rangle$ is the complex linear extension of the standard inner product on $\boldsymbol{R}^3$ (this definition is independent of z).

Then we set

$$\mathscr{B}(H,c)=\mathscr{A}(H,c)/\mathscr{K}$$

and

$$\mathscr{C}(H,c)=\{(A,a)\in\mathscr{A}(H,c);\ a\otimes a=0,\ a\otimes\bar{a}\neq 0\}/\mathscr{K}\,.$$

Note that $\mathscr{C}(H,c)$ is a subspace of $\mathscr{B}(H,c)$.

The space $\mathscr{B}(H,c)$ may be considered as a generalization of the space of isometric immersions with CMC$=H$ of M in $\mathfrak{M}^3(c)$. Indeed, we can prove the following:

PROPOSITION. *If M is simply connected, then*

(i) $\mathscr{C}(H,c)\cong\{$*conformal immersions with* CMC$=H$ *of* M *in* $\mathfrak{M}^3(c)\}/\mathrm{Iso}_0(\mathfrak{M}^3(c))$ *and*

(ii) $\mathscr{B}(0,c)\cong\{$*harmonic maps from* M *into* $\mathfrak{M}^3(c)\}/\mathrm{Iso}_0(\mathfrak{M}^3(c))$,
where $\mathrm{Iso}_0(\mathfrak{M}^3(c))$ *is the identity component of the group of isometries of* $\mathfrak{M}^3(c)$.

Moreover in §3 we shall naturally define an S^1-action on $\mathscr{B}(H,c)$ leaving $\mathscr{C}(H,c)$

invariant. Then our main theorem is as follows:

THEOREM. *Let $H, H', c, c' \in \boldsymbol{R}$. If* $\operatorname{sign}(H^2+c)=\operatorname{sign}(H'^2+c')$, *then there exists a natural S^1-equivariant isomorphism*

$$F: \mathscr{B}(H, c) \to \mathscr{B}(H', c')$$

such that $F(\mathscr{C}(H, c))=\mathscr{C}(H', c')$.

We remark that if we restrict to the case that M is a simply connected Riemann surface, we obtain $\mathscr{C}(\sqrt{H^2-c}, c) \cong \mathscr{C}(\sqrt{H^2-c'}, c')$ for any $H, c, c' \in \boldsymbol{R}$ such that $H^2-c, H^2-c' \geq 0$ and $c \neq c'$. Since M is simply connected, by the Proposition, this isomorphism gives an S^1-equivariant bijective correspondence between the space of isometric immersions with CMC$=\sqrt{H^2-c}$ of M in $\mathfrak{M}^3(c)$ and the space of isometric immersions with CMC$=\sqrt{H^2-c'}$ of M in $\mathfrak{M}^3(c')$. We can check that this correspondence is just the one obtained by Lawson (see §3, Remark), although our proof is completely different from his. Originally the author studied the bijective correspondence above for a special case in [F]. This paper was inspired by Lawson's result and turned out that the previous paper [F] is a special case.

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1. Preliminaries. In this section we shall give basic facts which will be needed later.

Let $N=G/K$ be a reductive homogeneous space, $\mathfrak{g}$ (respectively $\mathfrak{k}$) the Lie algebra of G (respectively K) and $\pi: G \to N$ the natural projection. Since N is reductive, we have a decomposition $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{m}$ with an $\mathrm{Ad}_G K$-invariant summand $\mathfrak{m}$. We can naturally define a $\mathfrak{g}$-valued 1-form β on N, called the Maurer-Cartan form. If $P_{\mathfrak{m}}$ denotes the projection from $\mathfrak{g}$ to $\mathfrak{m}$, then

$$\beta(X)=\mathrm{Ad}\, g P_{\mathfrak{m}}(\mathrm{Ad}\, g^{-1} \xi),$$

where $g \in G$, $X=d/dt|_{t=0} \exp t\xi.x$, $\xi \in \mathfrak{g}$ and $x=\pi(g)$ (see [BR, p. 6]).

Now let M be a manifold.

DEFINITION. *A map $\Phi: M \to G$ is called a framing of $\varphi: M \to N$ if it satisfies $\pi \circ \Phi=\varphi$.*

We set $\alpha=\Phi^{-1} d\Phi$ for a framing of φ. Corresponding to the reductive decomposition $\mathfrak{g}=\mathfrak{k} \oplus \mathfrak{m}$, we have a decomposition of α,

$$\alpha=\alpha_{\mathfrak{k}}+\alpha_{\mathfrak{m}}.$$

Then we have:

PROPOSITION 1.1 (see [BP, p. 241]). *Let M be an oriented Riemannian manifold and Φ a framing of $\varphi: M \to N$. Then*

$$\beta(\mathrm{trace}\nabla d\varphi) = *\mathrm{Ad}\,\Phi\{d*\alpha_{\mathfrak{m}} + [\alpha \wedge *\alpha_{\mathfrak{m}}]\}\ ,$$

where $*$ is the Hodge star operator on M.

We shall now describe $\mathfrak{M}^3(c)$ as a (Riemannian symmetric) reductive homogeneous space.

We put

$$G_0 = SO(3) \ltimes \boldsymbol{R}^3 = \left\{ \begin{pmatrix} P & p \\ 0 & 1 \end{pmatrix} \in GL(4, \boldsymbol{R});\ P \in SO(3),\ p \in \boldsymbol{R}^3 \right\},$$

$$G_c = SO(4)$$

for $c > 0$, and

$$G_c = SO^+(3, 1) = \{X = (x_{ij}) \in GL(4, \boldsymbol{R});\ {}^tXJX = J,\ \det X = 1,\ x_{44} > 0\}$$

for $c < 0$, where $J = \mathrm{diag}(1, 1, 1, -1)$. Set

$$K_c = \left\{ \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \in GL(4, \boldsymbol{R});\ P \in SO(3) \right\}.$$

An involutive automorphism is defined by

$$\sigma_c(X) = JXJ\ ,$$

where $X \in G_c$.

The corresponding symmetric decomposition $\mathfrak{g}_c = \mathfrak{k}_c \oplus \mathfrak{m}_c$ is given by

$$\mathfrak{g}_c = \left\{ \begin{pmatrix} A & a \\ -\mathrm{sign}(c)^t a & 0 \end{pmatrix} \in M(4, \boldsymbol{R});\ A \in \mathfrak{so}(3),\ a \in \boldsymbol{R}^3 \right\},$$

$$\mathfrak{k}_c = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in M(4, \boldsymbol{R});\ A \in \mathfrak{so}(3) \right\}$$

and

$$\mathfrak{m}_c = \left\{ \begin{pmatrix} 0 & a \\ -\mathrm{sign}(c)^t a & 0 \end{pmatrix} \in M(4, \boldsymbol{R});\ a \in \boldsymbol{R}^3 \right\}.$$

An $\mathrm{Ad}_{G_c} K_c$-invariant metric on $\mathfrak{m}_c$ is defined by

$$g_c\left(\begin{pmatrix} 0 & a \\ -\mathrm{sign}(c)^t a & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ -\mathrm{sign}(c)^t b & 0 \end{pmatrix} \right) = L^2(c)\langle a, b\rangle\ ,$$

where $a, b \in \boldsymbol{R}^3$. Then $\mathfrak{M}^3(c)$ is a Riemannian symmetric space corresponding to $(G_c, K_c, \sigma_c, g_c)$.

2. A generalization of isometric immersions with constant mean curvature. In this section we shall prove the Proposition in the Introduction.

Let $\varphi: M \to \mathfrak{M}^3(c)$ be a map and Φ a framing of φ (M is not necessarily simply connected). It is not hard to see that such a framing always exists. We have a decomposition $\Phi^{-1}d\Phi =: \alpha = \alpha_{\mathfrak{k}_c} + \alpha_{\mathfrak{m}_c}$. Since M is a Riemann surface, we have a type decomposition

$$\alpha_{\mathfrak{m}_c} = \alpha'_{\mathfrak{m}_c} + \alpha''_{\mathfrak{m}_c},$$

where $\alpha'_{\mathfrak{m}_c}$ is an $\mathfrak{m}_c^{\boldsymbol{C}}$-valued $(1,0)$-form with complex conjugate $\alpha''_{\mathfrak{m}_c}$. We may consider $\alpha'_{\mathfrak{m}_c}$ and $\alpha''_{\mathfrak{m}_c}$ to be $\boldsymbol{C}^3$-valued. We denote the decomposition $\alpha = \alpha_{\mathfrak{k}_c} + \alpha'_{\mathfrak{m}_c} + \alpha''_{\mathfrak{m}_c}$ simply as $\alpha = \alpha_{\mathfrak{k}} + \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$.

LEMMA 2.1. *$\varphi: M \to \mathfrak{M}^3(c)$ is a conformal immersion if and only if $\alpha'_{\mathfrak{m}} \otimes \alpha'_{\mathfrak{m}} = 0$ and $\alpha'_{\mathfrak{m}} \otimes \alpha''_{\mathfrak{m}} \neq 0$.*

PROOF. Let $\{z\}$ be a holomorphic local coordinate on M. Since

$$d\varphi(X) = \pi_*(\mathrm{Ad}\,\Phi\alpha_{\mathfrak{m}}(X))$$

for any tangent vector X on M, we have

$$g_c\left(d\varphi\left(\frac{\partial}{\partial z}\right), d\varphi\left(\frac{\partial}{\partial z}\right)\right)dz \otimes dz = L^2(c)\alpha'_{\mathfrak{m}} \otimes \alpha'_{\mathfrak{m}}$$

and

$$g_c\left(d\varphi\left(\frac{\partial}{\partial z}\right), d\varphi\left(\frac{\partial}{\partial \bar{z}}\right)\right)dz \otimes d\bar{z} = L^2(c)\alpha'_{\mathfrak{m}} \otimes \alpha''_{\mathfrak{m}}.$$

This completes the proof. ■

If we write $\alpha_{\mathfrak{m}}$ as

$$\alpha_{\mathfrak{m}} = \begin{pmatrix} 0 & a \\ -\mathrm{sign}(c)^t a & 0 \end{pmatrix}$$

for $a \in \Omega^1(M, \boldsymbol{R}^3)$, we can define an $\mathfrak{m}_c$-valued 2-form $\alpha_{\mathfrak{m}} \times \alpha_{\mathfrak{m}}$ by

$$\alpha_{\mathfrak{m}} \times \alpha_{\mathfrak{m}} = \begin{pmatrix} 0 & a \times a \\ -\mathrm{sign}(c)^t(a \times a) & 0 \end{pmatrix}.$$

We set

$$n = \frac{L}{2}\pi_*(\mathrm{Ad}\,\Phi * (\alpha_{\mathfrak{m}} \times \alpha_{\mathfrak{m}})).$$

If $\varphi: M \to \mathfrak{M}^3(c)$ is a conformal immersion, then there exists a function $h \in C^\infty(M)$ such that

$$\varphi^* g_c = e^h g\ ,$$

where g is a Riemannian metric on M.

Let $\{x, y\}$ be an isothermal coordinate system on M. It is easy to see the following:

LEMMA 2.2. *If φ is a conformal immersion, then*

$$g_c\left(n, d\varphi\left(\frac{\partial}{\partial x}\right)\right) = g_c\left(n, d\varphi\left(\frac{\partial}{\partial y}\right)\right) = 0$$

and

$$g_c(n, n) = e^{2h}\ .$$

It is obvious that $\mathscr{K}$ acts on $C^\infty(M, G_c)$ from the right by

$$\Phi k = \Phi\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix},$$

where $\Phi \in C^\infty(M, G_c)$ and $k \in \mathscr{K}$. Note that if Φ is a framing of $\varphi : M \to \mathfrak{M}^3(c)$, so is Φk. We write $\alpha = \Phi^{-1} d\Phi$ as

$$\alpha = \begin{pmatrix} A & a \\ -\operatorname{sign}(c)^t a & 0 \end{pmatrix},$$

where A (respectively a) is an $\mathfrak{so}(3)$- (respectively $\boldsymbol{R}^3$-) valued 1-form on M. Then direct computation gives the following:

LEMMA 2.3. *Let Φ be a framing of $\varphi : M \to \mathfrak{M}^3(c)$ and $k \in \mathscr{K}$. Then*

$$(\Phi k)^{-1} d(\Phi k) = \begin{pmatrix} k^{-1} A k + k^{-1} dk & k^{-1} a \\ -\operatorname{sign}(c)^t (k^{-1} a) & 0 \end{pmatrix}.$$

Now we are in a position to prove the Proposition in the Introduction.

PROOF. It is enough to prove (i).

Let $\varphi : M \to \mathfrak{M}^3(c)$ be a conformal immersion with CMC $= H$ of M in $\mathfrak{M}^3(c)$. Then φ satisfies

$$\frac{1}{2}\operatorname{trace}\nabla d\varphi = Hn\ .$$

Let Φ be a framing of φ. Then by Proposition 1.1, Lemma 2.1 and Lemma 2.2, we have

$$d * \alpha_{\mathrm{m}} + [\alpha \wedge * \alpha_{\mathrm{m}}] = HL(c)\alpha_{\mathrm{m}} \times \alpha_{\mathrm{m}}\ , \quad \alpha'_{\mathrm{m}} \otimes \alpha'_{\mathrm{m}} = 0 \quad \text{and} \quad \alpha'_{\mathrm{m}} \otimes \alpha''_{\mathrm{m}} \neq 0\ .$$

Direct computation shows that

$$(2.1)\qquad (d\alpha'_{\mathfrak{m}}+[\alpha_{\mathfrak{k}}\wedge\alpha'_{\mathfrak{m}}])-(d\alpha''_{\mathfrak{m}}+[\alpha_{\mathfrak{k}}\wedge\alpha''_{\mathfrak{m}}])=2\sqrt{-1}HL(c)\alpha'_{\mathfrak{m}}\times\alpha''_{\mathfrak{m}}\,.$$

Taking the $\mathfrak{m}_c$- and $\mathfrak{k}_c$-parts of the Maurer-Cartan equations for α, we have

$$(2.2)\qquad (d\alpha'_{\mathfrak{m}}+[\alpha_{\mathfrak{k}}\wedge\alpha'_{\mathfrak{m}}])+(d\alpha''_{\mathfrak{m}}+[\alpha_{\mathfrak{k}}\wedge\alpha''_{\mathfrak{m}}])=0$$

and

$$(2.3)\qquad d\alpha_{\mathfrak{k}}+\frac{1}{2}[\alpha_{\mathfrak{k}}\wedge\alpha_{\mathfrak{k}}]+[\alpha'_{\mathfrak{m}}\wedge\alpha''_{\mathfrak{m}}]=0\,.$$

The equations (2.1) and (2.2) are equivalent to

$$(2.4)\qquad d\alpha'_{\mathfrak{m}}+[\alpha_{\mathfrak{k}}\wedge\alpha'_{\mathfrak{m}}]=\sqrt{-1}HL(c)\alpha'_{\mathfrak{m}}\times\alpha''_{\mathfrak{m}}\,.$$

If we write $\alpha'_{\mathfrak{m}}$ as

$$\alpha'_{\mathfrak{m}}=\begin{pmatrix} 0 & a \\ -\operatorname{sign}(c)^t a & 0\end{pmatrix}$$

for $a\in\Omega^{1,0}(M, \boldsymbol{C}^3)$, we can see that (2.3) and (2.4) give $(*_{H,c})$.

Moreover the conformality condition is $a\otimes a=0$ and $a\otimes\bar{a}\neq 0$.

Conversely let A and a satisfy $(*_{H,c})$, $a\otimes a=0$ and $a\otimes\bar{a}\neq 0$. Then since M is simply connected, there exists a map $\Phi\colon M\to G_c$ such that

$$\Phi^{-1}d\Phi=\begin{pmatrix} A & a+\bar{a} \\ -\operatorname{sign}(c)^t(a+\bar{a}) & 0\end{pmatrix}$$

and $\varphi=\pi\circ\Phi$ gives a conformal immersion with CMC$=H$ of M in $\mathfrak{M}^3(c)$ (see [BP, p. 230]).

By Lemma 2.3 this correspondence gives the isomorphism (i). ∎

REMARK. (i) In the case $H=0$, (2.4) and (2.5) are the harmonic map equations obtained by Burstall and Pedit [BP].

(ii) In the case $c<0$ and $H=0$, $(*_{0,c})$ are equivalent to Hitchin's self-duality equations for a trivial bundle $M\times SO(3)$. In the case $c>0$ and $H=0$, $(*_{0,c})$ are obtained by dimensionally reducing the self-duality equations in $\boldsymbol{R}^4$ with signature (2, 2). For more information about the self-duality equations, we refer to Hitchin [H].

3. Harmonic maps and associated maps. In this section we shall prove our main theorem.

We define a bijective map $\iota\colon \boldsymbol{R}^3\to\mathfrak{so}(3)$ by

$$\iota(p)=\begin{pmatrix} 0 & -p^3 & p^2 \\ p^3 & 0 & -p^1 \\ -p^2 & p^1 & 0\end{pmatrix}\qquad\text{for}\quad p=\begin{pmatrix} p^1 \\ p^2 \\ p^3\end{pmatrix}\in\boldsymbol{R}^3\,.$$

The map ι extends naturally to a bijective map from the space of $\boldsymbol{C}^3$-valued forms to that of $\mathfrak{so}(3)$-valued forms.

Let $H\geq 0$, $c\in\boldsymbol{R}$ such that $H^2-c\geq 0$. For $(A, a)\in\mathscr{A}(H, 0)$, we define a $\mathscr{K}$-equivariant bijective map

$$S(H; 0, c): \mathscr{A}(H, 0)\to\mathscr{A}(\sqrt{H^2-c}, c)$$

by

$$S(H; 0, c)(A, a)=\left(A+\sqrt{-1}(H-\sqrt{H^2-c})\iota(a-\bar{a}), \frac{1}{L(c)}a\right).$$

Then we can define a $\mathscr{K}$-equivariant bijective map

$$S(H; c, c'): \mathscr{A}(\sqrt{H^2-c}, c)\to\mathscr{A}(\sqrt{H^2-c'}, c')$$

by $S(H; c, c')=S(H; 0, c')\circ S^{-1}(H; 0, c)$, where $H\geq 0$ and $c, c'\in\boldsymbol{R}$ such that H^2-c, $H^2-c'\geq 0$ and $c\neq c'$.

$S^1=\{\lambda\in\boldsymbol{C}; |\lambda|=1\}$ acts on $\mathscr{A}(H, 0)$ by

$$\lambda(A, a)=(A-\sqrt{-1}H(\bar{\lambda}-1)\iota(a)+\sqrt{-1}H(\lambda-1)\iota(\bar{a}), \bar{\lambda}a)$$

for $\lambda\in S^1$ and $(A, a)\in\mathscr{A}(H, 0)$. Then we can define an S^1-action on $\mathscr{A}(\sqrt{H^2-c}, c)$ by

$$\lambda(A, a)=S(H; 0, c)\circ\lambda\circ S^{-1}(H; 0, c)(A, a)$$

for $\lambda\in S^1$ and $(A, a)\in\mathscr{A}(\sqrt{H^2-c}, c)$.

Direct computation gives the following:

LEMMA 3.1. (i) *For any* $\lambda\in S^1$,

$$S(H; c, c')\circ\lambda=\lambda\circ S(H; c, c').$$

(ii) *For any* $k\in\mathscr{K}$, $\lambda\in S^1$ *and* $(A, a)\in\mathscr{A}(\sqrt{H^2-c}, c)$,

$$(k\circ\lambda)(A, a)=(\lambda\circ k)(A, a).$$

Then we have the following:

THEOREM 3.2. $S(H; c, c')$ *defines the quotient map*

$$S_{\mathscr{B}}(H; c, c'): \mathscr{B}(\sqrt{H^2-c}, c)\to\mathscr{B}(\sqrt{H^2-c'}, c'),$$

which is S^1*-equivariant.*

Now let $H\geq 0$ and $c<0$ such that $H^2+c<0$.

For $(A, c)\in\mathscr{A}(0, c)$, we define a $\mathscr{K}$-equivariant bijective map

$$T(0, H; c): \mathscr{A}(0, c)\to\mathscr{A}(H, c)$$

by

$$T(0, H; c)(A, a)=\left(A-\frac{\sqrt{-1}H}{\sqrt{-(H^2+c)}}\iota(a-\bar{a}), \sqrt{\frac{-c}{-(H^2+c)}}a\right).$$

S^1 acts on $\mathscr{A}(0, c)$ by $\lambda(A, a)=(A, \bar{\lambda}a)$ for $\lambda\in S^1$ and $(A, a)\in\mathscr{A}(0, c)$. (If M is simply connected, this action coincides with the action on the space of harmonic maps from a simply connected Riemann surface into a Riemannian symmetric space defined by Burstall and Pedit [BP].)

Let $H, H'\in\boldsymbol{R}$ and $c<0$ such that H^2+c, $H'^2+c<0$.

Similar to the above case, we can define a bijective map

$$T(H, H'; c): \mathscr{A}(H, c)\to\mathscr{A}(H', c)$$

and an S^1-action on $\mathscr{A}(H, c)$, where $H, H'\geq 0$ and $c<0$ such that H^2+c, $H'^2+c<0$ and $H\neq H'$. Then we have the following:

THEOREM 3.3. *$T(H, H'; c)$ defines the quotient map*

$$T_{\mathscr{B}}(H, H'; c): \mathscr{B}(H, c)\to\mathscr{B}(H', c),$$

which is S^1-equivariant.

Combining Theorem 3.2 with Theorem 3.3, we obtain our main theorem.

REMARK. Let $H, c\in\boldsymbol{R}$ such that $H^2-c\geq 0$ and $(A, a)\in\mathscr{A}(\sqrt{H^2-c}, c)$. If M is simply connected and (A, a) corresponds to an isometric immersion $\varphi: M\to\mathfrak{M}^3(c)$ with CMC$=\sqrt{H^2-c}$, then Theorem 3.2 gives us an S^1-family of isometric immersions $\{\varphi_{\lambda,\sqrt{H^2-c'}}\}_{\lambda\in S^1}$ in $\mathfrak{M}^3(c')$ with CMC$=\sqrt{H^2-c'}$ for any c' such that $H^2-c'\geq 0$. This family is just the one obtained by Lawson [L]. In order to see this, we have only to calculate the second fundamental forms of $\varphi_{\lambda,\sqrt{H^2-c'}}$.

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