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ON $\boldsymbol{Q}$-STRUCTURES OF QUASISYMMETRIC DOMAINS

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Abstract. We will give a complete classification of $\boldsymbol{Q}$-structures of quasisymmetric domains. In the standard case, it will be shown that there are only very natural $\boldsymbol{Q}$-structures coming from semisimple $\boldsymbol{Q}$-algebras with positive involutions. As is shown in the Appendix, when the domain is symmetric, any $\boldsymbol{Q}$-structure of it as a quasisymmetric domain can uniquely be extended to one as a symmetric domain.

The purpose of this note is to determine the $\boldsymbol{Q}$-structures of quasisymmetric domains.

The notion of a quasisymmetric domain was introduced in [S3] (cf. also [S6, Ch. V]). It was shown that, among Siegel domains (of the second kind), the symmetric domains were characterized by three conditions (i), (ii), (iii). A Siegel domain is called *quasisymmetric* if it satisfies the conditions (i), (ii). It is known that any symmetric domain $\mathscr{D}$ with a fixed boundary component $\mathscr{F}$ has a natural structure of a fiber space (a Siegel domain of the third kind) over $\mathscr{F}$, in which the fiber over each point of $\mathscr{F}$ is a quasisymmetric domain. All quasisymmetric domains of "standard" type are obtained in this form (see §4), while there are quasisymmetric domains of non-standard (quadratic) type that are not obtained in this manner.

A quasisymmetric domain $\mathscr{S}_I$ is defined by a data $(U, V, A, \mathscr{C}, I)$, where U, (V, I) are real and complex vector spaces of finite dimension, I denoting a complex structure on V. $\mathscr{C}$ is a self-dual homogeneous cone in U (condition (i)) and A is an alternating bilinear map $V\times V\rightarrow U$ such that $A(v, Iv')$ $(v, v'\in V)$ is "$\mathscr{C}$-positive" (see 1.1). In §§1, 2 we summarize basic definitions and properties concerning quasisymmetric domains. Here we give the condition (ii) in the form independent of the complex structure I, viewing I as a point in the parameter space $\mathfrak{S}=\mathfrak{S}(V, A, \mathscr{C})$. To give a $\boldsymbol{Q}$-structure of $\mathscr{S}_I$ is, roughly speaking, equivalent to giving a $\boldsymbol{Q}$-structure of (U, V) such that the affine automorphism group $\mathrm{Aff}\,\mathscr{S}_I$ is defined over $\boldsymbol{Q}$. By virtue of the complete reducibility of quasisymmetric domains (see 2.5), our problem of determining $\boldsymbol{Q}$-structures of $\mathscr{S}_I$ is reduced to the $\boldsymbol{Q}$-irreducible case. A general method of determining $\boldsymbol{Q}$-irreducible $\boldsymbol{Q}$-structures of $\mathscr{S}_I$ with $V\neq 0$ is given in §3. In particular, it will be shown that a $\boldsymbol{Q}$-structure of $\mathscr{S}_I$ is essentially determined by that of the enveloping algebra of the representation of $\mathrm{Lie\,Aut}\,\mathscr{C}$ on V, which is a ($\boldsymbol{Q}$-simple) $\boldsymbol{Q}$-algebra with positive involution. Applying this method to the standard and non-standard cases, in §§4, 5,

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respectively, one can easily classify all possible $\boldsymbol{Q}$-structures of $\mathscr{S}_I$. We also give an explicit expression of A in each case.

In the simplest case, where $\mathscr{C}=\mathscr{P}_{\nu_1}(\boldsymbol{R})$, a $\boldsymbol{Q}$-structure of $\mathscr{S}_I$, denoted as $(\mathrm{III}^{(1)}_{\nu_1;\nu_2/2})$, is given as follows. One takes a pair of $\boldsymbol{Q}$-structures of U and V, for which there exist two $\boldsymbol{Q}$-vector spaces V_1 and V_2 such that one has

$$U(\boldsymbol{Q})=\mathrm{S}(V_1\otimes V_1)\,,\qquad V(\boldsymbol{Q})=V_1\otimes V_2\,,$$

S denoting the symmetrizer and $\dim_{\boldsymbol{Q}} V_i=\nu_i$ $(i=1, 2)$. Then the alternating bilinear map A and the complex structure I are given in the form

$$A(v_1\otimes v_2, v'_1\otimes v'_2)=\mathrm{S}(v_1\otimes v'_1)a_2(v_2, v'_2)$$
$$(v_i, v'_i\in V_i, i=1, 2)\,,$$
$$I=1_{V_1}\otimes I_2\,,$$

a_2 denoting a non-degenerate alternating $\boldsymbol{Q}$-bilinear form on $V_2\times V_2$ and I_2 denoting a "rational" point in the Siegel space $\mathfrak{S}=\mathfrak{S}(V_2(\boldsymbol{R}), a_2)$. It will be shown in §4 that, in the standard case, one can obtain all $\boldsymbol{Q}$-structures of $\mathscr{S}_I$, generalizing this construction to vector spaces over a division algebra over $\boldsymbol{Q}$ with positive involution.

In the Appendix, we will show that, when the domain $\mathscr{S}_I$ is symmetric, any $\boldsymbol{Q}$-structure of $\mathscr{S}_I$ as a quasisymmetric domain can be extended (uniquely) to a $\boldsymbol{Q}$-structure of it as a symmetric domain.

One of the motivations of this study is to construct a new kind of cusp singularities (cf. [S9]). Cusps of the arithmetic quotients of symmetric tube domains have been studied by many mathematicians. Especially, a generalization of the Hirzebruch conjecture, which relates the zero value of the zeta functions $Z_{\mathscr{C}}$ associated with the cone $\mathscr{C}$ with some geometric invariants of the cusp, was recently established by Ogata [O2] and Ishida [I2] (see also [SO]). In the case of quasisymmetric domains with $V\neq 0$, for which $\boldsymbol{Q}$-rank Aut$\,\mathscr{C}$ is $=1$, one can obtain similar cusps, which we propose to call *cusps of the second kind*; in the notation of §4, this occurs only in the following three cases:

$$R_{F/\boldsymbol{Q}}(\mathrm{III}^{(1)}_{1;\nu_2/2})_I\,,\quad R_{F/\boldsymbol{Q}}(\mathrm{III}^{(2)}_{2;\nu_2}, D_0, h_2)_I\,,$$
$$R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{\delta_0;(p,q)}, D_0/Z, h_2)_I\qquad(\delta_0\geq 2)\,.$$

It is expected that one can further generalize the result of Ogata and Ishida to the case of the cusps of the second kind to obtain a geometric interpretation of the values of the zeta functions $Z_{\mathscr{C}}$ at *negative* integers.

1. Siegel domains.

1.1. *Siegel domains* (*of the second kind*) (cf. [PS], [S6, Ch. III, §§5–6]). A Siegel domain is defined by the following data $(U, V, A, \mathscr{C}, I)$. U and V are finite-dimensional

real vector spaces and $A: V\times V\to U$ is an alternating bilinear map. $\mathscr{C}$ is an open convex cone in U, which is "non-degenerate" in the sense that $\bar{\mathscr{C}}\cap -\bar{\mathscr{C}}=\{0\}$. I is a complex structure on V satisfying the following condition:
(1) $A(v, Iv')$ is symmetric and "$\mathscr{C}$-positive", i.e. one has

$$A(v, Iv)\in\bar{\mathscr{C}}-\{0\} \qquad \text{for all} \quad v\in V, v\neq 0 .$$

This implies that A is non-degenerate, i.e. if $A(v, v')=0$ for all $v'\in V$, then $v=0$.

We set

$$V(\boldsymbol{C})=V\otimes_{\boldsymbol{R}}\boldsymbol{C}=V_+\oplus V_-$$

with $V_\pm=\{v\in V(\boldsymbol{C})\,|\,Iv=\pm iv\}$ and extend A in a natural manner to a $\boldsymbol{C}$-bilinear map $V(\boldsymbol{C})\times V(\boldsymbol{C})\to U(\boldsymbol{C})$, denoted again by the same letter. Then one has $A(V_+, V_+)=A(V_-, V_-)=0$ and

$$2iA(v_-, v'_+)=A(v, Iv')+iA(v, v')$$

for $v, v'\in V$, $v_\pm$ denoting the $V_\pm$-part of v.

A *Siegel domain* $\mathscr{S}_I=\mathscr{S}(U, V, A, \mathscr{C}, I)$ is defined by

$$\mathscr{S}_I=\left\{(u, w)\in U(\boldsymbol{C})\times V_+\,\middle|\,\operatorname{Im}u-\frac{i}{2}A(\bar{w}, w)\in\mathscr{C}\right\}. \tag{2}$$

When $V=\{0\}$, one obtains a tube domain $\mathscr{S}_0=U+i\mathscr{C}$.

We denote by $\mathfrak{S}=\mathfrak{S}(V, A, \mathscr{C})$ the set of all complex structures I on V satisfying the condition (1); by the assumption one has $\mathfrak{S}\neq\varnothing$. In what follows, it will be convenient to consider the complex structure I to be a point in the parameter space $\mathfrak{S}$, rather than fixing it once and for all. Then the total space $\tilde{\mathscr{S}}=\{(u, w, I)\,|\,(u, w)\in\mathscr{S}_I, I\in\mathfrak{S}\}$ is a so-called "Siegel domain of the third kind".

1.2. *Automorphism groups.* We first define the (generalized) *Heisenberg group* $\tilde{V}=H(U, V, A)$. By definition $\tilde{V}$ is the direct product $U\times V$ endowed with a multiplication

$$(u, v)(u', v')=\left(u+u'-\frac{1}{2}A(v, v'), v+v'\right) \tag{3}$$

for $(u, v), (u', v')\in\tilde{V}$. It is clear that with the natural homomorphisms one has an exact sequence

$$1\to U\to\tilde{V}\to V\to 1\,, \tag{4}$$

in which U is central. It is known that, conversely, all central extension $\tilde{V}$ of V by U (as Lie groups) is obtained in this manner with a (uniquely determined) alternating bilinear map A. In our case, A being non-degenerate, U coincides with the center of $\tilde{V}$.

We set

(5) $\mathrm{Aut}(U, V, A)=\{g=(g_1, g_2)\,|\,g_1\in GL(U), g_2\in GL(V),\ g_1\circ A=A\circ g_2\times g_2\}$,

and write $g_i=\rho_i(g)$ for $g=(g_1, g_2)\in\mathrm{Aut}(U, V, A)$. We are concerned with the following automorphism groups:

$$\begin{aligned} G_1&=\mathrm{Aut}(U, \mathscr{C})=\{g_1\in GL(U)\,|\,g_1\mathscr{C}=\mathscr{C}\}\,, \\ G&=\mathrm{Aut}(U, V, A, \mathscr{C})=\{g\in\mathrm{Aut}(U, V, A)\,|\,\rho_1(g)\in G_1\}\,, \\ G_2&=Sp(V, A)=\{g_2\in GL(V)\,|\,A\circ g_2\times g_2=A\}\,. \end{aligned} \tag{6}$$

Note that one has $\mathrm{Ker}\,\rho_1=1\times G_2$ and $\mathfrak{S}(V, A, \mathscr{C})\subset G_2$. It is known that G_2 is a reductive algebraic group of hermitian type and $\mathfrak{S}(V, A, \mathscr{C})$ is the associated symmetric domain (see 2.3 and [S5]). Since $G\subset\mathrm{Aut}\,\tilde{V}$, one can construct a semidirect product $\tilde{G}=G\cdot\tilde{V}$.

For $v\in V$ and $w\in V_+$, one defines an automorphy factor by

$$\mathscr{J}(v, w)=A\left(w+\frac{1}{2}v_+, v_-\right),$$

which satisfies the relation

$$\mathscr{J}(v+v', w)=\mathscr{J}(v, w+v'_+)+\mathscr{J}(v', w)+\frac{1}{2}A(v, v')\,.$$

Then the Heisenberg group $\tilde{V}$ acts on $\mathscr{S}_I$ by

$$(a, b)((u, w))=(u+a+\mathscr{J}(b, w), w+b_+) \quad \text{for} \quad (a, b)\in\tilde{V} \quad\text{and}\quad (u, w)\in\mathscr{S}_I\,. \tag{7}$$

On the other hand, for $I\in\mathfrak{S}(V, A, \mathscr{C})$, one puts

$$\begin{aligned} G_I&=\mathrm{Aut}(U, V, A, \mathscr{C}, I)=\{g\in G\,|\,\rho_2(g)\in GL(V, I)\}\,, \\ G_{2I}&=\mathrm{Aut}(V, A, I)=Sp(V, A)\cap GL(V, I)\,. \end{aligned}$$

Then G_I acts linearly on $\mathscr{S}_I$, and the semidirect product $\tilde{G}_I=G_I\cdot\tilde{V}$ acts affinely on $\mathscr{S}_I$. G_{2I} is a maximal compact subgroup of G_2. It is known ([PS], [S6, p. 129, Prop. 6.2]) that the affine automorphism group $\mathrm{Aff}\,\mathscr{S}_I$ of $\mathscr{S}_I$ coincides with $\tilde{G}_I$.

2. Quasisymmetric domains.

2.1. *Quasisymmetric case.* A Siegel domain $\mathscr{S}_I=\mathscr{S}(U, V, A, \mathscr{C}, I)$ is called *quasisymmetric* if two conditions (i), (ii) below are satisfied. (For the meaning of these conditions, see [S3, Prop. 1], or [S6, Ch. V, §§3, 4, especially, Prop. 4.1]. Here we state the condition (ii) in the form independent of the complex structure I. For the classification of quasisymmetric domains, see [S2] and [S3], or [S6, Ch. V, §5].)

(i) There exists a (positive definite) inner product $\langle\ \rangle$ on U such that, defining the dual of $\mathscr{C}$ by

$$\mathscr{C}^* = \{u \in U \mid \langle u, u' \rangle > 0 \text{ for all } u' \in \overline{\mathscr{C}} - \{0\}\},$$

one has $\mathscr{C} = \mathscr{C}^*$. Moreover, the automorphism group $G_1 = \mathrm{Aut}(U, \mathscr{C})$ is transitive on $\mathscr{C}$.

When this condition is satisfied, $\mathscr{C}$ is called a *self-dual homogeneous cone*. One then has $G_1 = {}^tG_1$, t denoting the transpose with respect to $\langle\ \rangle$. This implies that G_1 is a reductive "algebraic" group (in a weaker sense that the identity connected component G_1° coincides with that of the real points of a linear algebraic group defined over $\boldsymbol{R}$). The map $\theta_1 : x \mapsto -{}^tx$ is a Cartan involution of the Lie algebra $\mathfrak{g}_1$ of G_1. Let $\mathfrak{g}_1 = \mathfrak{k}_1 + \mathfrak{p}_1$ be the corresponding Cartan decomposition. Then it is known that for a suitable choice of a point e in $\mathscr{C}$ one has

$$\mathfrak{k}_1 = \{x \in \mathfrak{g}_1 \mid xe = 0\}.$$

It follows that, for each $u \in U$, there exists a uniquely determined element T_u in $\mathfrak{g}_1$ such that ${}^tT_u = T_u$ and $T_u e = u$; in particular, $T_e = 1_U$. The map $u \mapsto T_u$ gives a linear isomorphism $U \simeq \mathfrak{p}_1$.

It is well known that the vector space U endowed with a product $uu' = T_u u'$ $(u, u' \in U)$ is a formally real Jordan algebra with unit element e (cf. e.g. [S6, p. 33, Th. 8.5]). In what follows, we will normalize the inner product $\langle\ \rangle$ by setting

$$\langle u, u' \rangle = \mathrm{tr}(\kappa T_{uu'}), \tag{8}$$

where in the notation of 2.5 below $\kappa = \sum (r_i/n_i) 1_{U^{(i)}}$ with $n_i = \dim U^{(i)}$ and $r_i = \boldsymbol{R}\text{-rank}\, \mathfrak{g}_1^{(i)}$. By this relation e and $\langle\ \rangle$ determine each other uniquely.

2.2. We now state the second condition:

(ii) The homomorphism $\rho_1 : G \to G_1$ is "almost surjective", i.e. one has $\rho_1(G^\circ) = G_1^\circ$.

In what follows, we assume that the conditions (i), (ii) are satisfied. Then with the natural homomorphisms one has an exact sequence

$$1 \to G_2 \to G \to G_1 \to (\text{finite}). \tag{9}$$

Since G_1 and G_2 are reductive "algebraic", so is G. Hence there exists a connected normal "algebraic" subgroup G_1' of G such that

$$G^\circ = G_1' \cdot (1 \times G_2^\circ), \qquad G_1' \cap (1 \times G_2^\circ) = (\text{finite}). \tag{10}$$

Then the restriction of ρ_1 on G_1' gives an isogeny $G_1' \to G_1$. (Such a subgroup G_1' is uniquely determined, because G_1' is of cone type and G_2 is of hermitian type.) Note that, since I is contained in G_2°, one has $G_1' \subset G_I^\circ$ and hence $\rho_1(G_I^\circ) = G_1^\circ$. It follows that the domain $\mathscr{S}_I$ is affinely homogeneous.

Let $\mathfrak{g}$, $\mathfrak{g}_i$ $(i = 1, 2)$, and $\mathfrak{g}_1'$ denote the Lie algebras of G, G_i, and G_1', respectively. Then $\rho_1 | \mathfrak{g}_1' : \mathfrak{g}_1' \to \mathfrak{g}_1$ is an isomorphism; we put $\beta = \rho_2 \circ (\rho_1 | \mathfrak{g}_1')^{-1}$. Then β is a representation of $\mathfrak{g}_1$ on V and one has

(11) $$\mathfrak{g}'_1 = \{(x, \beta(x)) \mid x \in \mathfrak{g}_1\}\ .$$

Since $G'_1 \subset G_I$, β is actually a representation of $\mathfrak{g}_1$ in $\mathfrak{gl}(V, I)$.

2.3. *Reformulations.* For $u \in U$ and $v, v' \in V$, we set

(12) $$A_u(v, v') = \langle u, A(v, v')\rangle\ ,$$

(13) $$a(v, v') = A_e(v, v')\ .$$

Clearly a is an alternating bilinear form on $V \times V$ and for $I \in \mathfrak{S}$ the bilinear form $a(v, Iv')$ is symmetric and positive definite; in other words, if one puts

$$h_I(v, v') = a(v, Iv') + ia(v, v')\ ,$$

then h_I is a positive definite hermitian form (which is $\boldsymbol{C}$-linear in v') on the complex vector space (V, I). Let V^* and $\mathrm{Alt}(V)$ denote the dual space of V and the space of all alternating bilinear forms on $V \times V$, respectively. $\mathrm{Alt}(V)$ may be identified with the subspace of $\mathrm{Hom}_{\boldsymbol{R}}(V, V^*)$ formed of all skewsymmetric elements. We define an involution $\iota = \iota(a)$ of $\mathrm{End}_{\boldsymbol{R}} V$ by

(14) $$\iota : y \mapsto a^{-1}\,{}^t y a \qquad (y \in \mathrm{End}_{\boldsymbol{R}} V)\ .$$

Clearly, for $y \in \mathrm{End}_{\boldsymbol{R}} V$, one has $y^\iota = y$ if and only if $ay \in \mathrm{Alt}(V)$ and, for $y \in \mathrm{End}_{\boldsymbol{C}}(V, I)$, y^ι is the adjoint of y with respect to the hermitian form h_I. One sets

$$\mathrm{Her}(V, a, I) = \{y \in \mathrm{End}_{\boldsymbol{C}}(V, I) \mid y^\iota = y\}$$

and denote by $\mathscr{P}(V, a, I)$ the cone of all positive definite elements in $\mathrm{Her}(V, a, I)$ with respect to h_I.

For $u \in U$ there corresponds uniquely an element $\varphi(u)$ in $\mathrm{End}_{\boldsymbol{R}} V$ such that

(15) $$A_u(v, v') = a(v, \varphi(u)v') \qquad (v, v' \in V)\ ;$$

in particular, one has $\varphi(e) = 1_V$. Then the condition (1) is equivalent to

(16) $$\varphi(U) \subset \mathrm{Her}(V, a, I)\ , \qquad \varphi(\mathscr{C}) \subset \mathscr{P}(V, a, I)\ .$$

Note also that in this notation one has

(17) $$\begin{aligned} G_2 = Sp(V, A) &= \{g_2 \in Sp(V, a) \mid [g_2, \varphi(U)] = 0\}\ , \\ \mathfrak{S}(V, A, \mathscr{C}) &= \mathfrak{S}(V, a) \cap G_2\ , \end{aligned}$$

$\mathfrak{S}(V, a)$ denoting the "Siegel space" associated with $Sp(V, a)$ (i.e. the space of all complex structures I on V such that $a(v, Iv')$ is symmetric and positive definite). This implies that G_2 is a reductive algebraic group of hermitian type with a Cartan involution

$$\theta_2 : g_2 \mapsto I^{-1} g_2 I\ ,$$

and $\mathfrak{S}(V, A, \mathscr{C})$ is the associated symmetric domain (cf. [S5]).

Now, in the quasisymmetric case, one has for $x \in \mathfrak{g}_1$

$$xA(v, v') = A(\beta(x)v, v') + A(v, \beta(x)v') \qquad (v, v' \in V),$$

or equivalently,

$$(\beta 1) \qquad \varphi({}^t x u) = \beta(x)^\iota \varphi(u) + \varphi(u)\beta(x) \qquad (u \in U).$$

LEMMA 1. *The representation* $\beta: \mathfrak{g}_1 \to \mathfrak{gl}(V, I)$ *defined by* (11) *satisfies the relation*

$$(\beta 2) \qquad \beta({}^t x) = \beta(x)^\iota \quad \textit{for} \quad x \in \mathfrak{g}_1,$$

where $\iota = \iota(a)$.

PROOF. Putting $u = e$ in $(\beta 1)$ one sees that $x \in \mathfrak{k}_1$ implies $\beta(x) \in i\,\mathrm{Her}(V, a, I)$. It follows ([S2, p. 127]) that β can be written as a commutative sum of two representations β_0, $\beta_1: \mathfrak{g}_1 \to \mathfrak{gl}(V, I)$ such that

$$(*) \qquad \begin{aligned} &\beta_0(\mathfrak{g}_1) \subset i\,\mathrm{Her}(V, a, I), \\ &\beta_1({}^t x) = \beta_1(x)^\iota \qquad (x \in \mathfrak{g}_1). \end{aligned}$$

Since G'_1 is "algebraic" and $\rho_i \mid G'_1$ $(i = 1, 2)$ are rational, all eigenvalues of $\beta(x)$ $(x \in \mathfrak{p}_1)$ are real. On the other hand, $(*)$ implies that for x in $\mathfrak{p}_1$ all eigenvalues of $\beta_0(x)$, resp. $\beta_1(x)$ are purely imaginary, resp. real. Hence one has $\beta_0(\mathfrak{p}_1) = 0$ and, since $\mathfrak{g}_1$ is generated by $\mathfrak{p}_1$, one has $\beta_0 = 0$. Thus $\beta = \beta_1$ satisfies $(\beta 2)$. q.e.d.

By $(\beta 1)$ and $(\beta 2)$ one has

$$(**) \qquad \varphi(T_u u') = \beta(T_u)\varphi(u') + \varphi(u')\beta(T_u).$$

Hence putting $u' = e$, one has

$$(18) \qquad \beta(T_u) = \frac{1}{2}\varphi(u) \qquad \text{for} \quad u \in U;$$

in particular, $\beta(1_U) = (1/2)1_V$. Since $\mathfrak{g}_1$ is generated by $\mathfrak{p}_1$, the relation (18) shows that β is uniquely determined by φ. (This gives another proof for the uniqueness of G'_1.)

[Note that the relations $(**)$ and (18) imply

$$(19) \qquad \varphi(uu') = \frac{1}{2}\{\varphi(u)\varphi(u') + \varphi(u')\varphi(u)\} \qquad (u, u' \in U),$$

which means that the map φ is a unital Jordan algebra homomorphism of (U, e) into $\mathrm{Her}(V, a, I)$ (cf. [S6, loc. cit.]).]

2.4. *Admissible triples.* Let $(U, V, A, \mathscr{C})$ be a data satisfying the conditions (i), (ii). In general, a triple (e, a, β) formed of $e \in \mathscr{C}$, a non-degenerate alternating bilinear form a on $V \times V$, and a representation $\beta: \mathfrak{g}_1 \to \mathfrak{gl}(V)$ is called an *admissible triple* belonging to $(U, V, \mathscr{C})$, if there exists a linear map $\varphi: U \to \mathrm{End}_{\mathbf{R}} V$ with $\varphi(e) = 1_V$ such that the conditions (β1), (β2) are satisfied with $\iota = \iota(a)$. Since these conditions imply (18), β and φ determine each other uniquely. They also imply that $a\varphi(U) \subset \mathrm{Alt}(V)$. For an admissible triple (e, a, β) one sets

$$\mathfrak{S}(V, a, \beta) = \{I \in \mathfrak{S}(V, a) \mid [I, \beta(\mathfrak{g}_1)] = 0\} . \tag{20}$$

If an admissible triple (e, a, β) comes from the data $(U, V, A, \mathscr{C})$ as explained in 2.3, then it is said to be belonging to $(U, V, A, \mathscr{C})$. In that case, one has by (17)

$$\mathfrak{S}(V, A, \mathscr{C}) = \mathfrak{S}(V, a, \beta) .$$

In general, two admissible triples (e, a, β) and (e', a', β') are called *equivalent* if $\beta = \beta'$ and if there exists $g'_1 \in G'_1$ such that one has $e' = \rho_1(g'_1)e$ and $a' = a \circ \beta(g_1'^{-1}) \times \beta(g_1'^{-1})$. Clearly, two admissible triples belonging to the same $(U, V, A, \mathscr{C})$ are equivalent.

Conversely, suppose that one has $(U, \mathscr{C})$ satisfying the condition (i), a real vector space V, and an admissible triple (e, a, β) belonging to $(U, V, \mathscr{C})$. Then, it is easy to see that, if $I \in \mathscr{S}(V, a, \beta)$, then the linear map $\varphi: U \to \mathrm{End}_{\mathbf{R}} V$ associated with β satisfies the condition (16). Hence, if one defines a bilinear map $A: V \times V \to U$ by (12) and (15), then A is an alternating bilinear map satisfying the condition (1). In this manner, one recovers the data $(U, V, A, \mathscr{C})$ satisfying (i), (ii), to which the triple (e, a, β) is belonging. Clearly equivalent admissible triples give rise to one and the same data $(U, V, A, \mathscr{C})$.

Thus we have shown that *to give a data* $(U, V, A, \mathscr{C})$ (*with* $\mathfrak{S}(V, A, \mathscr{C}) \neq \emptyset$) *satisfying* (i), (ii) *is equivalent to giving* $(U, \mathscr{C})$ *satisfying* (i), *a real vector space* V, *and an equivalence class of admissible triples* (e, a, β) *belonging to* $(U, V, \mathscr{C})$ (*for which* $\mathfrak{S}(V, a, \beta) \neq \emptyset$).

2.5. *Complete reducibility.* Let $(U, V, A, \mathscr{C}, I)$ be a data satisfying the conditions (i), (ii), and let (e, a, β) be an admissible triple belonging to it. Let

$$(U, \mathscr{C}) = \prod_{i=1}^{l} (U^{(i)}, \mathscr{C}^{(i)}) \tag{21}$$

be the direct decomposition of $(U, \mathscr{C})$ into irreducible factors. Then each $\mathscr{C}^{(i)}$ is an irreducible self-dual homogeneous cone in $U^{(i)}$. If one sets

$$G_1^{(i)} = \mathrm{Aut}(U^{(i)}, \mathscr{C}^{(i)}) , \qquad \mathfrak{g}_1^{(i)} = \mathrm{Lie}\, G_1^{(i)} ,$$

then one has

$$\mathfrak{g}_1 = \bigoplus_{i=1}^{l} \mathfrak{g}_1^{(i)} , \qquad \mathfrak{g}_1^{(i)} = \{1_{U^{(i)}}\}_{\mathbf{R}} \oplus \mathfrak{g}_1^{(i)s} , \tag{22}$$

where $\mathfrak{g}_1^{(i)s}$ (the semisimple part of $\mathfrak{g}_1^{(i)}$) is simple or reduces to $\{0\}$. One has

$$e = \sum_{i=1}^{l} e^{(i)} , \qquad e^{(i)} \in \mathscr{C}^{(i)} .$$

One also has the following decomposition of the representation space ([S2] or [S6, p. 237, Prop. 5.2]):

$$V=\bigoplus_{i=1}^{l} V^{(i)}, \qquad \beta=\bigoplus \beta^{(i)},$$
(23)
$$a=\sum a^{(i)}, \qquad I=\sum I^{(i)},$$

where $V^{(i)}=\beta(1_{U^{(i)}})V$, $(e^{(i)}, a^{(i)}, \beta^{(i)})$ is an admissible triple belonging to $(U^{(i)}, V^{(i)}, \mathscr{C}^{(i)})$, and $I^{(i)}\in\mathfrak{S}(V^{(i)}, a^{(i)}, \beta^{(i)})$.

It follows that one has $A=\sum A^{(i)}$ with

$$A^{(i)}\colon V^{(i)}\times V^{(i)}\to U^{(i)},$$

each $(U^{(i)}, V^{(i)}, A^{(i)}, \mathscr{C}^{(i)}, I^{(i)})$ $(1\leq i\leq l)$ being a data satisfying the conditions (i), (ii), to which the triple $(e^{(i)}, a^{(i)}, \beta^{(i)})$ is belonging.

Thus one obtains the direct decompositions of the domains:

$$\mathscr{S}(U, V, A, \mathscr{C}, I)=\prod_{i=1}^{l} \mathscr{S}(U^{(i)}, V^{(i)}, A^{(i)}, \mathscr{C}^{(i)}, I^{(i)}), \tag{24}$$

$$\mathfrak{S}(V, A, \mathscr{C})=\prod_{i=1}^{l} \mathfrak{S}(V^{(i)}, A^{(i)}, \mathscr{C}^{(i)}), \tag{25}$$

which are known to be the unique irreducible decompositions of $\mathscr{S}_I$ and $\mathfrak{S}$ (as complex manifolds) ([S6, p. 237, Th. 5.3]).

3. Q-structures of a quasisymmetric domain.

3.1. *Definition of a Q-structure.* Let $(U, V, A, \mathscr{C}, I)$ be a data defining a quasisymmetric domain $\mathscr{S}_I$ and (e, a, β) an admissible triple belonging to it. By a *Q-structure* of $\mathscr{S}_I$ we mean a pair of Q-structures of U, V, i.e., a pair of Q-vector spaces U_0, V_0 such that $U=U_0\otimes_Q R$, $V=V_0\otimes_Q R$, satisfying the conditions (Q1), (Q2) below.

(Q1) *The Lie algebra $\mathfrak{g}_1$ and the bilinear map A are defined over Q.*

This condition implies that the groups $\tilde{V}$, G, and G_i $(i=1, 2)$ are defined over Q; hence so is the "algebraic" subgroup G_1' in (10). It follows that the representation $\beta\colon \mathfrak{g}_1\to\mathfrak{gl}(V)$ is also defined over Q.

Under the condition (Q1), we can always choose e in $U_0=U(Q)$. Then the corresponding Cartan involution θ_1 of $\mathfrak{g}_1$ and hence $\mathfrak{k}_1$, $\mathfrak{p}_1$, the linear map $u\mapsto T_u$ (hence the normalized inner product $\langle\ \ \rangle$) are defined over Q. The bilinear form $a=A_e$ is also defined over Q. Conversely, if the triple (e, a, β) is defined over Q, then so is A. Thus we can rephrase the condition (Q1) as

(Q1′) *The Lie algebra $\mathfrak{g}_1$ is defined over Q, and the triple (e, a, β) can be taken to be defined over Q.*

Next we state the condition (Q2):

(Q2) *The Cartan involution of* $\mathfrak{g}_2$ *defined by* I *is* $\boldsymbol{Q}$*-rational.*

This means that the point I in the symmetric domain $\mathfrak{S}=\mathfrak{S}(V, A, \mathscr{C})$ is "rational" (with respect to the given $\boldsymbol{Q}$-structure) in the sense of [S8]. It follows that G_I and G_{2I} are defined over $\boldsymbol{Q}$. [Note that (Q2) does not necessarily imply that $GL(V, I)$ or $\mathrm{Her}(V, a, I)$ are defined over $\boldsymbol{Q}$, and that under (Q1) there may be no rational points in $\mathfrak{S}$.]

3.2. $\boldsymbol{Q}$*-irreducible* $\boldsymbol{Q}$*-forms.* We assume that a $\boldsymbol{Q}$-structure (U_0, V_0) satisfying the conditions (Q1), (Q2) is given. By virtue of the complete reducibility we may (hence will) further assume, without any loss of generality, that $(U, \mathscr{C})$ is $\boldsymbol{Q}$-irreducible, i.e. no proper partial product in the direct decomposition (21) is defined over $\boldsymbol{Q}$. The $\boldsymbol{Q}$-structure of $\mathscr{S}_I$ is then called $\boldsymbol{Q}$*-irreducible.*

In the case $V=0$, the domain $\mathscr{S}_I$ is a symmetric tube domain, for which our problem of classifying $\boldsymbol{Q}$-structures becomes trivial. Hence, in what follows, *we will always assume that* U *is* $\boldsymbol{Q}$*-irreducible and* $V\neq 0$. Then the representation β is faithful and φ is injective. Note that, if $\dim U^{(1)}=1$, one has $\mathfrak{g}_1^{(1)}=0$ and our theory becomes also trivial.

The Galois group $\mathscr{G}=\mathrm{Gal}\,\bar{\boldsymbol{Q}}/\boldsymbol{Q}$ acts transitively on the set $\{U^{(i)}\ (1\leq i\leq l)\}$. Hence, if one puts $\mathscr{G}_1=\{\sigma\in\mathscr{G}\mid U^{(1)\sigma}=U^{(1)}\}$, then the field $F\subset\bar{\boldsymbol{Q}}$ corresponding to $\mathscr{G}_1$ by Galois theory is a totally real number field of degree l. If one sets $\mathscr{G}=\coprod_{i=1}^{l}\mathscr{G}_1\sigma_i$ with a set of representatives $\{\sigma_i\}$ $(\sigma_1=1)$ for $\mathscr{G}_1\diagdown\mathscr{G}$, then one has $U^{(i)}=U^{(1)\sigma_i}$. In the notation of 2.5, $\mathfrak{g}_1^{(1)}$ and $e^{(1)}$ are then defined over F. Moreover, $V^{(1)}=\beta(1_{U^{(1)}})V$ is defined over F and hence so are also $a^{(1)}$, $\beta^{(1)}$, $A^{(1)}$, etc. and the Cartan involution of $\mathfrak{g}_2^{(1)}$ defined by $I^{(1)}$. The corresponding objects $\mathfrak{g}_1^{(i)}$, etc. for $2\leq i\leq l$ are obtained from these by the conjugation σ_i. By abuse of notation, we sometimes express this situation by writing $\mathfrak{g}_1=R_{F/\boldsymbol{Q}}(\mathfrak{g}_1^{(1)})$, etc. Note that if $\dim U^{(1)}>1$, $\mathfrak{g}_1^s$ (the semisimple part of $\mathfrak{g}_1$) is $\boldsymbol{Q}$-simple and "pure" (i.e. all $\boldsymbol{R}$-simple factors $\mathfrak{g}_1^{(i)s}$ are mutually $\boldsymbol{R}$-isomorphic). The representations $\beta^{(i)}$ are also mutually $\boldsymbol{R}$-equivalent in an obvious sense.

By the above observation, we see that the problem of determining all $\boldsymbol{Q}$-structures of $\mathscr{S}_I$ satisfying the conditions (Q1), (Q2) can be solved in the following steps.

0. Fix a totally real number field F of degree l.

1. Find all F-structures of $(U^{(1)}, V^{(1)})$ such that $\mathfrak{g}_1^{(1)}$ and the faithful representation $\beta^{(1)}$ are defined over F. Such an F-structure of $(U^{(1)}, V^{(1)})$ will be called *admissible.* Then we set $U=R_{F/\boldsymbol{Q}}U^{(1)}$, $\mathfrak{g}_1=R_{F/\boldsymbol{Q}}(\mathfrak{g}_1^{(1)})$, and $(V, \beta)=R_{F/\boldsymbol{Q}}(V^{(1)}, \beta^{(1)})$. The $(U^{(i)}, V^{(i)})$ $(i\geq 2)$ are given the conjugate admissible F^{σ_i}-structures.

2. Choose $e\in\mathscr{C}\cap U(\boldsymbol{Q})$ and find a non-degenerate alternating bilinear form $a^{(1)}$ on $V^{(1)}\times V^{(1)}$ defined over F such that $(e^{(1)}, a^{(1)}, \beta^{(1)})$ is admissible. Then all the conjugates $(e^{(1)\sigma_i}, a^{(1)\sigma_i}, \beta^{(1)\sigma_i})$ $(2\leq i\leq l)$ are automatically admissible.

In this way one obtains an admissible triple (e, a, β) defined over $\boldsymbol{Q}$, which determines an alternating bilinear map A defined over $\boldsymbol{Q}$. Thus one has a $\boldsymbol{Q}$-structure of $\mathscr{S}_I$ satisfying (Q1).

3. Finally, find all rational points I in the symmetric domain $\mathfrak{S}=\mathfrak{S}(V, A, \mathscr{C})$ with respect to the given $\boldsymbol{Q}$-structure.

The solution of the step 3 was already given in [S8]. We give solutions of the steps 1 and 2 in the succeeding sections.

3.3. *The $\boldsymbol{R}$-primary case.* For simplicity, in the rest of this section, we assume that the representation $(V^{(1)}, \beta^{(1)})$ is $\boldsymbol{R}$-primary, i.e. a direct sum of mutually equivalent $\boldsymbol{R}$-irreducible representations. Actually, it is known ([S2]) that this is the case except for the case where $\mathscr{C}^{(1)}$ is a quadratic cone $\mathscr{P}(1, n_1-1)$ with $n_1\equiv 2 \pmod 4$.

In what follows, a division $\boldsymbol{R}$-algebra D_1 is always endowed with its standard involution $\xi\mapsto\bar{\xi}$. We denote by δ_1 and d_1 the degree of D_1 over its center and the degree of the center over $\boldsymbol{R}$, respectively; i.e., $\delta_1=1$ for $D_1=\boldsymbol{R}$, $\boldsymbol{C}$ and $\delta_1=2$ for $D_1=\boldsymbol{H}$, and $d_1=1$ for $D_1=\boldsymbol{R}$, $\boldsymbol{H}$ and $d_1=2$ for $D_1=\boldsymbol{C}$.

Let $(V_1^{(1)}, \beta_1^{(1)})$ be an $\boldsymbol{R}$-irreducible representation of $\mathfrak{g}_1^{(1)}$ contained in $(V^{(1)}, \beta^{(1)})$ and put $V_2^{(1)}=\mathrm{Hom}_{\mathfrak{g}_1^{(1)}}(V_1^{(1)}, V^{(1)})$. Then there exists a uniquely determined division $\boldsymbol{R}$-algebra D_1 such that $V_1^{(1)}$ is a *right* D_1-module and the $\mathfrak{g}_1^{(1)}$-endomorphisms of $V_1^{(1)}$ are given by the right multiplication μ_ξ ($\xi\in D_1$). Then $V_2^{(1)}$ has a natural structure of a *left* D_1-module defined by $\xi v_2=v_2\circ\mu_\xi$, and one has a tensor product decomposition:

$$V^{(1)}=V_1^{(1)}\otimes_{D_1}V_2^{(1)}, \tag{26a}$$

$$\beta^{(1)}=\beta_1^{(1)}\otimes 1. \tag{26b}$$

Suppose that $(U^{(1)}, V^{(1)})$ is given an admissible F-structure. Then, $(V^{(1)}(F), \beta^{(1)})$ is F-primary. Hence, in a manner similar to the above, one has an F-irreducible representation (V_1, β_1) over F, $V_2=\mathrm{Hom}_{\mathfrak{g}_1^{(1)}(F)}(V_1, V^{(1)}(F))$, and a division F-algebra D_0, such that V_1 and V_2 are right and left D_0-modules, respectively, and

$$V^{(1)}(F)=V_1\otimes_{D_0}V_2, \tag{27a}$$

$$\beta^{(1)}\mid V^{(1)}(F)=\beta_1\otimes 1, \tag{27b}$$

(cf. [S1, pp. 230–231, Prop. 1, 2], or [S6, Ch. IV, §1]).

Since $\mathfrak{g}_1^{s}$ is pure, one has decompositions of $V^{(i)}=V^{(1)\sigma_i}$ similar to (26a) with the *same* D_1 for all $1\le i\le l$. To be more precise, let $c_1^{(i)}$ be a primitive idempotent in $D_0^{\sigma_i}(\boldsymbol{R})=D_0^{\sigma_i}\otimes_{F^{\sigma_i}}\boldsymbol{R}$ and fix an $\boldsymbol{R}$-isomorphism

$$\psi_1^{(i)}: D_1 \xrightarrow{\sim} c_1^{(i)}D_0^{\sigma_i}(\boldsymbol{R})c_1^{(i)}.$$

Then the D_1-module $V_1^{(i)}=(V_1^{\sigma_i}(\boldsymbol{R})c_1^{(i)}, \psi_1^{(i)})$ gives an $\boldsymbol{R}$-irreducible representation of $\mathfrak{g}_1^{(i)}$ contained in $(V^{(i)}, \beta^{(i)})$. (In particular, one may assume that $V_1^{(1)}$ is given in this manner.) Hence, putting $V_2^{(i)}=(c_1^{(i)}V_2^{\sigma_i}(\boldsymbol{R}), \psi_1^{(i)})$, one has

$$V^{(i)}=V_1^{(i)}\otimes_{D_1}V_2^{(i)}, \tag{28a}$$

$$\beta^{(i)}=\beta_1^{(i)}\otimes 1 \qquad (1\le i\le l). \tag{28b}$$

One denotes the degree of D_0 over its center Z by δ_0, and the D_0-rank of V_j $(j=1, 2)$ by v_j. Let $D_0(\boldsymbol{R})\simeq M_{s_1}(D_1)$; then one has $\delta_0=\delta_1 s_1$ and

$$\dim_{\boldsymbol{R}} V_j^{(i)}=v_j s_1 \delta_1^2 d_1\,, \quad \dim_{\boldsymbol{R}} V^{(i)}=v_1 v_2 \delta_0^2 d_1 \qquad (1\leq i\leq l,\ j=1, 2)\,. \tag{29}$$

Since one has $Z^{\sigma_i}(\boldsymbol{R})=\boldsymbol{R}$ or $\simeq \boldsymbol{C}$ simultaneously for $1\leq i\leq l$, according as $d_1=1$ or 2, Z is either $=F$ or a totally imaginary quadratic extension of F.

3.4. *The algebra* $\mathscr{A}_1$. Let $\mathscr{A}_1$ denote the $\boldsymbol{R}$-subalgebra of $\mathrm{End}_{\boldsymbol{R}} V^{(1)}$ generated by $\beta^{(1)}(\mathfrak{g}_1^{(1)})$. Then $\mathscr{A}_1$ is $\boldsymbol{R}$-simple and $\mathscr{A}_1\simeq \mathrm{End}_{D_1}(V_1^{(1)})\sim D_1$. Moreover, $\mathscr{A}_1$ is defined over F and $\mathscr{A}_1(F)\simeq \mathrm{End}_{D_0}(V_1)\sim D_0$. $\mathscr{A}_1$ is of degree $v_1\delta_0 d_1=v_1 s_1\delta_1 d_1$ over $\boldsymbol{R}$.

LEMMA 2. *For each Cartan involution θ_1 of $\mathfrak{g}_1^{(1)}$ there exists a uniquely determined involution ι_1 of $\mathscr{A}_1$ such that one has*

$$\beta^{(1)}(\theta_1 x)=-\beta^{(1)}(x)^{\iota_1}\,. \tag{30}$$

Such an involution ι_1 is positive.

PROOF. Let θ_1 be a Cartan involution of $\mathfrak{g}_1^{(1)}$. Then θ_1 extends to a Cartan involution θ'_1 of $(\mathscr{A}_1)_{\mathrm{Lie}}$, which is reductive. Then there exists a positive involution ι_1 of $\mathscr{A}_1$ such that one has $\theta'_1 y=-y^{\iota_1}$ for $y\in\mathscr{A}_1$. This ι_1 satisfies (30). Since $\mathscr{A}_1$ is generated by $\beta^{(1)}(\mathfrak{g}_1^{(1)})$, ι_1 is uniquely determined. q.e.d.

It follows that, if one has an admissible F-structure on $(U^{(1)}, V^{(1)})$ and if $e\in\mathscr{C}\cap U(\boldsymbol{Q})$, then the involution ι_1 corresponding to θ_1 determined by $e^{(1)}$ is defined over F, and for each i the conjugate $\iota_1^{\sigma_i}$ corresponds to the Cartan involution $\theta_1^{\sigma_i}$ of $\mathfrak{g}_1^{(i)}$ determined by $e^{(i)}=e^{(1)\sigma_i}\in\mathscr{C}^{(i)}$. Thus ι_1 is *totally positive*, i.e., all the conjugates $\iota_1^{\sigma_i}$ are positive. Otherwise expressed, $R_{F/\boldsymbol{Q}}(\iota_1)$ is a positive involution of the simple $\boldsymbol{Q}$-algebra $R_{F/\boldsymbol{Q}}(\mathscr{A}_1)(\boldsymbol{Q})$. It follows that D_0 has also a totally positive involution ι_0 such that $\iota_0|Z=\iota_1|Z$.

As is well known, for the algebra D_0 with a totally positive involution one has only the following four possibilities:

(Type 1.1) $D_0=F$; $\delta_0=1$, $D_1=\boldsymbol{R}$,

(Type 1.2) D_0 is a totally indefinite quaternion algebra over F; $\delta_0=2$, $D_1=\boldsymbol{R}$,

(Type 2) D_0 is a totally definite quaternion algebra over F; $\delta_0=2$, $D_1=\boldsymbol{H}$,

(Type 3) D_0 is a central division algebra over a CM-field Z with an involution of the second kind with respect to Z/F; $\delta_0\geq 1$, $D_1=\boldsymbol{C}$.

Note that in case $\delta_0=\delta_1$ the (unique) positive involution ι_0 of D_0 is induced by the canonical involution of D_1.

We identify $\mathscr{A}_1(F)$ with $\mathrm{End}_{D_0}(V_1)$ and set

$$\varphi_1(u)=2\beta_1(T_u) \qquad \text{for} \quad u\in U^{(1)}\,. \tag{31}$$

Then φ_1 *is a linear map*: $U^{(1)}\to \mathrm{Her}(\mathscr{A}_1, \iota_1)$ *and the pair* (β_1, φ_1) *satisfies the relations similar to* $(\beta 1)$, $(\beta 2)$:

$$\varphi_1(x(u)) = \beta_1(x)\varphi_1(u) + \varphi_1(u)\beta_1(x)^{\iota_1}, \quad (32)$$
$$\beta_1(\theta_1(x)) = -\beta_1(x)^{\iota_1}, \quad \varphi_1(e^{(1)}) = 1 .$$

One notes that, given a "base point" $e^{(1)} \in \mathscr{C}^{(1)}$, the involution ι_1 and the map φ_1 are uniquely characterized by (32). These relations also imply that φ_1 is a Jordan algebra homomorphism of $U^{(1)}$ into $(\mathscr{A}_1)_{\mathrm{Jordan}}$ and that $\varphi_1(\mathscr{C}^{(1)})$ is contained in the cone of all positive elements in $\mathrm{Her}(\mathscr{A}_1, \iota_1)$.

PROPOSITION 1. *The normalized inner product of $U^{(1)}$ corresponding to $e^{(1)}$ is given by*

$$\langle u, u' \rangle = r_1(v_1\delta_0 d_1)^{-1}\operatorname{tr}(\varphi_1(u)\varphi_1(u')) \quad (u, u' \in U^{(1)}), \quad (33)$$

where $r_1 = \boldsymbol{R}$-rank $\mathfrak{g}_1^{(1)}$ and tr *denotes the reduced trace* $\operatorname{tr}_{\mathscr{A}_1/\boldsymbol{R}}$.

Put $\langle u, u' \rangle' = \operatorname{tr}(\varphi_1(u)\varphi_1(u'))$. Then by (32) one has

$$\langle xu, u' \rangle' = -\langle u, \theta_1(x)u' \rangle' \qquad \text{for} \quad x \in \mathfrak{g}_1^{(1)} .$$

Hence one has $\langle \ \rangle' = c\langle \ \rangle$ with a real constant c. Putting $u = u' = e^{(1)}$, one has by (8) $c = r_1^{-1}\operatorname{tr}(1) = r_1^{-1}v_1\delta_0 d_1$, as desired.

3.5. We shall now show that, conversely, one can obtain admissible F-structures of $(U^{(1)}, V^{(1)})$ from an F-algebra structure of $\mathscr{A}_1$.

THEOREM 1. *Let $\mathscr{A}_1$ be the subalgebra of $\mathrm{End}_{\boldsymbol{R}} V^{(1)}$ generated by $\beta^{(1)}(\mathfrak{g}_1^{(1)})$. Then an F-algebra structure of $\mathscr{A}_1$ gives rise to an admissible F-structure of $(U^{(1)}, V^{(1)})$ if and only if the following conditions* (a), (b), (c) *are satisfied*:

(a) *$\beta^{(1)}(\mathfrak{g}_1^{(1)})$ is a linear subspace of $\mathscr{A}_1$ defined over F.*

(b) *There exists a totally positive involution ι_1 of $\mathscr{A}_1(F)$ leaving $\beta^{(1)}(\mathfrak{g}_1^{(1)})(F)$ invariant.*

(c) *Let $\mathscr{A}_1(F) \sim D_0$, $\mathscr{A}_1 \sim D_1$ and let δ_0 and δ_1 be the degree of D_0 and D_1 over the center. Then the multiplicity of the $\boldsymbol{R}$-irreducible representation $\beta_1^{(1)}$ in $\beta^{(1)}$ is divisible by $s_1 = \delta_0/\delta_1$.*

PROOF. The "only if" part is clear from what we said in 3.4. To prove the "if" part, we construct an admissible F-structure of $(U^{(1)}, V^{(1)})$, starting from an F-algebra structure of $\mathscr{A}_1$ satisfying the conditions (a), (b), (c).

Take a primitive idempotent c_1 in $\mathscr{A}_1(F)$ and fix an F-isomorphism

$$\psi_1 : D_0 \xrightarrow{\sim} c_1\mathscr{A}_1(F)c_1 .$$

Then $V_1 = (\mathscr{A}_1(F)c_1, \psi_1)$ is a (right) D_0-module of rank v_1 and one can make an identification $\mathscr{A}_1(F) = \mathrm{End}_{D_0}(V_1)$. By the condition (a) one has an F-Lie algebra structure on $\mathfrak{g}_1^{(1)}$ such that $\beta_1 = \beta^{(1)} | \mathfrak{g}_1^{(1)}(F)$ is an F-linear representation of $\mathfrak{g}_1^{(1)}(F)$ in $\mathscr{A}_1(F) = \mathrm{End}_{D_0}(V_1)$. Then, defining $V_j^{(1)}$ $(j = 1, 2)$ as explained in 3.3, one obtains the decomposition (26a), (26b). By the condition (c), the multiplicity of $\beta_1^{(1)}$ in $\beta^{(1)}$ can be written as $\nu_2 s_1$, and one has the relation (29) for $i=1$.

Now an F-structure of $V^{(1)}$ is defined as follows. Fix an $\boldsymbol{R}$-isomorphism $D_0(\boldsymbol{R}) \simeq M_{s_1}(D_1)$ and the matrix units $(e_{ij}^{(1)})_{1 \le i, j \le s_1}$ in $D_0(\boldsymbol{R})$ such that $c_1^{(1)} = e_{11}^{(1)}$. Then there exist injective $\mathfrak{g}_1^{(1)}$-equivariant linear maps

$$\phi_i : V_1(\boldsymbol{R}) = \bigoplus_{k=1}^{s_1} V_1^{(1)} e_{1k}^{(1)} \to V^{(1)} \qquad (1 \le i \le \nu_2)$$

such that one has $V^{(1)} = \bigoplus \phi_i(V_1(\boldsymbol{R}))$. Hence one can define an F-structure on $V^{(1)}$ so that

$$V^{(1)}(F) = \bigoplus_{i=1}^{\nu_1} \phi_i(V_1) .$$

Then, in the manner explained in 3.3, one obtains the decomposition (27a), (27b).

An F-structure of $U^{(1)}$ is defined as follows. Take a totally positive involution ι_1 of $\mathscr{A}_1(F)$ leaving $\beta_1(\mathfrak{g}_1^{(1)}(F))$ invariant. Let θ_1 be a Cartan involution of $\mathfrak{g}_1^{(1)}$ defined by (30) and let $e^{(1)}$ be the corresponding point in $U^{(1)}$ (determined up to a scalar multiplication). One defines an F-structure of $U^{(1)}$ so that

$$U^{(1)}(F) = \{u \in U^{(1)} \mid T_u \in \mathfrak{p}_1^{(1)}(F)\} .$$

Then, clearly, $U^{(1)}(F)$ is invariant under $\mathfrak{g}_1^{(1)}(F)$, and one has $e^{(1)} \in U^{(1)}(F)$, $\varphi_1(U^{(1)}(F)) \subset \mathrm{Her}(\mathscr{A}_1(F), \iota_1)$. Thus one obtains an admissible F-structure of $(U^{(1)}, V^{(1)})$. q.e.d.

In the above notation, since $\theta_1^{(i)} = \theta_1^{\sigma_i}$ is a Cartan involution of $\mathfrak{g}_1^{(i)}$, one may, replacing $e^{(1)}$ by $\alpha e^{(1)}$ with $\alpha \in F^\times$ if necessary, assume that $e^{(i)} = e^{(1)\sigma_i} \in \mathscr{C}^{(i)}$ for all $1 \le i \le l$, i.e. $e = \sum e^{(i)} \in \mathscr{C}$.

REMARK. The F-algebra structure of $\mathscr{A}_1$ satisfying (a) is uniquely determined by that of $\mathfrak{g}_1^{(1)}$. The admissible F-structure of $(U^{(1)}, V^{(1)})$ compatible with a given F-structure of $\mathfrak{g}_1^{(1)}$ is uniquely determined up to $\mathfrak{g}_1^{(1)}$-automorphisms of $(U^{(1)}, V^{(1)})$.

3.6. *Determination of* $a^{(1)}$. Let $\varepsilon \in \{\pm 1\}$. In general, by a (D_0, ι_0)-ε-*hermitian form* h_1 on a right D_0-module V_1 we mean an F-bilinear map $h_1 : V_1 \times V_1 \to D_0$ satisfying the following conditions:

$$h_1(v_1, v_1'\xi) = h_1(v_1, v_1')\xi , \quad h_1(v_1', v_1) = \varepsilon h_1(v_1, v_1')^{\iota_0}$$
$$\text{for} \quad v_1, v_1' \in V_1 , \quad \xi \in D_0 .$$

The dual V_1^* of V_1 (as an F-vector space) is viewed as a left D_0-module in a natural manner. Then the hermitian form h_1 may be identified with an ε-symmetric (D_0, ι_0)-semilinear map $h_1 : V_1 \to V_1^*$ by the relation

$$\mathrm{tr}_{D_0/F}(h_1(v_1, v_1')) = \langle v_1, h_1(v_1') \rangle . \tag{34}$$

Similarly, a (D_0, ι_0)-ε'-hermitian form h_2 on a left D_0-module V_2 (satisfying this time $h_2(\xi v_2, v_2')=\xi h_2(v_2, v_2')$, etc.) is identified with an ε'-symmetric (D_0, ι_0)-semilinear map $h_2\colon V_2 \to V_2^*$ by a relation similar to (34), V_2^* being viewed as a right D_0-module.

Now suppose one has an admissible F-structure on $(U^{(1)}, V^{(1)})$ and $e \in \mathscr{C} \cap U^{(1)}(\boldsymbol{Q})$. Let ι_1 be the totally positive involution of $\mathscr{A}_1(F)=\mathrm{End}_{D_0}(V_1)$ corresponding to $e^{(1)}$ in the sense of Lemma 2. Then ι_1 can be written in the form

$$(35) \qquad \iota_1=\iota_1(h_1)\colon y \mapsto h_1^{-1}\,{}^t y h_1$$

with a (D_0, ι_0)-η-hermitian form h_1 on V_1 $(\eta=\pm 1)$ uniquely determined up to a scalar multiplication of $F^\times$. (In the case of Type 3, one may, hence will, assume that $\eta=1$.)

The hermitian form h_1 can be taken to be "totally positive (definite)". To be more precise, let $c_1^{(i)}$, $\psi_1^{(i)}$, $V_1^{(i)}$ be as defined in 3.3 and extend $\iota_0^{\sigma_i}$ to an $\boldsymbol{R}$-linear involution of $D_0^{\sigma_i}(\boldsymbol{R})$. Then as is easily seen, there exist $b_1^{(i)} \in D_0^{\sigma_i}(\boldsymbol{R})^\times$ $(1 \le i \le l)$ such that one has

$$(36) \qquad \psi_1^{(i)}(\xi)^{\iota_0^{\sigma_i}}=b_1^{(i)^{-1}}\psi_1^{(i)}(\bar{\xi})b_1^{(i)} \qquad (\xi \in D_1)\,;$$

in particular, one has

$$c_1^{(i)\iota_0^{\sigma_i}}=b_1^{(i)^{-1}}c_1^{(i)}b_1^{(i)}\,.$$

The elements $c_1^{(i)}b_1^{(i)}=b_1^{(i)}c_1^{(i)\iota_0^{\sigma_i}}$ are uniquely determined by the $c_1^{(i)}$ up to scalar multiplications of $Z(\boldsymbol{R})^\times$. In particular, one has

$$(37) \qquad b_1^{(i)\iota_0^{\sigma_i}}c_1^{(i)\iota_0^{\sigma_i}}=\eta_i c_1^{(i)}b_1^{(i)} \qquad \text{with} \quad \eta_i=\pm 1\,.$$

(In the case of Type 3, one chooses $b_1^{(i)}$ so that $\eta_i=1$.) Then there exist D_1-$\eta\eta_i$-hermitian forms $h_1^{(i)}$ on $V_1^{(i)}$ determined by the relation

$$(38) \qquad \psi_1^{(i)}(h_1^{(i)}(v_1 c_1^{(i)}, v_1' c_1^{(i)}))=c_1^{(i)}b_1^{(i)}h_1^{\sigma_i}(v_1, v_1')c_1^{(i)} \qquad \text{for} \quad v_1, v_1' \in V_1^{\sigma_i}\,.$$

Since ι_1 is totally positive, one has $\eta\eta_i=1$ $(1 \le i \le l)$ and the $h_1^{(i)}$'s are definite. Hence one has $\eta=-1$ for Type 1.2 and $\eta=1$ for all other cases. For the given choice of $b_1^{(i)}$'s one may choose h_1 in such a way that all the $h_1^{(i)}$ are positive definite.

REMARK. The above definition of the "positivity" of h_1 depends on the choice of the $b_1^{(i)}$'s, which is usually made in the following manner. Fix isomorphisms $M^{(i)}\colon D_0^{\sigma_i}(\boldsymbol{R}) \xrightarrow{\sim} M_{s_1}(D_1)$ and the matrix units $(\varepsilon_{jk}^{(i)})_{1 \le j,k \le s_1}$ in $D_0^{\sigma_i}(\boldsymbol{R})$ in such a way that

$$M^{(i)}(\psi_1^{(i)}(\xi))=\xi M^{(i)}(\varepsilon_{11}^{(i)}) \qquad \text{for} \quad \xi \in D_1\,;$$

in particular, $\varepsilon_{11}^{(i)}=c_1^{(i)}$. Then one chooses $b_1^{(i)}$ so that

$$\varepsilon_{kj}^{(i)\iota_0^{\sigma_i}}=b_1^{(i)^{-1}}\varepsilon_{jk}^{(i)}b_1^{(i)}\,;$$

then by (37) one has $b_1^{(i)\iota_0^{\sigma_i}}=\eta_1 b_1^{(i)}$. By these conditions the $b_1^{(i)}$ are uniquely determined up to scalar multiplications of $\boldsymbol{R}^\times$. Now, for Type 1.1 and 2 one has $s_1=1$, $c_1^{(i)}=1$, so that one may put $b_1^{(i)}=1$. For Type 1.2, one has $s_1=2$, $\eta_i=-1$, and one takes $b_1^{(i)}$ so that

$$M^{(i)}(b_1^{(i)})=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For Type 3, one chooses $b_1^{(i)}$ so that $M^{(i)}(b_1^{(i)})$ is positive definite. We also note that in this notation (38) is equivalent to saying that

$$(38') \qquad M^{(i)}(b_1^{(i)}h_1^{\sigma_i}(v_1, v_1'))=(h_1^{(i)}(v_1\varepsilon_{j1}^{(i)}, v_1'\varepsilon_{k1}^{(i)}))_{1\le j,k\le s_1} \qquad \text{for} \quad v_1, v_1'\in V_1^{\sigma_i}$$

(cf. [S6, Ch. IV, §3]).

THEOREM 2. *Suppose that $(U^{(1)}, V^{(1)})$ is given an admissible F-structure, $e\in\mathscr{C}\cap U^{(1)}(\boldsymbol{Q})$, and h_1 is a totally positive (D_0, ι_0)-η-hermitian form on V_1 such that $\iota_1=\iota_1(h_1)$ is the involution corresponding to $e^{(1)}$. Then $(e^{(1)}, a^{(1)}, \beta^{(1)})$ is an admissible triple belonging to $(U^{(1)}, V^{(1)}, \mathscr{C}^{(1)})$ defined over F if and only if $a^{(1)}$ is of the form*

$$(39) \quad a^{(1)}(v_1\otimes_{D_0}v_2, v_1'\otimes_{D_0}v_2')=\mathrm{tr}_{D_0/F}(h_1(v_1, v_1')^{\iota_0}h_2(v_2, v_2')) \quad \textit{for } v_j, v_j'\in V_j,\ j=1, 2,$$

where h_2 is a (D_0, ι_0)-$(-\eta)$-hermitian form on V_2.

(Cf. [S1, p. 234, Prop. 3], or [S6, Ch. IV, §2].)

PROOF. Assume that $(e^{(1)}, a^{(1)}, \beta^{(1)})$ is an admissible triple defined over F. Then by (β2) and (30) the involution $\iota=\iota(a^{(1)})$ leaves $\mathscr{A}_1$ invariant and $\iota|\mathscr{A}_1=\iota_1$. Since one has

$$\mathrm{End}_F(V^{(1)}(F))=\mathrm{End}_{D_0}(V_1)\otimes_Z\mathrm{End}_{D_0}(V_2),$$

there exists an involution ι_2 of $\mathrm{End}_{D_0}(V_2)$ such that $\iota_2|Z=\iota_0|Z$ and

$$(y_1\otimes_Z y_2)^{\iota}=y_1^{\iota_1}\otimes_Z y_2^{\iota_2} \qquad (y_j\in\mathrm{End}_{D_0}(V_j),\quad j=1, 2).$$

Hence, making the natural identification $V^{(1)}(F)^*=V_2^*\otimes_{D_0}V_1^*$, one has a (D_0, ι_0)-$(-\eta)$-hermitian map $h_2\colon V_2\to V_2^*$ such that

$$a^{(1)}(v_1\otimes_{D_0}v_2)=h_1(v_1)\otimes_{D_0}h_2(v_2),$$

which is equivalent to (39). The converse is clear. q.e.d.

With the same notation as in Theorem 2, let $(e^{(i)}, a^{(i)}, \beta^{(i)})=(e^{(1)}, a^{(1)}, \beta^{(1)})^{\sigma_i}$ $(1\le i\le l)$; then they are admissible triples belonging to $(U^{(i)}, V^{(i)}, \mathscr{C}^{(i)})$ defined over F^{σ_i}. Let $c_1^{(i)}$, $\psi_1^{(i)}$, $b_1^{(i)}$ be as above. Then for each $1\le i\le l$ there is D_1-skew-hermitian form $h_2^{(i)}$ on the left D_1-module $V_2^{(i)}$ determined by the relation

$$(40) \qquad \psi_1^{(i)}(h_2^{(i)}(c_1^{(i)}v_2, c_1^{(i)}v_2'))=c_1^{(i)}h_2^{\sigma_i}(v_2, v_2')b_1^{(i)^{-1}}c_1^{(i)} \qquad \text{for} \quad v_2, v_2'\in V_2^{\sigma_i},$$

and one has

$$(41) \qquad a^{(i)}(v_1\otimes_{D_1}v_2, v_1'\otimes_{D_1}v_2')=\mathrm{tr}_{D_1/\boldsymbol{R}}(\overline{h_1^{(i)}(v_1, v_1')h_2^{(i)}(v_2, v_2')}) \quad \text{for } v_j, v_j'\in V_j^{(i)},\ j=1, 2$$

(cf. [S6, Ch. IV, §3]).

3.7. *The description of* $\mathfrak{S}$. Let

$$I \in \mathfrak{S} = \mathfrak{S}(V, a, \beta), \qquad I = \sum_{i=1}^{l} I^{(i)},$$

$$I^{(i)} \in \mathfrak{S}^{(i)} = \mathfrak{S}(V^{(i)}, a^{(i)}, \beta^{(i)}).$$

Then, since $I^{(i)}$ is $\beta^{(i)}(\mathfrak{g}_1^{(i)})$-invariant, one has

$$I^{(i)} = 1 \otimes_{D_1} I_2^{(i)} \qquad (1 \le i \le l), \tag{42}$$

with a complex structure $I_2^{(i)} \in \mathrm{End}_{D_1}(V_2^{(i)})$, which by (41) satisfies the condition

$$h_2^{(i)}(v_2, I^{(i)}v_2') \quad (v_2, v_2' \in V_2^{(i)}) \quad \text{is } D_1\text{-hermitian and positive definite}. \tag{43}$$

Let $\mathfrak{S}(V_2^{(i)}, h_2^{(i)})$ denote the space of D_1-linear complex structures on $V_2^{(i)}$ satisfying the condition (43). Then one has

$$\mathfrak{S}(V^{(i)}, a^{(i)}, \beta^{(i)}) \simeq \mathfrak{S}(V_2^{(i)}, h_2^{(i)}). \tag{44}$$

This implies, in particular, that for any Q-rational admissible triple (e, a, β) one has

$$\mathfrak{S}(V, a, \beta) \simeq \prod_{i=1}^{l} \mathfrak{S}(V_2^{(i)}, h_2^{(i)}) \neq \varnothing.$$

The symmetric domain $\mathfrak{S}$ (with the given Q-structure) is denoted as $R_{F/Q}\mathfrak{S}(V_2, D_0, h_2)$. In the case where D_0 is of Type 1.1, Type 1.2, and Type 2, $\mathfrak{S}$ is also written as $R_{F/Q}(\mathrm{III}_{v_2/2}^{(1)})$, $R_{F/Q}(\mathrm{III}_{v_2}^{(2)}, D_0, h_2)$, and $R_{F/Q}(\mathrm{II}_{v_2}^{(2)}, D_0, h_2)$, respectively.

Note that the corresponding group G_2 has no compact factors (and hence determined uniquely by $\mathfrak{S}$) except for the following two cases. The group G_2 corresponding to $R_{F/Q}(\mathrm{II}_1^{(2)}, D_0, h_2)$ is compact, so that the corresponding domain $\mathfrak{S}$ reduces to a point. The group G_2 corresponding to $R_{F/Q}(\mathrm{II}_2^{(2)}, D_0, h_2)$ (under the assumption that $\mathfrak{S}$ has rational points) is isogenous to the direct product of two Q-simple groups G_2', G_2'', one of which is compact and the other is isomorphic to the group corresponding to $R_{F/Q}(\mathrm{III}_1^{(1)})$. (These cases are usually excluded from the classification.)

3.8. In the case where D_0 is of Type 3, one has to determine furthermore the signature of $h_2^{(i)}$. For that purpose, let σ_i' and σ_i'' denote two imbeddings of the center Z of D_0 into C extending $\sigma_i\colon F \to R$; then one has $\sigma_i'' = \sigma_0 \circ \sigma_i'$, σ_0 denoting the complex conjugation of C. We determine $\psi_1^{(i)}$ and (σ_i', σ_i'') in such a way that

$$\psi_1^{(i)}(\alpha^{\sigma_i'}) = \bar{\psi}_1^{(i)}(\alpha^{\sigma_i''}) = \alpha^{\sigma_i} c_1^{(i)} \qquad (\alpha \in Z). \tag{45}$$

Then we say that the $\psi_1^{(i)}$ are *compatible with* the "CM-type" (σ_i') of the CM-field Z.

In this case, since $D_1 = C$ is commutative, we don't distinguish left and right C-vector spaces. Then, the $(V_j^{(i)}, \psi_j^{(i)})$ being C-vector spaces, one has direct decompositions

$$V_j^{(i)} \otimes_R C = V_j^{(i)\prime} \oplus V_j^{(i)\prime\prime}, \quad V_j^{(i)\prime\prime} = V_j^{(i)\prime\sigma_0}, \qquad (1 \le i \le l,\ j = 1, 2), \tag{46}$$

where

$$V_j^{(i)\prime}=\{v\in V_j^{(i)}\otimes_{\boldsymbol R}\boldsymbol C\,|\,v\psi_1^{(i)}(\xi)=\xi v \text{ for } \xi\in\boldsymbol C\}\,,$$
$$V_j^{(i)\prime\prime}=\{v\in V_j^{(i)}\otimes_{\boldsymbol R}\boldsymbol C\,|\,v\psi_1^{(i)}(\xi)=\bar\xi v \text{ for } \xi\in\boldsymbol C\}\,,$$

and $\dim_{\boldsymbol C} V_j^{(i)\prime}=\dim_{\boldsymbol C} V_j^{(i)\prime\prime}=\nu_j\delta_0$.

Let $\beta_1^{(i)\prime}$ and $\beta_1^{(i)\prime\prime}=\beta_1^{(i)\prime\sigma_0}$ denote the restrictions to $V_1^{(i)\prime}$ and $V_1^{(i)\prime\prime}$ of the natural extension of the representation $\beta_1^{(i)}$ to $V_1^{(i)}\otimes_{\boldsymbol R}\boldsymbol C$. Then they are absolutely irreducible and the primary decomposition of $(V^{(i)}\otimes_{\boldsymbol R}\boldsymbol C, \beta^{(i)})$ is given by

$$V^{(i)}\otimes_{\boldsymbol R}\boldsymbol C=V_1^{(i)\prime}\otimes_{\boldsymbol C}V_2^{(i)\prime}\oplus V_1^{(i)\prime\prime}\otimes_{\boldsymbol C}V_2^{(i)\prime\prime}\,. \tag{47}$$

Now, for the given complex structure $I^{(i)}$ on $V^{(i)}$, set

$$V_+^{(i)}=\{v\in V^{(i)}\otimes_{\boldsymbol R}\boldsymbol C\,|\,I^{(i)}v=\sqrt{-1}\,v\}\,.$$

Then $V_+^{(i)}$ is $\beta^{(i)}(\mathfrak g_1^{(i)})$-invariant, and the primary decomposition of it is of the form

$$V_+^{(i)}=V_1^{(i)\prime}\otimes_{\boldsymbol C}W_2^{(i)\prime}\oplus V_1^{(i)\prime\prime}\otimes_{\boldsymbol C}W_2^{(i)\prime\prime}\,, \tag{48}$$

where $W_2^{(i)\prime}$ and $W_2^{(i)\prime\prime}$ are complex subspaces of $V_2^{(i)\prime}$ and $V_2^{(i)\prime\prime}$ of dimension p_i and q_i, respectively. Since one has

$$V^{(i)}\otimes_{\boldsymbol R}\boldsymbol C=V_+^{(i)}\oplus V_+^{(i)\sigma_0}\,,$$

one has

$$V_2^{(i)\prime}=W_2^{(i)\prime}\oplus W_2^{(i)\prime\prime\sigma_0}\,; \tag{49}$$

in particular, $p_i+q_i=\nu_2\delta_0$ $(1\le i\le l)$. Thus one has

$$(V^{(i)}, I^{(i)}, \beta^{(i)})\simeq(V_+^{(i)}, p_i\beta_1^{(i)\prime}\oplus q_i\beta_1^{(i)\prime\prime})\,. \tag{50}$$

Otherwise expressed, one has

$$\begin{aligned}V^{(i)}&=R_{\boldsymbol C/\boldsymbol R}(V_1^{(i)\prime}\otimes_{\boldsymbol C}V_2^{(i)\prime})\,,\\ I^{(i)}&=R_{\boldsymbol C/\boldsymbol R}(1\otimes_{\boldsymbol C}I_2^{(i)\prime})\,,\end{aligned} \tag{51}$$

where $I_2^{(i)\prime}$ is a complex structure on $V_2^{(i)\prime}$, defined by

$$I_2^{(i)\prime}=\begin{cases}\sqrt{-1} & \text{on } W_2^{(i)\prime}\,,\\ -\sqrt{-1} & \text{on } W_2^{(i)\prime\prime\sigma_0}\,.\end{cases} \tag{51a}$$

Let $h_j^{(i)\prime}$ denote the $(-1)^{j-1}$-hermitian forms on $V_j^{(i)\prime}$ obtained from $h_j^{(i)}$ by the $\boldsymbol C$-isomorphism $(V_j^{(i)}, \psi_1^{(i)})\simeq V_j^{(i)\prime}$; then $h_2^{(i)\prime}(w_2, w_2')$ $(w_2, w_2'\in V_2^{(i)})$ is $\boldsymbol C$-linear in w_2. For the sake of consistency, we set

$$\tilde h_2^{(i)\prime}(w_2, w_2')=\overline{h_2^{(i)\prime}(w_2, w_2')}\,,$$

to obtain a skew-hermitian form which is $\boldsymbol{C}$-linear in w'_2. Then by (41) one has

$$a^{(i)}(v_1 \otimes_{\boldsymbol{C}} v_2, v'_1 \otimes_{\boldsymbol{C}} v'_2) = 2\,\mathrm{Re}(h_1^{(i)\prime}(w_1, w'_1)\tilde{h}_2^{(i)\prime}(w_2, w'_2)), \tag{52}$$

where

$$v_j = w_j + \bar{w}_j, \quad v'_j = w'_j + \bar{w}'_j, \quad v_j, v'_j \in V_j^{(i)}, \quad w_j, w'_j \in V_j^{(i)\prime} \qquad (1 \le i \le l,\ j = 1, 2),$$

and the symbol $\otimes_{\boldsymbol{C}}$ in (52) stands for the tensor product over $\psi_1^{(i)}(\boldsymbol{C})$. Since $a^{(i)}I^{(i)}$ and the hermitian form $h_1^{(i)\prime}$ are positive definite, one has by (51), (51a) and (52) that *the hermitian form* $\sqrt{-1}\,\tilde{h}_2^{(i)\prime}$ *on* $V_2^{(i)\prime}$ *is of signature* (p_i, q_i). In this sense, we say that h_2 (or I_2) is of signature $(p_i, q_i)_{1 \le i \le l}$ with respect to the given "CM-type" (σ'_i). In this case $\mathfrak{S}$ is written as

$$\mathfrak{S} = \prod \mathfrak{S}(V_2^{(i)\prime}, \tilde{h}_2^{(i)\prime}) = R_{F/\boldsymbol{Q}}\mathfrak{S}(V_2, D_0/Z, h_2). \tag{53}$$

For the given skew-hermitian form h_2, the CM-type $(\sigma'_i)_{1 \le i \le l}$ can be so chosen that one has $p_i \ge q_i$ for $1 \le i \le l$. When $\mathfrak{S}$ has rational points, the reductive group G_2 is (strictly) pure, so that there exist integers p, q such that $p_i = p$, $q_i = q$ $(1 \le i \le l)$. Then the symmetric domain $\mathfrak{S}$ in (53) is denoted as

$$R_{F/\boldsymbol{Q}}(\mathrm{I}_{p,q}^{(\delta_0)}, D_0/Z, h_2).$$

The corresponding group G_2 has no compact factors, except for the case $q = 0$, in which case the group G_2 itself is compact. Note also that the group corresponding to $R_{F/\boldsymbol{Q}}(\mathrm{I}_{3,1}^{(1)}, Z, h_2)$ is $\boldsymbol{Q}$-isogenous to the one corresponding to $R_{F/\boldsymbol{Q}}(\mathrm{II}_3^{(2)}, D_0, h'_2)$ for a suitable totally definite quaternion algebra D_0 over F and a D_0-skew-hermitian form h'_2 of 3 variables.

REMARK. When $p > q$, there exist rational points in $\mathfrak{S}$ if and only if one has $\delta_0 | q$ and $\boldsymbol{Q}$-rank $G_2 = q/\delta_0$. If this is the case, I is rational, if and only if there exists a D_0-submodule W_2 of V_2 of rank q/δ_0 such that

$$W_2^{(i)\prime} = (W_2^{\perp})^{\sigma'_i}(\boldsymbol{C}) \cap V_2^{(i)\prime}, \qquad W_2^{(i)\prime\prime} = W_2^{\sigma''_i}(\boldsymbol{C}) \cap V_2^{(i)\prime\prime}.$$

$^\perp$ denoting the orthogonal complement with respect to h_2. When $p = q$, the situation is a little more complicated ([S8]).

4. The standard case.

4.1. *Admissible F-structures of* $(U^{(1)}, V^{(1)})$. According to the classification theory of irreducible self-dual homogeneous cones, $\mathscr{C}^{(1)}$ is isomorphic to one of the following cones:

$$\mathscr{P}_{r_1}(\boldsymbol{R})\ (r_1 \ge 1), \quad \mathscr{P}_{r_1}(\boldsymbol{C})\ (r_1 \ge 2), \quad \mathscr{P}_{r_1}(\boldsymbol{H})\ (r_1 \ge 3), \quad \mathscr{P}(1, n_1 - 1)\ (n_1 \ge 3).$$

We call the first three cases *standard* and the fourth *non-standard* or *quadratic*. Note that $\mathscr{P}_1(\boldsymbol{R})$ is the unique case for which $r_1 = n_1 = 1$ and that the quadratic case is characterized by $r_1 = 2$; in particular, one has the isomorphisms $\mathscr{P}_2(\boldsymbol{R}) \simeq \mathscr{P}(1, 2)$,

$\mathscr{P}_2(\boldsymbol{C})\simeq\mathscr{P}(1,3)$. (For convenience, we exclude $\mathscr{P}_2(\boldsymbol{H})\simeq\mathscr{P}(1,5)$ from the standard case. Because of the assumption $V\neq 0$, the exceptional case $\mathscr{P}_3(\boldsymbol{O})$ is also excluded.)

In the standard case, one has

$$\mathfrak{g}_1\simeq(\mathfrak{g}_1^{(1)})^l\,,\qquad \mathfrak{g}_1^{(1)}=\{1_{U^{(1)}}\}_{\boldsymbol{R}}\oplus\mathfrak{g}_1^{(1)s}\,,$$
$$\mathfrak{g}_1^{(1)s}\simeq\mathfrak{sl}_{r_1}(D_1)\,,\qquad D_1=\boldsymbol{R},\boldsymbol{C},\boldsymbol{H}\,. \tag{54}$$

We know ([S2]) that the representation $(V^{(1)},\beta^{(1)})$ is $\boldsymbol{R}$-primary. In (26a, b) $V_1^{(1)}$ is a D_1-module of rank r_1 and $\beta_1^{(1)}$ is a Lie algebra isomorphism

$$\beta_1^{(1)}:\mathfrak{g}_1^{(1)}\xrightarrow{\ \sim\ }\{y\in\operatorname{End}_{D_1}(V_1^{(1)})\,|\,\operatorname{tr}y\in\boldsymbol{R}\}\,, \tag{55}$$

tr denoting here the reduced trace of $\operatorname{End}_{D_1}(V^{(1)})$ over its center. Thus one has $\mathscr{A}_1\simeq\operatorname{End}_{D_1}(V_1^{(1)})\simeq M_{r_1}(D_1)$ and $r_1=\nu_1\delta_0/\delta_1$.

It follows that, if one has an F-algebra structure on $\mathscr{A}_1$ with a totally positive involution ι_1, then the conditions (a), (b) in Proposition 2 are automatically satisfied. Hence, *in the standard case, an F-algebra structure of $\mathscr{A}_1$ gives rise to an admissible F-structure of $(U^{(1)},V^{(1)})$ if and only if there exists a totally positive involution ι_1 of $\mathscr{A}_1(F)$ and the condition* (c) *in Proposition* 2 *is satisfied.*

Now, suppose one has an F-algebra structure on $\mathscr{A}_1$ satisfying these conditions and fix an admissible F-structure of $(U^{(1)},V^{(1)})$ compatible with it. Then one has (27a, b) with

$$\mathscr{A}_1(F)=\operatorname{End}_{D_0}V_1\,,$$
$$\beta_1:\mathfrak{g}_1^{(1)s}(F)\xrightarrow{\ \sim\ }\mathfrak{sl}(V_1/D_0)\,. \tag{56}$$

Hence in this case one has F-rank $\mathfrak{g}_1^{(1)}=\nu_1$.

REMARK. Our argument shows that, in our case, the F-forms of $\mathfrak{g}_1^{(1)}$ corresponding to the unitary groups do not occur. (In fact, for such an F-form the representation $\beta^{(1)}$ is not defined over F).

On the other hand, one has

$$U^{(1)}=\mathrm{S}(V_1^{(1)}\otimes_{D_1}V_1^{(1)})\,, \tag{57}$$

where S denotes the symmetrizer and the second factor $V_1^{(1)}$ in the right hand side is viewed as a left D_1-space by $\xi v_1=v_1\bar{\xi}$ $(v_1\in V_1^{(1)},\ \xi\in D_1)$. $U^{(1)}$ is also identified with the space of all symmetric D_1-semilinear maps: $V_1^{(1)*}\to V_1^{(1)}$. Then the action of $\mathfrak{g}_1^{(1)}$ on $U^{(1)}$ is given by

$$x(u)=\beta_1^{(1)}(x)\circ u+u\circ{}^t\beta_1^{(1)}(x) \tag{58}$$

for $x\in\mathfrak{g}_1^{(1)}$ and $u\in U^{(1)}$.

From (57) one also has an F-structure of $U^{(1)}$ such that

$$U^{(1)}(F)=\mathrm{S}_\eta(V_1\otimes_{D_0}V_1)\,, \tag{59}$$

S_η denoting the η-symmetrizer $\mathrm{S}_\eta=(1/2)(1+\eta\tau)$, where τ is the transposition and $\eta=-1$ if D_0 is of Type 1.2 and $\eta=1$ otherwise. Thus $U^{(1)}(F)$ is identified with the space of all η-symmetric (D_0, ι_0)-semilinear maps: $V_1^*\to V_1$. Then the action of $\mathfrak{g}_1^{(1)}(F)$ on $U^{(1)}(F)$ is given by a formula similar to (58).

4.2. Now let $e\in\mathscr{C}\cap U(\boldsymbol{Q})$, $e=(e^{(i)})$, and consider $e^{(1)}$ as a (D_0, ι_0)-semilinear isomorphism $V_1^*\xrightarrow{\sim}V_1$. Then its inverse $e^{(1)^{-1}}\colon V_1\to V_1^*$ may be viewed as a (D_0, ι_0)-η-hermitian form on V_1, which we denote by h_1, i.e.,

$$\mathrm{tr}_{D_0/F}(h_1(v_1, v_1'))=\langle v_1, e^{(1)^{-1}}v_1'\rangle \qquad (v_1, v_1'\in V_1)\,. \tag{60}$$

PROPOSITION 2. *Let φ_1 and ι_1 be as defined in* 3.4. *Then, for $u\in U^{(1)}(F)$ and $y\in\mathscr{A}_1(F)$, one has*

$$\varphi_1(u)=u\circ e^{(1)^{-1}}\,, \tag{61}$$

$$y^{\iota_1}=e^{(1)}\circ{}^t y\circ e^{(1)^{-1}}\,. \tag{62}$$

(*Thus one has $\iota_1=\iota_1(h_1)$, i.e., our notation is consistent.*)

PROOF. For the proof, we denote the right hand sides of (61) and (62) by $\varphi_1'(u)$ and $y^{\iota_1'}$, respectively. Then it is clear that one has $\varphi_1'(u)\in\mathrm{Her}(\mathscr{A}_1, \iota_1')$ and, for $x\in\mathfrak{g}_1^{(1)}(F)$,

$$\varphi_1'(x(u))=(\beta_1(x)\circ u+u\circ{}^t\beta_1(x))\circ e^{(1)^{-1}}=\beta_1(x)\circ\varphi_1'(u)+\varphi_1'(u)\circ\beta_1(x)^{\iota_1'}\,.$$

Hence φ_1' is an F-isomorphism $U^{(1)}\simeq\mathrm{Her}(\mathscr{A}_1, \iota_1')$ satisfying the first and the third equations in (32). In particular, one has

$$x(e^{(1)})=0\Longleftrightarrow\beta_1(x)+\beta_1(x)^{\iota_1'}=0\,,$$

which shows that the map $y\mapsto -y^{\iota_1'}$ $(y\in\mathscr{A}_1)$ induces the Cartan involution θ_1 of $\mathfrak{g}_1^{(1)}$ corresponding to $e^{(1)}$. Thus the second equation in (32) is also satisfied. Hence by the uniqueness of ι_1 and φ_1 one has $\varphi_1'=\varphi_1$, $\iota_1'=\iota_1$. q.e.d.

By (19) and (61) the Jordan product in $U^{(1)}$ is given by

$$uu'=\frac{1}{2}(u\circ e^{(1)^{-1}}\circ u'+u'\circ e^{(1)^{-1}}\circ u)\,,$$

and by (33) the normalized inner product on $U^{(1)}$ corresponding to $e^{(1)}$ is given by

$$\langle u, u'\rangle=(\delta_1 d_1)^{-1}\,\mathrm{tr}_{\mathscr{A}_1/\boldsymbol{R}}(ue^{(1)^{-1}}u'e^{(1)^{-1}})\,. \tag{63}$$

Finally one obtains the following

PROPOSITION 3. *Suppose we are in the standard case. Let $(e^{(1)}, a^{(1)}, \beta^{(1)})$ be an admissible triple defined over F belonging to $(U^{(1)}, V^{(1)}, \mathscr{C}^{(1)})$, $h_1=e^{(1)^{-1}}$, and let h_2 be a*

(D_0, ι_0)-$(-\eta)$-hermitian form on V_2 satisfying (39). *Then the corresponding alternating bilinear map $A^{(1)}: V^{(1)} \times V^{(1)} \to U^{(1)}$ is given as follows:*

$$A^{(1)}(v_1 \otimes_{D_0} v_2, v_1' \otimes_{D_0} v_2') = \eta \delta_1 d_1 \mathrm{S}_\eta(v_1 h_2(v_2, v_2') \otimes_{D_0} v_1') \quad \text{for } v_1, v_1' \in V_1 \text{ and } v_2, v_2' \in V_2 . \tag{64}$$

PROOF. For $u \in U^{(1)}(F)$ one has

$$\begin{aligned}\langle u, A^{(1)}(v_1 \otimes_{D_0} v_2, v_1' \otimes_{D_0} v_2')\rangle &= A_u(v_1 \otimes_{D_0} v_2, v_1' \otimes_{D_0} v_2') \\ &= a^{(1)}(v_1 \otimes_{D_0} v_2, (ue^{(1)^{-1}})v_1' \otimes_{D_0} v_2') \\ &= \mathrm{tr}_{D_0/F}(h_1(v_1, (ue^{(1)^{-1}})v_1')^{\iota_0} h_2(v_2, v_2')) \\ &= \mathrm{tr}_{D_0/F}(h_1(v_1 h_2(v_2, v_2'), (ue^{(1)^{-1}})v_1')) \\ &= \langle v_1 h_2(v_2, v_2'), (e^{(1)^{-1}} ue^{(1)^{-1}})v_1' \rangle \\ &= \eta\, \mathrm{tr}_{\mathscr{A}(F)/F}((v_1 h_2(v_2, v_2') \otimes_{D_0} v_1')e^{(1)^{-1}} ue^{(1)^{-1}}) \\ &= \eta \delta_1 d_1 \langle u, \mathrm{S}_\eta(v_1 h_2(v_2, v_2') \otimes_{D_0} v_1')\rangle ,\end{aligned}$$

whence follows (64). q.e.d.

4.3. *Classification.* In the classification theory, the quasisymmetric domain $\mathscr{S}_I$ with a $\boldsymbol{Q}$-structure described above is expressed by the following symbols, according as D_0 is of Type 1.1, 1.2, 2, or 3.

$$R_{F/\boldsymbol{Q}}(\mathrm{III}^{(1)}_{\nu_1;\nu_2/2})_I , \qquad R_{F/\boldsymbol{Q}}(\mathrm{III}^{(2)}_{2\nu_1;\nu_2}, D_0, h_2)_I ,$$
$$R_{F/\boldsymbol{Q}}(\mathrm{II}^{(2)}_{\nu_1;\nu_2}, D_0, h_2)_I \qquad (\nu_1 \geq 3) ,$$
$$R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{\nu_1\delta_0;(p,q)}, D_0/Z, h_2)_I \qquad (\nu_1\delta_0 \geq 2) .$$

In the standard case, *the total space $\tilde{\mathscr{S}}$ is always symmetric.* For $R_{F/\boldsymbol{Q}}(\mathrm{III}^{(1)}_{\nu_1;\nu_2/2})_I$, the space $\tilde{\mathscr{S}}$ may be identified with the Siegel domain (of the third kind) expression of $R_{F/\boldsymbol{Q}}(\mathrm{III}^{(1)}_{\nu_1+\nu_2/2})$ over the ν_1-th rational boundary component $\mathfrak{S} = R_{F/\boldsymbol{Q}}(\mathrm{III}^{(1)}_{\nu_2/2})$. In the case of $R_{F/\boldsymbol{Q}}(\mathrm{III}^{(2)}_{2\nu_1;\nu_2}, D_0, h_2)_I$, resp. $R_{F/\boldsymbol{Q}}(\mathrm{II}^{(2)}_{\nu_1;\nu_2}, D_0, h_2)_I$ $(\nu_1 \geq 3)$, let h_2' denote a D_0-hermitian, resp. D_0-skew-hermitian, form of $2\nu_1 + \nu_2$ variables in the same Witt class as h_2. Then $\tilde{\mathscr{S}}$ may be identified with the Siegel domain expression of $R_{F/\boldsymbol{Q}}(\mathrm{III}^{(2)}_{2\nu_1+\nu_2}, D_0, h_2')$, resp. $R_{F/\boldsymbol{Q}}(\mathrm{II}^{(2)}_{2\nu_1+\nu_2}, D_0, h_2')$ over the ν_1-th rational boundary component $\mathfrak{S} = R_{F/\boldsymbol{Q}}(\mathrm{III}^{(2)}_{\nu_2}. D_0, h_2)$, resp. $R_{F/\boldsymbol{Q}}(\mathrm{II}^{(2)}_{\nu_2}, D_0, h_2)$. In particular, $R_{F/\boldsymbol{Q}}(\mathrm{II}^{(2)}_{\nu_1;1}, D_0, h_2)_I$ $(\nu_1 \geq 3)$ *is identified with the symmetric domain* $R_{F/\boldsymbol{Q}}(\mathrm{II}^{(2)}_{2\nu_1+1}, D_0, h_2')$. In the case $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{\nu_1\delta_0;(p,q)}, D_0/Z, h_2)_I$ $(\nu_1\delta_0 \geq 2, p+q = \nu_2\delta_0)$, let h_2' denote a (D_0, ι_0)-skew-hermitian form of $2\nu_1 + \nu_2$ variables in the same Witt class as h_2. Then the total space $\tilde{\mathscr{S}}$ may be identified with the Siegel domain expression of $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{\nu_1\delta_0+p,\nu_1\delta_0+q}, D_0/Z, h_2')$ over the ν_1-th rational boundary component $\mathfrak{S} = R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{p,q}, D_0/Z, h_2)$. In particular, $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{\nu_1\delta_0;(\nu_2\delta_0,0)}, D_0/Z, h_2)_I$ *is identified with the symmetric domain* $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{(\nu_1+\nu_2)\delta_0,\nu_1\delta_0},$

$D_0/Z, h'_2)$.

In general, it is known that, for any boundary point p of an irreducible symmetric domain $\mathscr{D}$, the "fiber" over p, i.e., the union of all geodesic lines in $\mathscr{D}$ tending to p, is an irreducible quasisymmetric domain and, if p belongs to the *first* boundary component, it is of type $(\mathrm{III}^{(1)}_{1;\nu_2/2})_I$. For instance, for the symmetric domain $\tilde{\mathscr{S}}=R_{F/\boldsymbol{Q}}(\mathrm{II}^{(2)}_{2+\nu'_2}, D_0, h'_2)$, resp. $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(1)}_{1+p,1+q}, Z, h'_2)$ $(p+q=\nu'_2)$, the fiber over a rational point I in the first rational boundary component $\mathfrak{S}=R_{F/\boldsymbol{Q}}(\mathrm{II}^{(2)}_{\nu'_2}, D_0, h_2)$, resp. $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(1)}_{p,q}, Z, h_2)$ is of type $R_{F/\boldsymbol{Q}}(\mathrm{III}^{(1)}_{1;\nu_2/2})_I$ $(\nu_2=2\nu'_2)$. [But, because of the existence of compact factors in $GL_1(\boldsymbol{H})$ and $GL_1(\boldsymbol{C})$, the automorphism group of the fiber induced by the paraboric subgroup is, in general, smaller than $\mathrm{Aff}(R_{F/\boldsymbol{Q}}(\mathrm{III}^{(1)}_{1;\nu_2/2})_I$.] In particular, *the domain* $R_{F/\boldsymbol{Q}}(\mathrm{III}^{(1)}_{1;\nu_2/2})_I$ *can be identified with the symmetric domain* $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(1)}_{1+\nu'_2,1}, Z, h'_2)$ (along with the automorphism group), where Z, h'_2 are determined as follows. Let a_2 be a non-degenerate alternating bilinear form on $V_2=V^{(1)}(F)$, $I\in R_{F/\boldsymbol{Q}}\mathscr{S}(V_2, a_2)$, and let Z be the CM-field attached to I, i.e., $Z=F(\sqrt{-\alpha_1})$, where α_1 is a totally positive element in F such that $\sum\sqrt{\alpha_1^{\sigma_i}}I^{(i)}$ is $\boldsymbol{Q}$-rational. Then h_2 is a Z-skew-hermitian form on V_2 given by

$$h_2(v, v')=a_2(v, v')-\sqrt{-1}\,a_2(v, I^{(1)}v'),$$

which is totally positive with respect to the CM-type (σ'_i) determined by $\sqrt{-\alpha_1}^{\sigma'_i}=\sqrt{-1}\sqrt{\alpha_1^{\sigma_i}}$, and h'_2 is a Z-skew-hermitian form of $2+\nu'_2$ variables in the same Witt class as h_2.

5. The quadratic case.

5.1. *F-structures of* $(U^{(1)}, \mathfrak{g}^{(1)}_1)$. We keep the notation of §3. In the quadratic case, one has

$$\mathscr{C}^{(1)}\simeq\mathscr{P}(1, n_1-1)=\left\{(\xi_i)\in\boldsymbol{R}^{n_1}\,\middle|\,\xi_1^2-\sum_{i=2}^{n_1}\xi_i^2>0\right\},$$

$$\mathfrak{g}_1\simeq(\mathfrak{g}^{(1)}_1)^l,\quad \mathfrak{g}^{(1)s}_1\simeq\mathfrak{so}(1, n_1-1),\tag{65}$$

where $n_1=\dim U^{(1)}\geq 3$. In this case, $r_1=\boldsymbol{R}\text{-rank}\,\mathfrak{g}^{(1)}_1=2$.

One obtains all F-forms of $\mathfrak{g}^{(1)}_1$ in the following manner. F is a totally real number field of degree l. Suppose that $U^{(1)}$ is given an F-structure and $S^{(1)}$ is a symmetric bilinear form on $U^{(1)}\times U^{(1)}$ defined over F. Put $(U, S)=R_{F/\boldsymbol{Q}}(U^{(1)}, S^{(1)})$. We assume that all $S^{(i)}=S^{(1)\sigma_i}$ $(1\leq i\leq l)$ are of signature $(1, n_1-1)$. Then one has an F-structure of $\mathfrak{g}^{(1)}_1$ given by

$$\mathfrak{g}^{(1)s}_1(F)=\mathfrak{so}(U^{(1)}(F), S^{(1)})=\{x\in\mathfrak{gl}(U^{(1)}(F))\,|\,{}^t xS^{(1)}+S^{(1)}x=0\}.$$

For convenience, one fixes an F-rational orthogonal basis $\{e_i\}$ of $U^{(1)}$ such that

$$S^{(1)}\sim\mathrm{diag}(\alpha_1, \ldots, \alpha_{n_1}),$$

where α_1 is totally positive and $\alpha_2, \dots, \alpha_{n_1}$ are totally negative.

REMARK. When n_1 is even, there is a possibility of F-forms of $\mathfrak{g}_1^{(1)}$ defined by a quaternion skew-hermitian form h of $n_1/2$ variables with respect to a totally indefinite quaternion algebra over F. However, since h should give rise to a symmetric bilinear form of signature $(1, n_1-1)$ at every real place, an easy observation of the root diagrams shows that $\mathfrak{g}_1^{(1)}$ is F-anisotropic. By a theorem of Kneser ([Sc, Lem. 10.3.5, Th. 10.4.1]), this can happen only for $n_1 \leq 6$. For $n_1=4$, by virtue of the isomorphism $\mathscr{P}(1, 3) \simeq \mathscr{P}_2(\boldsymbol{C})$, the F-forms of this type were already treated in §4, so that we may exclude them from the general discussion of the quadratic case. For $n_1=6$, such F-forms come from a central division algebra of degree 4, which can not have positive involutions. Hence F-forms of this type do not occur. For $n_1=8$, there is also a possibility of F-forms of $\mathfrak{g}_1^{(1)}$ coming from the triality. But, for the reason similar to the one given in [S1, p. 270], such F-forms do not occur either.

5.2. *The Clifford algebras.* Let $C=C(U^{(1)}, S^{(1)})$ denote the Clifford algebra of $S^{(1)}$ and let C^+ denote its even part. C and C^+ are semisimple $\boldsymbol{R}$-algebra defined over F. Put

$$\tilde{e}=e_1 \cdots e_{n_1} \in C(F),$$

$$\Delta=\tilde{e}^2=(-1)^{n_1(n_1-1)/2}\alpha_1 \cdots \alpha_{n_1} \in F^\times \qquad (66)$$

(the discriminant of $S^{(1)}$).

By our assumption, Δ is totally positive (resp. totally negative) for $n_1 \equiv 1, 2$ (resp. $\equiv 0, 3$) (mod 4).

When n_1 is odd, C^+ is a central simple $\boldsymbol{R}$-algebra of degree $2^{(n_1-1)/2}$ defined over F. When n_1 is even, the center of C^+ is $\{1, \tilde{e}\}_{\boldsymbol{R}}$. Hence, if $n_1 \equiv 0 \pmod 4$, the center Z of $C^+(F)$ is a totally imaginary quadratic extension of F, isomorphic to $F(\sqrt{\Delta})$ with $\Delta \ll 0$ (totally negative). Thus C^+ is simple and of degree $2^{n_1/2-1}$ over its center $Z(\boldsymbol{R}) \simeq \boldsymbol{C}$. If $n_1 \equiv 2 \pmod 4$, one has $\Delta \gg 0$ (totally positive) and

$$C^+=C_1^+ \oplus C_2^+, \qquad \frac{1}{2}(1+(-1)^{i-1}\sqrt{\Delta}^{-1}\tilde{e}) \in C_1^+ \qquad (67)$$

with central simple $\boldsymbol{R}$-algebras C_i^+ $(i=1, 2)$ of degree $2^{n_1/2-1}$. (The ordering of C_1^+, C_2^+ may be determined by the orientation of $U^{(1)}$.) If, moreover, $\Delta \sim 1$ over F (i.e., $\Delta \in (F^\times)^2$), then each C_i^+ is defined over F and one has $C_1^+(F) \simeq C_2^+(F)$ (by the map $x \mapsto e_1^{-1} x e_1$). If $n_1 \equiv 2 \pmod 4$ *and* $\Delta \nsim 1$, $C^+(F)$ is simple with center $Z \simeq F(\sqrt{\Delta})$, which is a totally real quadratic extension of F. In this case, one has $C^+(F) \simeq C_i^+(F(\sqrt{\Delta}))$ $(i=1, 2)$.

Let ρ denote the canonical involution of C^+ (i.e., one has $(e_{i_1} \cdots e_{i_k})^\rho = e_{i_k} \cdots e_{i_1}$). Then it is easy to see that

$$\rho' : x \mapsto e_1 x^\rho e_1^{-1} \qquad (68)$$

is a totally positive involution of C^+; when n_1 is even and $\Delta\sim 1$, we mean by this that ρ' induces a totally positive involution on each simple factor C_i^+ $(i=1, 2)$ ([S6, p. 282, Prop. 5.1]).

Let D_0 be a division algebra over F such that $C^+(F)$ (or $C_i^+(F))\sim D_0$. Then the degree δ_0 of D_0 (over its center) is ≤ 2. One has F-rank $\mathfrak{g}_1^{(1)}=1$ if $\delta_0=2$ and $n_1\leq 4$, and F-rank $\mathfrak{g}_1^{(1)}=2$ otherwise. One has

$$D_0(\boldsymbol{R})\sim D_1=\begin{cases}\boldsymbol{R} & \text{if } n_1\equiv 1, 2, 3 \pmod 8,\\ \boldsymbol{C} & \text{if } n_1\equiv 0, 4 \pmod 8,\\ \boldsymbol{H} & \text{if } n_1\equiv 5, 6, 7 \pmod 8.\end{cases} \tag{69}$$

Thus D_0 is of Type 1, if $n_1\equiv 1, 3 \pmod 8$ or $\equiv 2 \pmod 8$ and $\Delta\sim 1$, of Type 2, if $n_1\equiv 5$, $7 \pmod 8$ or $\equiv 6 \pmod 8$ and $\Delta\sim 1$, and of Type 3, if $n_1\equiv 0 \pmod 4$. When $n_1\equiv 2 \pmod 4$ and $\Delta\nsim 1$, D_0 is of Type 1 or 2 over $F(\sqrt{\Delta})$ according as $n_1\equiv 2$ or $6 \pmod 8$.

5.3. *F-structures of* $(V^{(1)}, \beta^{(1)})$: *the case* $n_1\not\equiv 2 \pmod 4$. In this case $\beta^{(1)}$ is $\boldsymbol{R}$-primary and the $\boldsymbol{R}$-irreducible factor is given by the spin representation. As is well known, there exists a canonical F-isomorphism

$$\beta_1:\mathfrak{g}_1^{(1)}\xrightarrow{\sim}\beta_1(\mathfrak{g}_1^{(1)})\subset(C^+)_{\mathrm{Lie}}$$

such that one has

$$x(u)=[\beta_1(x), u] \qquad \text{for} \quad x\in\mathfrak{g}_1^{(1)} \quad \text{and} \quad u\in U^{(1)}, \tag{70}$$

$$\beta_1(\mathfrak{g}_1^{(1)})=\{y\in C^+\,|\, y+y^\rho\in\boldsymbol{R},\ [y, U^{(1)}]\subset U^{(1)}\}. \tag{71}$$

If one denotes by κ the unique $\boldsymbol{R}$-irreducible representation of the simple $\boldsymbol{R}$-algebra C^+, then the spin representation of $\mathfrak{g}_1^{(1)}$ is given by $\kappa\circ\beta_1$. Therefore, identifying $\beta^{(1)}(x)$ $(x\in\mathfrak{g}_1^{(1)})$ with $\beta_1(x)$, one may make an identification $\mathscr{A}_1=C^+$. It is then clear that the natural F-algebra strucrure of $\mathscr{A}_1=C^+$ (which is the unique F-algebra structure making $\beta^{(1)}$ and β_1 defined over F) satisfies the conditions (a), (b) in Proposition 2 with $\iota_1=\rho'$. Hence *the natural F-algebra structure of* $\mathscr{A}_1$ *gives rise to an admissible F-structure of* $(U^{(1)}, V^{(1)})$, *if and only if the condition* (c) *in Proposition* 2 *is satisfied.* For simplicity, one puts $e^{(1)}=e_1$; then one recovers the same F-structure of $U^{(1)}$ given in 5.1.

In the notation of §3, one has

$$\nu_1\delta_0=\begin{cases}2^{(n_1-1)/2}\\ 2^{n_1/2-1}\end{cases} \qquad d_1=\begin{cases}1 & \text{if } n_1 \text{ is odd},\\ 2 & \text{if } n_1\equiv 0 \pmod 4.\end{cases}$$

5.4. Now, fix $e^{(1)}=e_1\in U^{(1)}(F)$ with $\alpha_1=S^{(1)}(e_1, e_1)\gg 0$. Then one has

PROPOSITION 4. *For* $u\in U^{(1)}(F)$ *and* $y\in C^+(F)$, *one has*

$$\varphi_1(u)=ue_1^{-1}, \tag{72}$$

$$y^{\iota_1}=e_1y^\rho e_1^{-1}. \tag{73}$$

PROOF. We know (73) already ([S6, Prop. 5.1]). To prove (72), define φ_1 by (72) for a moment. Then it is enough to show that $\varphi_1(u) \in \mathrm{Her}(C^+, \rho')$, $\varphi_1(e_1)=1$, and that φ_1 satisfies the first relation in (32), because these properties characterize φ_1. The first two properties of φ_1 are obvious. From (70) one has

$$\varphi_1(x(u))=(\beta_1(x)u-u\beta_1(x))e_1^{-1}=\beta_1(x)\varphi_1(u)+\varphi_1(x)e_1\beta_1(x)^{\rho}e_1^{-1},$$

which proves the first relation in (32). q.e.d.

By an easy computation, one has

$$\frac{1}{2}(\varphi_1(u)\varphi_1(u')+\varphi_1(u')\varphi_1(u))=S(e_1, e_1)^{-1}(S(u, e_1)\varphi_1(u')+S(u'_1, e_1)\varphi_1(u)-S(u, u')).$$

This shows that the Jordan product in $U^{(1)}$ is given by

$$u\circ u'=S(e_1, e_1)^{-1}(S(u, e_1)u'+S(u', e_1)u-S(u, u')e_1).$$

It follows that the normalized inner product on $U^{(1)}$ is given by

$$\langle u, u'\rangle=2S(u, e_1)S(u', e_1)-S(u, u')S(e_1, e_1). \tag{74}$$

On the other hand, let c_1 be a primitive idempotent of $C^+(F)$ and ψ_1 an F-isomorphism: $D_0 \xrightarrow{\sim} c_1C^+(F)c_1$. Then the (D_0, ι_0)-η-hermitian form h_1 on $V_1=(C^+(F)c_1, \psi_1)$ is given by

$$h_1(v_1, v'_1)=\psi_1^{-1}(b_1e_1v_1^{\rho}e_1^{-1}v'_1) \qquad (v_1, v'_1\in V_1), \tag{75}$$

where b_1 is an element of $C^+(F)^{\times}$ such that

$$\psi_1(\xi_1)^{\iota_1}=b_1^{-1}\psi_1(\xi_1^{\iota_0})b_1, \qquad b_1^{\iota_1}=\eta b_1.$$

Finally to obtain an explicit form of $A^{(1)}$, let $\langle\ \rangle_{C^+}$ denote the inner product on C^+ defined by

$$\langle x, y\rangle_{C^+}=\mathrm{tr}_{C^+/\mathbf{R}}(x^{\iota_1}y).$$

For $x\in C^+$, let $[x]_U$ denote the element of $U^{(1)}$ such that $\varphi_1([x]_U)$ coincides with the $\varphi_1(U^{(1)})$-component of x with respect to the inner product $\langle\ \rangle_{C^+}$.

PROPOSITION 5. *Suppose we are in the quadratic case with $n_1\not\equiv 2 \pmod 4$. Let $(e^{(1)}, a^{(1)}, \beta^{(1)})$ be an admissible triple with $e^{(1)}=e_1$ defined over F belonging to $(U^{(1)}, V^{(1)}, \mathscr{C}^{(1)})$ and let h_1 and h_2 be as given in* (75) *and* (39). *Then the corresponding alternating bilinear map $A^{(1)}$: $V^{(1)}\times V^{(1)}\to U^{(1)}$ is given as follows*:

$$A^{(1)}(v_1\otimes_{D_0}v_2, v'_1\otimes_{D_0}v'_2)=\frac{1}{2}\eta v_1\delta_0 d_1[v_1\psi_1(h_2(v_2, v'_2))b_1v_1'^{\iota_1}]_U. \tag{76}$$

PROOF. For $u\in U^{(1)}(F)$, $v_1, v'_1\in V_1=C^+(F)c_1$, $v_2, v'_2\in V_2$, one has

$$
\begin{aligned}
A_u(v_1 \otimes_{D_0} v_2, v_1' \otimes_{D_0} v_2') &= a^{(1)}(v_1 \otimes_{D_0} v_2, (ue_1^{-1})v_1' \otimes_{D_0} v_2') \\
&= \mathrm{tr}_{D_0/F}(h_1(v_1, (ue_1^{-1})v_1')^{\iota_0} h_2(v_2, v_2')) \\
&= \mathrm{tr}_{D_0/F}(h_1(v_1 h_2(v_2, v_2'), (ue_1^{-1})v_1')) \\
&= \mathrm{tr}_{C^+/\boldsymbol{R}}(b_1 \psi_1(h_2(v_2, v_2'))^{\iota_1} v_1^{\iota_1} u e_1^{-1} v_1') \\
&= \langle ue_1^{-1}, \eta v_1 \psi_1(h_2(v_2, v_2')) b_1 v_1'^{\iota_1} \rangle_{C^+} \\
&= \frac{1}{2} \eta \nu_1 \delta_0 d_1 \langle u, [v_1 \psi_1(h_2(v_2, v_2')) b_1 v_1'^{\iota_1}]_U \rangle ,
\end{aligned}
$$

which proves our assertion. q.e.d.

5.5. *Classification.* In the classification theory, the domains $\mathscr{S}_I$ and $\mathfrak{S}$ in the present case are denoted as

$$
R_{F/\boldsymbol{Q}}(\mathrm{IV}_{n_1;\nu_2}, S^{(1)}, h_2)_I \quad (n_1 \geq 3, \not\equiv 2 \,(\mathrm{mod}\, 4)), \quad R_{F/\boldsymbol{Q}}\mathfrak{S}(V_2, D_0, h_2) .
$$

(When D_0 is of Type 1.1, i.e., when $D_0 = F$, one omits h_2.)

The total space $\tilde{\mathscr{S}}$ is symmetric for the following three cases. For $n_1 = 3$, by virtue of the isomorphism $\mathscr{P}(1, 2) \simeq \mathscr{P}_2(\boldsymbol{R})$, the domain $R_{F/\boldsymbol{Q}}(\mathrm{IV}_{3;\nu_2}, S^{(1)}, h_2)_I$ is identified with $R_{F/\boldsymbol{Q}}(\mathrm{III}^{(1)}_{2,\nu_2/2})_I$ or $R_{F/\boldsymbol{Q}}(\mathrm{III}^{(2)}_{2,\nu_2}, D_0, h_2)_I$ $(D_0 = C^+(F))$ according as $D_0 = F$ or not. Hence the corresponding $\tilde{\mathscr{S}}$ is the Siegel domain expression of $R_{F/\boldsymbol{Q}}(\mathrm{III}^{(\delta_0)}_{2+\nu_2\delta_0/2})$ over the $2/\delta_0$-th rational boundary component $\mathfrak{S} = R_{F/\boldsymbol{Q}}(\mathrm{III}^{(\delta_0)}_{\nu_2\delta_0/2})$. For $n_1 = 4$, by virtue of the isomorphism $\mathscr{P}(1, 3) \simeq \mathscr{P}_2(\boldsymbol{C})$, the domain $R_{F/\boldsymbol{Q}}(\mathrm{IV}_{4;\nu_2}, S^{(1)}, h_2)_I$ is identified with $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{2;(p,q)}, D_0/Z, h_2)_I$ $(D_0 = C^+(F), Z = F(\sqrt{\Delta}), p+q = \delta_0\nu_2)$, so that the corresponding $\tilde{\mathscr{S}}$ is the Siegel domain expression of $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{2+p,2+q}, D_0/Z, h_2')$ over the $2/\delta_0$-th boundary component $\mathfrak{S} = R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{p,q}, D_0/Z, h_2)$. In particular, $R_{F/\boldsymbol{Q}}(\mathrm{IV}_{4;\nu_2}, S^{(1)}, h_2)_I$ *with* $q = 0$ *is identified with the symmetric domain* $R_{F/\boldsymbol{Q}}(\mathrm{I}^{(\delta_0)}_{2+\nu_2\delta_0,2}, D_0/Z, h_2')$. In the case $R_{F/\boldsymbol{Q}}(\mathrm{IV}_{8;1}, S^{(1)}, h_2)_I$, the domain $\mathfrak{S}$ reduces to a point I $(I = \sum |\Delta^{\sigma_i}|^{-1/2} \tilde{e}^{(i)})$ and $\tilde{\mathscr{S}} = \mathscr{S}_I$ *is a symmetric domain of the exceptional type* $(\mathrm{V})^l$ with a $\boldsymbol{Q}$-structure of $\boldsymbol{Q}$-rank 2.

5.6. *The case* $n_1 \equiv 2$ (mod 4). In this case, there exist two $\boldsymbol{R}$-irreducible (spin) representations of $\mathfrak{g}_1^{(1)}$. Let π_i denote the projection $C^+ \to C_i^+$ and κ_i the $\boldsymbol{R}$-irreducible representation of C_i^+ $(i = 1, 2)$. Define the injective homomorphism $\beta_1 : \mathfrak{g}_1^{(1)} \to C^+$ as in 5.3. Then the two spin representations of $\mathfrak{g}_1^{(1)}$ are given by $\kappa_i \circ \pi_i \circ \beta_1$ $(i = 1, 2)$. In general, the representation $(V^{(1)}, \beta^{(1)})$ has two $\boldsymbol{R}$-primary components corresponding to these $\boldsymbol{R}$-irreducible representations.

Let $\mathscr{A}_1$ denote the enveloping algebra of $\beta^{(1)}(\mathfrak{g}_1^{(1)})$ in $\mathrm{End}_{\boldsymbol{R}} V^{(1)}$. Then there exists a uniquely determined (algebra) homomorphism $\lambda : C^+ \to \mathscr{A}_1$ such that one has $\beta^{(1)} = \lambda \circ \beta_1$. Suppose that the F-structure of $(U^{(1)}, S^{(1)})$ is extended to an admissible F-structure of $(U^{(1)}, V^{(1)})$ (under the condition similar to the condition (c) in Theorem 1). Then C^+ and $\mathscr{A}_1$ have natural F-algebra structures such that β_1 and λ are defined over F.

When $\Delta \nsim 1$ over F, the F-algebra $C^+(F)$ is F-simple, and λ gives an F-isomorphism $C^+(F) \simeq \mathscr{A}_1(F)$. The center Z of $\mathscr{A}_1(F)$ is a totally real quadratic extension of F, isomorphic to $F(\sqrt{\Delta})$. Hence $\beta^{(1)}$ is F-primary, but *not* $\boldsymbol{R}$-primary, and we obtain a result similar to the one given in §3 with some modifications. For instance, (27a), (26a) must be modified in the form:

$$V^{(1)}(F) = R_{Z/F}(V_1 \otimes_{D_0} V_2)\,,$$
$$V^{(1)} = V_1^{(1)\prime} \otimes_{D_1} V_2^{(1)\prime} \oplus V_1^{(1)\prime\prime} \otimes_{D_1} V_2^{(1)\prime\prime}\,,$$

where V_1, $V_1^{(1)\prime}$, and $V_1^{(1)\prime\prime}$ are simple left ideals of $C^+(F)$, C_1^+, and C_2^+, respectively. In this case, $v_1\delta_0 = 2^{n_1/2-1}$, and one has

$$\dim_{\boldsymbol{R}} V_j^{(i)\prime} = \dim_{\boldsymbol{R}} V_j^{(i)\prime\prime} = v_j s_1 \delta_1^2\,,$$
$$\dim_{\boldsymbol{R}} V^{(i)} = 2 v_1 v_2 \delta_0^2\,.$$

In the classification theory, the domains $\mathscr{S}_I$ and $\mathfrak{S}$ are denoted as

$$R_{F(\sqrt{\Delta})/\boldsymbol{Q}}(\mathrm{IV}_{n_1;\,v_2,v_2}, S^{(1)}, h_2)_I \quad (n_1 \geq 6,\ \equiv 2(4))\,,$$
$$R_{F(\sqrt{\Delta})/\boldsymbol{Q}}\mathfrak{S}(V_2, D_0, h_2)\,.$$

When $\Delta \sim 1$ over F, $C^+(F)$ is decomposed as (67), in which each simple component $C_i^+(F)$ is invariant under ρ'. Hence one has either $\mathscr{A}_1(F) \simeq C^+(F)$ or $C_i^+(F)$ $(i=1, 2)$, according as $\beta^{(1)}$ has two or one F-primary component(s). For each F-primary component (which is also $\boldsymbol{R}$-primary) one has formulas similar to the ones given in the F-primary case, replacing β_1, φ_1 by $\pi_i \circ \beta_1$, $\pi_i \circ \varphi_1$. Thus in this case, (27a), (26a) should be modified as follows:

$$V^{(1)}(F) = V_1' \otimes_{D_0} V_2' \oplus V_1'' \otimes_{D_0} V_2''\,,$$
$$V^{(1)} = V_1^{(1)\prime} \otimes_{D_1} V_2^{(1)\prime} \oplus V_1^{(1)\prime\prime} \otimes_{D_1} V_2^{(1)\prime\prime}\,,$$

V_1', V_1'', $V_1^{(1)\prime}$, and $V_1^{(1)\prime\prime}$ being simple left ideals of $C_1^+(F)$, $C_2^+(F)$, C_1^+, and C_2^+, respectively. Denoting the ranks of D_0-modules V_j' and V_j'' $(i=1, 2)$ by v_j' and v_j'', one has

$$v_1' = v_1'' = 2^{n_1/2-1}\delta_0^{-1}\,, \quad v_2',\ v_2'' \geq 0\,,$$

and

$$\dim_{\boldsymbol{R}} V_j^{(i)\prime} = v_j' s_1 \delta_1^2\,, \quad \dim_{\boldsymbol{R}} V_j^{(i)\prime\prime} = v_j'' s_1 \delta_1^2\,,$$
$$\dim_{\boldsymbol{R}} V^{(i)} = v_1'(v_2' + v_2'')\delta_0^2\,.$$

In this case, the domains $\mathscr{S}_I$ and $\mathfrak{S}$ are denoted as

$$R_{F/\boldsymbol{Q}}(\mathrm{IV}_{n_1;\,v_2',v_2''}, S^{(1)}, h_2', h_2'')_I \qquad (n_1 \geq 6,\ \equiv 2(4))\,,$$
$$R_{F/\boldsymbol{Q}}\mathfrak{S}(V_2', D_0, h_2') \times R_{F/\boldsymbol{Q}}\mathfrak{S}(V_2'', D_0, h_2'')\,.$$

[One may choose the orientation of $U^{(1)}$ so that $\nu_2' \geq \nu_2''$ and, when $\nu_2''=0$, one omits the second factor $R_{F/\boldsymbol{Q}}\mathfrak{S}(V_2'', D_0, h_2'')$.]

In general, if p is a point in the *second* boundary component of an irreducible symmetric doamin, then the fiber over p is an irreducible quasisymmetric domain of type $(\mathrm{IV}_{n_1;\nu_2})$ or $(\mathrm{IV}_{n_1;\nu_2,0})$. Thus, for $n_1=6$, by virtue of the isomorphism $\mathscr{P}(1,5)\simeq\mathscr{P}_2(\boldsymbol{H})$, the domain $R_{F/\boldsymbol{Q}}(\mathrm{IV}_{6;\nu_2,0}, S^{(1)}, h_2)_I$ $(\Delta\sim 1)$ is identified (through the first spin representation) with the fiber over a rational point I in the second rational boundary component $\mathfrak{S}=R_{F/\boldsymbol{Q}}(\mathrm{II}_{\nu_2}^{(2)}, D_0, h_2)$ in the Siegel domain expression of $\tilde{\mathscr{S}}=R_{F/\boldsymbol{Q}}(\mathrm{II}_{4+\nu_2}^{(2)}, D_0, h_2')$, where $D_0=C_1^+(F)$ is a totally definite quaternion algebra over F. In particular, $R_{F/\boldsymbol{Q}}(\mathrm{IV}_{6;1,0}, S^{(1)}, h_2)$ *is identified with the symmetric domain* $R_{F/\boldsymbol{Q}}(\mathrm{II}_5^{(2)}, D_0, h_2')$. For $n_1=10$, the domain $R_{F/\boldsymbol{Q}}(\mathrm{IV}_{10;2/\delta_0,0}, S^{(1)}, h_2)_I$ $(\Delta\sim 1)$ is identified with the fiber over a rational point I in the second rational boundary component $\mathfrak{S}=R_{F/\boldsymbol{Q}}(\mathrm{III}_1^{(\delta_0)}, D_0, h_2)$ in the Siegel domain expression of a symmetric domain of the *exceptional* type $(\mathrm{VI})^l$ with a $\boldsymbol{Q}$-structure of $\boldsymbol{Q}$-rank $1+2/\delta_0$.

Appendix: The symmetric case.

A.1. *The condition* (iii). First we introduce some notation. For $v, v' \in V$, set

$$\varphi H_I(v, v') = \varphi(A(v, v'))I + \varphi(A(v, Iv')) . \tag{76}$$

Then one has

$$I\cdot\varphi H_I(v, v') = -\varphi H_I(Iv, v') = \varphi H_I(v, Iv') = \varphi H_I(v, v')I .$$

Thus $\varphi H_I(v, v')$ is $\boldsymbol{C}$-linear in v' and $\boldsymbol{C}$-semilinear in v with respect to the complex structure of V defined by I. It follows that one has

$$\varphi H_I(v, v')v'' = 2i(\varphi(A(v_-, v'_+))v''_+ - \varphi(A(v_+, v'_-))v''_-) . \tag{77}$$

Moreover, for $g_2 \in G_2$, one has

$$g_2^{-1}\varphi H_I(g_2 v, g_2 v')g_2 = \varphi H_{g_2^{-1}Ig_2}(v, v') . \tag{78}$$

The following result is known (cf. [S6, p. 223–224, Th. 3.5]).

PROPOSITION 6. *A quasisymmetric domain $\mathscr{S}_I$ is symmetric if and only if the following condition is satisfied*:

$$A(v, \varphi H_I(v', v'')v'') = A(\varphi H_I(v'', v)v', v'') \qquad \text{for} \quad v, v', v'' \in V , \tag{iii}$$

or equivalently,

$$A(\bar{w}, \varphi(A(\bar{w}', w''))w'') = A(\varphi(A(\bar{w}, w'')\bar{w}', w'') \qquad \text{for} \quad w, w', w'' \in V_+ . \tag{iii'}$$

COROLLARY. *If $\mathscr{S}_I$ is symmetric for one $I \in \mathfrak{S}$, then $\mathscr{S}_I$ is symmetric for all $I \in \mathfrak{S}$.*

This follows from Proposition 9 and (78).

REMARK. It is known ([S6, p. 228, Lem. 4.6]) that (iii) is equivalent to any one of the following conditions.

$$\text{(iii}_1\text{)} \qquad \varphi H_I(v, \varphi(u)v')v' = \varphi(u)\varphi H_I(v, v')v' ,$$

$$\text{(iii}'_1\text{)} \qquad \varphi H_I(\varphi(u)v, v')v' = \varphi H_I(v, v')\varphi(u)v'$$

$$(v, v' \in V, u \in U) .$$

By the classification, we see that an irreducible domain $\mathscr{S}_I$ is symmetric if and only if either one has $\mathfrak{g}_1 = \{1_U\}_{\boldsymbol{R}}$ or $\mathfrak{g}_2$ is compact. Note that there are some discrepancy of the notation between this paper and [S6, Ch. V]. In the latter, the complex structure I on V is fixed, so that (V, I) is identified with V_+. One has the following dictionary (on the left hand side is the notation in [S6]):

$$4H(v, v') = A(v, Iv') + iA(v, v') , \qquad 2R_u = \varphi(u) ,$$

$$8R(H(v, v'))(\text{on } V_+) = \varphi H_I(v, v')(\text{on } V_+) = 2i\varphi(A(v_-, v'_+)) .$$

A.2. *Infinitesimal automorphisms of* $\mathscr{S}_I$. Let $\operatorname{Aut}\mathscr{S}_I$ denote the group of biholomorphic automorphisms of $\mathscr{S}_I$ and let $\mathfrak{G} = \operatorname{Lie}\operatorname{Aut}\mathscr{S}_I$. Then $X \in \mathfrak{G}$ can be expressed by the corresponding "infinitesimal automorphism" of $\mathscr{S}_I$, i.e. the differential operator $\tilde{X}$ on $C^\infty(\mathscr{S}_I)$ defined by

$$(\tilde{X}f)(u, w) = \frac{d}{dt} f(\exp(tX)^{-1}(u, w))\big|_{t=0} ;$$

in notation, we write $X \leftrightarrow \tilde{X}$. Let (e_α) and (e'_λ) be bases of U_C and V_+ over $\boldsymbol{C}$, respectively, and let (u_α) and (w_λ) the corresponding complex coordinates of U_C and V_+. Then $\tilde{X}$ is expressed in the form

$$\tilde{X} = \sum_{\alpha=1}^{n} p_\alpha(u, w) \frac{\partial}{\partial u_\alpha} + \sum_{\lambda=1}^{m} q_\lambda(u, w) \frac{\partial}{\partial w_\lambda} . \tag{79}$$

Setting $p(u, w) = \sum_{\alpha=1}^{n} p_\alpha(u, w)e_\alpha$, $q(u, w) = \sum_{\lambda=1}^{m} q_\lambda(u, w)e'_\lambda$, we write

$$\tilde{X} = p(u, w)\frac{\partial}{\partial u} + q(u, w)\frac{\partial}{\partial w} .$$

First, for the Heisenberg group $\tilde{V}$, the Lie algebra $\operatorname{Lie}\tilde{V}$ is naturally identified with $U \oplus V$ (as a vector space). Viewing $\operatorname{Lie}\tilde{V}$ as a subalgebra of $\mathfrak{G}$, one has by (7)

$$a + b \leftrightarrow -(a - A(b_-, w))\frac{\partial}{\partial u} - b_+\frac{\partial}{\partial w} \qquad (a \in U, b \in V) . \tag{80}$$

Clearly one has

$$[a + b, a' + b'] = -A(b, b') \qquad (a, a' \in U, b, b' \in V) . \tag{81}$$

For the linear group G_I, one embeds $\mathrm{Lie}\, G_I = \mathfrak{g}_1 \oplus \mathfrak{k}_2$ into $\mathfrak{gl}(U) \times \mathfrak{gl}(V)$. Then, for $(X_1, Y_1) \in \mathrm{Lie}\, G_I$, one has

$$(X_1, Y_1) \leftrightarrow -X_1 u \frac{\partial}{\partial u} - Y_1 w \frac{\partial}{\partial w}. \tag{82}$$

Clearly one has

$$[(X_1, Y_1), a+b] = X_1 a + Y_1 b,$$
$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2]).$$

When $\mathscr{S}_I$ is symmetric, let θ be the Cartan involution of $\mathfrak{G}$ at $(ie, 0) \in \mathscr{S}_I$. Then one has a gradation of $\mathfrak{G}$ according to $\mathrm{ad}(-1_U, (-1/2)1_V)$ of the following form:

$$\mathfrak{G} = \sum_{\nu=-2}^{2} \mathfrak{G}_{\nu/2}, \qquad \theta \mathfrak{G}_{\nu/2} = \mathfrak{G}_{-\nu/2}. \tag{83}$$

$$\mathfrak{G}_{-1} = U, \quad \mathfrak{G}_{-1/2} = V, \quad \mathfrak{G}_0 = \mathrm{Lie}\, G_I = \mathfrak{g}_1 \oplus \mathfrak{k}_2,$$

and θ induces the Cartan involution $\theta_1 \oplus \theta_2$ on $\mathfrak{G}_0$ (cf. [M], [S6, p. 211, (A), p. 220, Prop. 3.3]). In order to describe the action of θ on U, V, it is convenient to use the following notation:

$$(u \square u')u'' = \{u, u', u''\} = (uu')u'' + u(u'u'') - u'(uu''),$$
$$u \square u' = T_{uu'} + [T_u, T_{u'}].$$

By (18) and (19) one has

$$\varphi(\{u, u', u''\}) = \frac{1}{2}(\varphi(u)\varphi(u')\varphi(u'') + \varphi(u'')\varphi(u')\varphi(u)), \tag{84}$$

$$\{u, A(v, v'), u'\} = \frac{1}{2}(A(\varphi(u)v, \varphi(u')v') + A(\varphi(u')v, \varphi(u)v')). \tag{85}$$

PROPOSITION 7. *One has*

$$\theta a \leftrightarrow -\{u, a, u\} \frac{\partial}{\partial u} - \varphi(u)\varphi(a) w \frac{\partial}{\partial w}, \tag{86}$$

$$\theta b \leftrightarrow -iA(\varphi(u)b_-, w) \frac{\partial}{\partial u} - i(\varphi(u)b_+ + \varphi(A(b_-, w))w) \frac{\partial}{\partial w}. \tag{87}$$

This was given in [S6, p. 224, Th. 3.6]. A more direct proof can be given as follows. The symmetry at $(ie, 0)$, denoted also by θ, is given by

$$\theta: (u, w) \mapsto (-u^{-1}, -i\varphi(u)^{-1} w),$$

where u^{-1} denotes the inverse of u in the Jordan algebra (U, e) and one has

$\varphi(u^{-1})=\varphi(u)^{-1}$ (cf. [S6, p. 139, Exc. 3]). Hence, for $a\in U$, one has

$$(\exp\theta a)(u,w)=(\theta\circ(\exp a)\circ\theta)(u,w)=\theta(-u^{-1}+a,\ -i\varphi(u)^{-1}w)$$
$$=((u^{-1}-a)^{-1},\ \varphi(u^{-1}-a)^{-1}\varphi(u)^{-1}w)\,.$$

Here one has

$$(u^{-1}-a)^{-1}=(1-u\Box a)^{-1}u=u-\{u,a,u\}+\cdots$$
$$\varphi(u^{-1}-a)^{-1}\varphi(u)^{-1}=1-\varphi(u)\varphi(a)+\cdots$$

([S6, p. 26, Exc. 6] and (84)). Hence one obtains (86). The relation (87) is obtained similarly by using (iii_1), (77), (85).

By direct computations from (80), (86) and (87) one obtains

$$[a,\theta a']=(-2a\Box a',\ -\varphi(a)\varphi(a'))\,, \tag{88}$$

$$[a,\theta b]=-\varphi(a)Ib\,, \tag{89}$$

$$[b,\theta b']=(-4\Phi_{b,b'},\ -4\Psi_{b,b'})\,, \tag{90}$$

where

$$4\Phi_{b,b'}:\ u\mapsto A(b,\varphi(u)Ib')\,,$$

$$4\Psi_{b,b'}:\ v\mapsto\frac{1}{2}(\varphi H_I(b',v)b-\varphi H_I(b,v)b'+\varphi H_I(b',b)v)\,.$$

(For (90) one uses (iii′). Cf. [S6, p. 231–233, Exc. 5 and Rem.])

A.3. ***Q**-structures of* $\mathfrak{G}$. Now we assume that there is given a $\boldsymbol{Q}$-structure of the quasisymmetric domain $\mathscr{S}_I$ in the sense of 3.1. This means that one has a $\boldsymbol{Q}$-structure of $\mathfrak{G}_{\mathrm{Aff}}=\mathfrak{G}_{-1}+\mathfrak{G}_{-1/2}+\mathfrak{G}_0$ such that $(1_U,(1/2)1_V)\in\mathfrak{g}_1$ is $\boldsymbol{Q}$-rational. Then, since $I\in\mathfrak{S}$ is "rational", there exists a totally positive element $\alpha_1\in F$ such that $\sum_{i=1}^{l}\sqrt{\alpha_1^{\sigma_i}}I^{(i)}$ is $\boldsymbol{Q}$-rational. [We say that I is a rational point with CM-field $F(\sqrt{-\alpha_1})$, endowed with the standard CM-type (σ_i') defined by $\sqrt{-\alpha_1}^{\sigma_i'}=\sqrt{-1}\sqrt{\alpha_1^{\sigma_i}}$.] In what follows, for $\lambda_i\in\boldsymbol{R}$ $(1\le i\le l)$ and $x=\sum x^{(i)}$, we write

$$(\lambda_i)\cdot x=\sum_{i=1}^{l}\lambda_i x^{(i)}\,.$$

In this section, we don't assume that e is $\boldsymbol{Q}$-rational. e is called *semirational* if there exists a totally positive element $\alpha\in F$ such that $(\sqrt{\alpha^{\sigma_i}})\cdot e$ is $\boldsymbol{Q}$-rational. We say that e or θ is *compatible with* the complex structure I if $(\sqrt{\alpha_1^{\sigma_i}})\cdot e$ is $\boldsymbol{Q}$-rational.

LEMMA. *Let e, $e'\in U$, $e'=(\lambda_i)\cdot e$ and denote the symbols relative to e' by the corresponding symbols relative to e with a prime. Then one has*

$$T'_a = (\lambda_i)^{-1} \cdot T_a\,, \quad \varphi'(a) = (\lambda_i)^{-1} \cdot \varphi(a)\,,$$
$$\{u, u', u''\}' = (\lambda_i)^{-2} \cdot \{u, u', u''\}\,,$$
$$\theta' a = (\lambda_i)^{-2} \cdot \theta a\,, \quad \theta' b = (\lambda_i)^{-1} \cdot \theta b$$

for $a, u, u', u'' \in U$, $b \in V$.

The proof is straightforward.

THEOREM 3. *Assume that* $\mathcal{S}_I$ *is symmetric and let* θ *be the Cartan involution of* $\mathfrak{G}$ *at* $(ie, 0) \in \mathcal{S}_I$. *Then, there exists a unique* $\boldsymbol{Q}$*-structure of* $\mathfrak{G}$ *satisfying the following conditions*:
(α) *It extends the given* $\boldsymbol{Q}$*-structure of* $\mathfrak{G}_{\mathrm{Aff}}$.
(β) *Whenever e is semirational, the restriction* $\theta | U$ *is* $\boldsymbol{Q}$*-rational.*
The Cartan involution θ *is* $\boldsymbol{Q}$*-rational with respect to this* $\boldsymbol{Q}$*-structure of* $\mathfrak{G}$ *if and only if* θ *is compatible with I.*

PROOF. First we prove the uniqueness in the first statement. Suppose one has a $\boldsymbol{Q}$-structure of $\mathfrak{G}$ satisfying the conditions (α), (β). (Note that, by the above lemma, the condition (β) is satisfied if $\theta | U$ is $\boldsymbol{Q}$-rational for one semirational e.) Then the $\boldsymbol{Q}$-structures on the vector spaces $\mathfrak{G}_{\nu/2}$ are uniquely determined except for $\nu = 1$. As for $\mathfrak{G}_{1/2} = \theta V$, one has by (89)

$$\theta I b = -[\theta e, b] \qquad (b \in V)\,.$$

Hence, if $(\sqrt{\alpha^{\sigma_i}}) \cdot e$ is $\boldsymbol{Q}$-rational, then the map $b \mapsto (\sqrt{\alpha^{\sigma_i}}) \cdot \theta I b$ is $\boldsymbol{Q}$-rational. By this condition, which is independent of the choice of the semirational e by the above lemma, the $\boldsymbol{Q}$-structure of $\mathfrak{G}_{1/2}$ is also uniquely determined. Conversely, by virtue of (88), (89), (90) and the above lemma, one sees that, defining the $\boldsymbol{Q}$-structure of $\mathfrak{G}_{1/2}$ and $\mathfrak{G}_1$ as indicated above, one obtains a $\boldsymbol{Q}$-structure of $\mathfrak{G}$ satisfying the conditions (α), (β). From this and the definition the second statement is clear. q.e.d.

REMARK. The above theorem remains valid for the case $V = 0$. In that case, any Cartan involution with semirational e is $\boldsymbol{Q}$-rational.

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